Abstracts

Workshop on CR and Sasakian Geometry

University of Luxembourg, 24–26 March 2009

Dmitri Alekseevsky: "Para-CR structures and related structures"

A para-CR structure is a para-complex analogue of a CR structure. It is defined as a distribution H on a manifold M together with a para-complex structure K on H, i.e. a field of endomorphisms K such that $K^2 = \mathrm{Id}$ and the eigendistributions $H^{\pm}$ of K are involutive.

Many notions and results of CR geometry remain valid in para-CR case. We present a survey of basic facts of para-CR geometry. A description of maximally homogeneous para-CR manifolds of semisimple type will be given. We consider also some structures subordinated to para-CR structure, for example, quaternionic para-CR structure, which is a para-analogue of 3-Sasakian structure, and pseudo-conformal quaternionic para-CR structure and describe their relations with pseudo-hyperKähler structure and pseudo-quaternionic Kähler structure.

An interesting special case of para-CR structures consists of non degenerate codimension one para-CR structures. Such structure can be defined as a decomposition $H = H^+ + H^-$ of a contact distribution H into direct sum of two integrable Lagrangian subdistributions. We discuss relations of such structures with second order ODE discovered by P. Nurowski and G.A.J.Sparling and to parabolic Monge-Ampere equations.

Jesse Alt: *TBA*

Mohamed Belkhelfa: "Some intrinsic and extrinsic symmetries on generalized Sasakian space forms"

After an overview of some extrinsic symmetric properties of generalized Sasakian space forms, we give a result on the action of Weyl tensor and projective curvature on Riemannian curvature and Ricci curvature respectively and an intrinsic symmetry of a hypersurface in a generalized Sasakian space form.

Beniamino Cappelletti Montano: "The foliated structure of contact metric (κ, μ)-spaces"

A contact metric (κ, μ)-space is a contact metric manifold $(M, \varphi, \xi, \eta, g)$ such that the curvature tensor field satisfies, for all $X, Y \in \Gamma(TM)$,

$$R_{XY}\xi = \kappa\left(\eta\left(Y\right)X - \eta\left(X\right)Y\right) + \mu\left(\eta\left(Y\right)hX - \eta\left(X\right)hY\right), \qquad (1)$$

for some real numbers κ and μ, where $2h$ denotes the Lie derivative of φ in the direction of ξ. This definition, introduced by Blair, Kouforgiorgos and Papantoniou ([3]), can be regarded as a generalization both of the Sasakian condition $R_{XY}\xi = \eta\left(Y\right)X - \eta\left(X\right)Y$ and of those contact metric manifolds verifying $R_{XY}\xi = 0$ which were studied by D. E. Blair in [1].

One of the peculiarities of contact metric (κ, μ)-spaces is that they are foliated by three mutually orthogonal foliations $\mathcal{D}(\lambda)$, $\mathcal{D}(-\lambda)$ and $\mathbb{R}\xi = \mathcal{D}(0)$, corresponding to the eigenspaces λ, $-\lambda$ and 0 of the operator h, where $\lambda = \sqrt{1-\kappa}$. In particular $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ define two transverse Legendre foliations of M so that any contact metric (κ, μ)-manifold is canonically endowed with a *bi-Legendrian structure*.

In this talk some recent results ([8], [9], [10]) concerning the canonical bi-Legendrian structure of a contact metric (κ, μ)-space will be presented. More in particular we show how the geometry of the bi-Legendrian structure $(\mathcal{D}(\lambda), \mathcal{D}(-\lambda))$ characterizes, in some sense, the contact metric (κ, μ)-space in question. Using this foliated approach, we are able to find conditions ensuring the existence of a compatible contact metric (κ, μ)-structure on a contact manifold (M, η) and to prove that any contact metric (κ, μ)-space M whose Boeckx invariant I_M ([4]) verifies $|I_M| < 1$ admits a compatible Sasakian structure and any contact metric (κ, μ)-space such that $|I_M| > 1$ admits a compatible Tanaka-Webster parallel structure (cf. [5]). Furthermore, the relations between contact metric (κ, μ)-spaces and paracontact geometry will be illustrated.

References

[1] D. E. Blair, *Two remarks on contact metric structures*, Tôhoku Math. J. **28** (1976), 373–379.

[2] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, 203. Birkhäuser Boston, Inc., Boston, MA, 2002.

[3] D. E. Blair, T. Koufogiorgos, B. J. Papantoniou, *Contact metric manifolds satisfyng a nullity condition*, Israel J. Math. **91** (1995), 189–214.

[4] E. Boeckx, *A full classification of contact metric (κ, μ)-spaces*, Illinois J. Math. **44** (2000), 212–219.

[5] E. Boeckx, J. T. Cho, *Pseudo-Hermitian symmetries*, Israel J. Math. **166** (2008), 125–145.

[6] B. Cappelletti Montano, *Some remarks on the generalized Tanaka-Webster connection of a contact metric manifold*, Rocky Mountain J. Math., to appear.

[7] B. Cappelletti Montano, *Bi-Legendrian structures and paracontact geometry*, Int. J. Geom. Meth. Mod. Phys., to appear.

[8] B. Cappelletti Montano, L. Di Terlizzi, *Contact metric (κ, μ)-spaces as bi-Legendrian manifolds*, Bull. Austral. Math. Soc. **77** (2008), 373–386.

[9] B. Cappelletti Montano, L. Di Terlizzi, *The structure of contact metric (κ, μ)-spaces*, submitted.

[10] B. Cappelletti Montano, L. Di Terlizzi, M. M. Tripathi, *Invariant submanifolds of contact (κ, μ)-manifolds*, Glasgow Math. J. **50** (2008), 499–507.

Liana David: "The geometry of Bochner-flat Kähler manifolds and its interplay with Sasaki geometry"

The Bochner tensor of a Kähler manifold is the "biggest" irreducible component of the curvature tensor under the action of the unitary group. In this talk I will explain the interactions between the local geometry of Bochner-flat Kähler manifolds and Sasaki geometry, using the Webster's correspondence and a generalized Kähler cone construction. Using the Wester's correspondence, I will prove that locally, there are four types of Bochner-flat Kähler manifolds and I will briefly describe their geometry. As opposed to the Webster's correspondence, the usual Kähler cone construction is not well suited for Bochner-flat geometry: Kähler cones over Sasaki manifolds are "almost never" Bochner-flat. With this motivation, I will develop an alternative "generalized cone construction", which provides a Kähler structure on (an open subset of) the cone over a CR manifold, endowed with a family of compatible Sasaki-Reeb vector fields. I will prove that any Bochner-flat

Kähler manifold of complex dimension bigger than two is locally isomorphic to a Bochner-flat generalized Kähler cone.

Franki Dillen: "Lagrangian submanifolds of $\mathbb{CP}^n$"

Giulia Dileo: "A classification of spherical symmetric CR manifolds"

Using the associated contact metric structure [1], we characterize the strongly pseudoconvex CR manifolds of hypersurface type whose Webster metric is symmetric in the sense of the definition given by Kaup and Zaitsev in [6]. As a consequence, we obtain a classification of the simply connected, symmetric pseudohermitian manifolds, which are spherical CR-manifolds, i.e. the Chern-Moser-Tanaka pseudoconformal invariant tensor vanishes ([3], [8]).

In the non-Sasakian case the Webster metric of a pseudohermitian manifold is locally symmetric if and only if the underlying contact metric structure satisfies the (κ, μ)-nullity condition for some real numbers κ, μ [2]. Furthermore, the manifold is spherical if and only if it is a pseudohermitian space form, or equivalently $\mu = 2$, which also characterizes the vanishing of the Webster scalar curvature. In this case the manifold is locally homothetic to the tangent sphere bundle $T_1\mathbb{H}^{n+1}$ of the Riemannian space form of curvature -1.

In the Sasakian case, the Webster metric is symmetric if and only if the pseudohermitian manifold is a φ-symmetric space. Hence, if M is symply connected, then it is homothetic to a Sasakian space form, which fibers over a Kähler space form, or to a principal fiber bundle over the Riemannian product of two Kähler space forms with holomorphic curvatures 1 and -1.

References

[1] D.E. Blair: Riemannian geometry of contact and symplectic manifolds. Progress in Mathematics **203**, Birkhäuser, Boston, 2002.

[2] E. Boeckx: A full classification of contact metric (k, μ)-spaces, *Illinois J. Math.* **44** (2000), no. 1, 212–219.

[3] S.S. Chern, J.K. Moser: Real hypersurfaces in complex manifolds, *Acta Math.* **133** (1974), 219–271.

[4] G. Dileo, A. Lotta: A classification of spherical symmetric CR manifolds, *Bull. Aust. Math. Soc.*, to appear.

[5] J.A. Jiménez, O. Kowalski: The Classification of φ-symmetric Sasakian Manifolds, *Monatsh. Math.* **115** (1993), 83–98.

[6] W. Kaup, D. Zaitsev: On symmetric Cauchy-Riemann manifolds, *Adv. Math.* **149** (2000), 145–181.

[7] M. Matsumoto, S. Tanno: Kählerian spaces with parallel or vanishing Bochner curvature tensor, *Tensor (N.S.)* **27** (1973), 291–294.

[8] K. Sakamoto, Y. Takemura: Curvature invariants of *CR*-manifolds, *Kodai Math. J.* **4** (1981), 251-265.

Gregor Fels: "Algebraic methods in CR-geometry"

Anna Fino: "Hypo-contact and Sasakian structures on Lie groups"

Any oriented hypersurface of a 6-dimensional Calabi-Yau manifold has a hypo structure in the sense of Conti and Salamon.

In the first part of the talk I will consider manifolds of dimension 5 with a hypo-contact structure, i.e. with a contact form arising from a hypo structure. A hypo-contact structure is in general not Sasakian; if it is Sasakian, then it has to be α-Einstein. I will show a classification of 5-dimensional hypo contact solvable Lie groups and how the hypo-contact structures can be extended in any dimension $2n+1$, by using differential forms and spinors.

In the second part of the talk I will present some general results on Sasakian Lie algebras, showing a classification of Sasakian Lie algebras of dimension 5 and determine which of them carry a Sasakian α-Einstein structure.

Alan Huckleberry: "Fibrations and globalizations of compact homogeneous CR-manifolds"

Jean-Jacques Loeb: "Complex and CR structures on compact Lie groups and homogeneous spaces"

In the talk, we give a geometric and unified method for the classification of complex and special CR structures on compact Lie groups. The existence of complex structures on compact Lie groups of even dimension is an old result of Samelson.

More recently, Charbonnel and Kalgui gave an algebraic classification of all invariant complex and CR structures for compact Lie groups.

I will also explain how the geometric method can be extended to homogeneous spaces and I will briefly discuss the noninvariant case.

Antonio Lotta: "Generalized pseudohermitian geometry"

A pseudohermitian structure on a strongly pseudoconvex CR manifold (M, HM, J) of hypersurface type was defined by S. Webster as a choice of a section η of the annihilator $H^oM \subset TM^*$ of the holomorphic bundle HM, whose scalar Levi form $\mathcal{L}_\eta$ is positive definite [6]. Each pseudohermitian manifold (M, HM, J, η) carries a canonical Riemannian metric g_η extending $\mathcal{L}_\eta$ in such a way that $HM^\perp$ coincides with the subbundle spanned by the Reeb vector field ξ of the contact form η. Webster showed that the equivalence problem for pseudohermitian manifolds can be canonically reduced to the equivalence of absolute parallelisms in spaces of dimension $(n+1)^2$ where n is the CR dimension. This is gained by attaching to each pseudohermitian manifold a canonical linear connection which parallelizes both J and the metric g_η. This connection was also treated independently by N. Tanaka, who gave an axiomatic characterization of it [5]. After these papers, pseudohermitian geometry was developed by many authors, see for instance [3], [1]. In [4], C. Stanton treated a more general situation, where arbitrary Levi metrics are studied, i.e. Riemannian metrics g on TM whose restriction to HM is a suitable Levi form $\mathcal{L}_\eta$, but $HM^\perp$ does not necessarily coincide with the subbundle spanned by the Reeb vector field. Also in this case the equivalence problem is solved by means of a distinguished linear connection parallelizing J and g.

In this talk, we shall present an approach to a more general kind of structures on almost CR manifolds that we call *generalized pseudohermitian structures*. Given an almost CR manifold (M, HM, J), such a structure is a pair (h, P) where h is a positive definite fiber metric h on HM compatible with J, and $P : TM \to TM$ is a smooth projector such that $Im(P) = HM$. Here we allow arbitrary CR codimension s and we do not require the formal integrability of J, so that generalized pseudohermitian manifolds include almost Hermitian manifolds ($s = 0$) and almost contact metric manifolds ($s = 1$). We shall show that to each generalized pseudohermitian structure one can associate a canonical linear connection on the holomorphic bundle HM which is invariant under equivalence. Some geometric features and examples of generalized psedohermitian structures will be considered, in particular concerning the notion of pseudoholomorphic sectional curvature. The basic formulas for pseudohermitian immersions will also be presented in the attempt to enlarge the theory of [2].

References

[1] S. Dragomir: A survey of pseudo-Hermitian geometry. Proceedings of the Workshop on Differential Geometry and Topology (Palermo, 1996). Rend. Circ. Mat. Palermo (2) Suppl. No. 49 (1997), 101–112.

[2] S. Dragomir: On pseudo-Hermitian immersions between strictly pseudoconvex CR manifolds. Amer. J. Math. 117 (1995), no. 1, 169–202.

[3] J.M. Lee: Pseudo-Einstein structures on CR manifolds. Amer. J. Math. 110 (1988), no. 1, 157–178.

[4] C. Stanton: Intrinsic connections for Levi metrics, Manuscripta Math. 75 (1992), no. 4, 349–364.

[5] N. Tanaka: On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, *Japan. J. Math.* **20** (1976), 131–190.

[6] S.M. Webster: Pseudo-Hermitian structures on a real hypersurface, *J. Differential Geom.* **13** (1978), no. 1, 25–41.

Stefano Marchiafava: "Twistorial properties of quaternionic maps and CR quaternionic manifolds"

We report on some recent results, in collaboration with S. Ianus, L. Ornea, R. Pantilie, concerning twistorial properties of quaternionic and CR quaternionic maps, and as an application we give a description of such maps in some specific case. The extension of the results to paraquaternionic case is also discussed.

Costantino Medori: "Parabolic CR manifolds"

Marian-Ioan Munteanu: "On the geometry of CR submanifolds"

Let (M, g) be a Riemannian manifold isometrically immersed in an almost Hermitian manifold $(\tilde{M}, \tilde{g}, \tilde{J})$. and denote by $T(M)$ its tangent bundle. Two important situations occur:

- $T_x(M)$ is invariant under the action of $\tilde{J}$, namely we have $\tilde{J}(T_x(M)) = T_x(M)$ for all $x \in M$; in this case M is called *complex* submanifold or *holomorphic* submanifold

- $T_x(M)$ is anti-invariant under the action of $\tilde{J}$, i.e. $\tilde{J}(T_x(M)) \subset T(M)^{\perp}_x$, for all $x \in M$; in this case M is know as a *totally real* submanifold.

In 1978 A. Bejancu [1] started a study of the geometry of a class of submanifolds (in an almost Hermitian manifold) situated between the two classes mentioned above. Such submanifolds were named *CR–submanifolds.* More precisely we have: *M is a CR-submanifold of an almost Hermitian manifold $(\tilde{M}, \tilde{g}, \tilde{J})$ if there exists a holomorphic distribution $\mathcal{D}$ on M, i.e. $\tilde{J}\mathcal{D}_x = \mathcal{D}_x$, $\forall x \in M$ and such that its orthogonal complement $\mathcal{D}^{\perp}$ is anti-invariant, namely $\tilde{J}\mathcal{D}^{\perp}_x \subset T(M)^{\perp}_x$, $\forall x \in M$.*

Denote by s the complex dimension of each fibre of $\mathcal{D}$ (supposed to be constant) and by q the real dimension of each fibre of $\mathcal{D}^{\perp}$. If

1. $q = 0$: the CR-submanifold becomes an holomorphic submanifold;

2. $s = 0$: the CR-submanifold becomes a totally real submanifold;

3. $q = \dim T_x(M)^{\perp}$: M is called a *generic* submanifold;

4. $s, q \neq 0$: M is called a proper CR-submanifold.

An example of proper generic CR-submanifold is furnished by any hypersurface in $\tilde{M}$.

The notion of CR-submanifold appears independently of the theory of CR-manifolds and it was a result of D.E. Blair and B.Y. Chen saying that CR-submanifolds $(M, \mathcal{D})$ posses an integrable CR-structure $T_{1,0}(M) = \{X - \sqrt{-1}\tilde{J}X \ : \ X \in \mathcal{D}\}$ (provided they are proper) [3].

The study of CR submanifolds was developed when the ambient space is (almost) Kähler, l.c.K. (locally conformal Kähler), quasi and nearly Kähler or quaternionic Kähler manifold.

Another line of thought, similar to that concerning Sasakian geometry as an odd dimensional version of Kählerian geometry led to the concept of a *contact CR-submanifold*, that is a submanifold M of an almost contact Riemannian manifold $(\widetilde{M}, (\phi, \xi, \widetilde{\eta}, \widetilde{g}))$ carrying an invariant distribution $\mathcal{D}$, i.e. $\phi_x \mathcal{D}_x \subseteq \mathcal{D}_x$, for any $x \in M$, such that the orthogonal complement $\mathcal{D}^{\perp}$ of $\mathcal{D}$ in $T(M)$ is anti-invariant, i.e. $\phi_x \mathcal{D}^{\perp}_x \subseteq T(M)^{\perp}_x$, for any $x \in M$. This notion was already used by A.Bejancu & N.Papaghiuc in [2] by using the terminology of semi-invariant submanifold. It is customary to require that ξ be tangent to M (cf. K.Yano & M.Kon, [7]), rather than normal which is too restrictive (by Prop. 1.1 in [7], p. 43, M must be anti-invariant, i.e. $\phi_x T_x(M) \subseteq T(M)^{\perp}_x$, $x \in M$).

Even that it seems to be introduced as a formal analogue to the notion of CR-submanifold, contact CR-submanifolds have a precise geometric meaning by using a result of S. Ianuş saying that *any normal almost contact Riemannian manifold is actually a CR-manifold* and the observation that a contact CR-submanifold is a CR-manifold.

Also the theory of contact CR-submanifolds was built up in different ambient spaces, namely trans-Sasakian, quasi-Sasakian, nearly trans-Sasakian and (locally conformal) cosymplectic manifolds.

The study of the integrability of the two distributions yields the notions of (contact) CR-product, (contact) CR-warped product, (contact) CR-doubly warped product, twisted and doubly twisted CR-products.

In 2001, B.Y. Chen gave a general inequality involving the length of the second fundamental form of a CR-warped product in a Kähler manifold as well as the embedding equations in the case in which the equality is satisfied. Since then, a large number of Chen's type inequalities are given for different kinds of CR-submanifolds and respectively contact CR-submanifolds.

References

[1] A.Bejancu, *CR submanifolds of a Kähler manifold. I*, Proc. Amer. Math. Soc., 69 (1978), 135-142; *CR submanifolds of a Kähler manifold. II*, Trans. Amer. Math. Soc., 250 (1979), 333-345.

[2] A.Bejancu & N.Papaghiuc, *Semi-invariant submanifolds of a Sasakian manifold ,* An. Şt. Univ. "Al.I.Cuza" Iaşi, Matem., 1(1981), 163-170.

[3] D.E. Blair & B-Y. Chen, *On CR submanifolds of Hermitian manifolds*, Israel J. Math., 34(1979), 353-363.

[4] B-Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math., 133 (2001) 3, 177-195; *Geometry of warped product CR-submanifolds in Kaehler manifolds. II*, ibid., 134 (2001) 2, 103-119.

[5] S. Ianus, *Sulle varietà di Cauchy-Riemann*, Rend. dell'Accad. Sci. Fis. Mat., Napoli, 39(1972), 191-195.

[6] M.I. Munteanu, *Warped product contact CR-submanifolds of Sasakian space forms*, Publ. Math. Debrecen 66 (2005) 1-2, 75120.

[7] K.Yano & M.Kon, *CR submanifolds of Kaehlerian and Sasakian manifolds*, Progress in Math., vol. 30, Birkhäuser, 1983.

Massimilano Pontecorvo: "On conformal Killing forms in quaternionic geometry"

In a recent work with L. David we found a new way to characterize quaternionic projective space in terms of existence of non-parallel conformal Killing 2-forms.

Sibel Sular: "Pseudoparallel submanifolds of Kenmotsu manifolds"

We study pseudoparallel, Ricci generalized pseudoparallel and 2-pseudoparallel submanifolds of Kenmotsu manifolds.

We have obtained the following results:

Theorem: Let M be a submanifold of a Kenmotsu manifold $\widetilde{M}$ tangent to ξ. Then M is pseudoparallel if and only if either M is totally geodesic or $L_\alpha = -1$ holds on M.

Corollary: Let M be a submanifold of a Kenmotsu manifold. Then M is semiparallel if and only if M is totally geodesic.

Theorem: Let M be a submanifold of a Kenmotsu manifold $\widetilde{M}$ tangent to ξ . Then M is 2-pseudoparallel if and only if either M is totally geodesic or $L_\alpha = -1$ holds on M.

Theorem: Let M be a submanifold of a Kenmotsu manifold $\widetilde{M}$ tangent to ξ. Then M is generalized Ricci-pseudoparallel if and only if either M is totally geodesic or $L_S = -\frac{1}{2n}$ holds on M.

References

[1] A. C. Asperti, G. A. Lobos and F. Mercuri, *Pseudo-paralel immersions in space forms*, Math. Contemp. 17 (1999), 59-70.

[2] A. C. Asperti, G. A. Lobos and F. Mercuri, *Pseudo-parallel immersions of a space forms*, Adv. Geom. 2(2002) 17 (1999), 57-71.

[3] J. Deprez, *Semi-parallel surfaces in the Euclidean space*, Journal of Geometry, 25(1985), 192-200.

[4] J. Deprez, *Semi-parallel hypersurfaces*, Rend. Sem. Mat. Univ. Politechn. Torino 44(1986), 303-316.

[5] R. Deszcz, L.Verstraelen and S.Yaprak, *Pseudosymmetric hypersurfaces in 4-dimensional space of constant curvature*, Bull. Ins. Math. Acad. Sinica, 22 (1994), 167-179.

[6] F. Dillen, *Semi-parallel hypersurfaces of a real space form*, Israel J. Math., 75, (1991), 193-202.

[7] H. Endo, *Certain submanifolds of contact metric manifolds*, Tensor (N.S.) 47 (1988), no. 2, 198–202.

[8] M. Kon, *Invariant submanifolds of normal contact metric manifolds*, Kodai Math. Sem. Rep. 25 (1973), 330–336.

[9] Ü. Lumiste, *Semi-symmetric submanifolds as the second order envelope of symmetric submanifolds*, Proc. Estonian Acad. Sci. Phys. Math., 39 (1990), 1-8.

[10] C. Özgür, K. Arslan and C. Murathan, *On a class of surfaces in the Euclidean space*, Commun. Fac. Sci. Univ. Ank. Series A1, 51 (2002), 47-54.

Adriano Tomassini: "Special complex structures"

Luigi Vezzoni: "Contact connections"

It is well known that any symplectic connection ∇ induces a canonical almost complex structure $\mathcal{J}$ on the symplectic twistor space.

In [2] Vaisman used the argument of [1] to prove that $\mathcal{J}$ is integrable if and only if the curvature of ∇ is of Ricci-type.

In this talk I will introduce contact connections which are the analogues of symplectic connections in the contact case. I will show that a Sasakian structure always induces a canonic contact connection and I will generalize the result of Vaisman to the contact case.

References

[1] O'Brian, N. R., Rawnsley J. H.: Twistor spaces, *Ann. Global Anal. Geom.* **3** (1985), pp. 29–58.

[2] Vaisman, I.: Symplectic twistor spaces, *J. Geom. Phys.* **3** (1986), pp. 507–524.

[3] Vezzoni, L: Connections on contact manifolds and Contact twistor space, *Israel J. Math.*, to appear.
