Para-CR Geometry

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Para-complex structure

An almost paracomplex structure on a manifold M is a field of endomorphisms $K \in \mathrm{End}(TM)$ with $K^2 = \mathrm{id}$.
It is called an (almost) paracomplex structure in the strong sense if its ± 1-eigendistributions have the same rank. An almost paracomplex structure K is called a paracomplex structure, if it is integrable, i.e.

$$[X, Y] + [KX, KY] - K[X, KY] - K[KX, Y] = 0$$

$\forall X, Y \in \Gamma(TM)$.
This is equivalent to say that the distributions $T^{\pm}M$ are involutive.

Recall that almost CR-structure of codimension k on a $2n+k$-dimensional manifold M is a distribution $HM \subset TM$ of rank $2n$ together with a field of endomorphisms $J \in \mathrm{End}(HM)$ such that $J^2 = -\mathrm{id}$.
An almost CR-structure is called CR-structure, if the $\pm i$-eigenspace subdistributions $H_\pm M$ of the complexified tangent bundle $T^{\mathbb{C}}M$ are involutive.

Almost Para-CR structure

We define an (almost) para-CR structure in a similar way.

A almost CR-structure of codimensions k (in the weak sense) on a $2n+k$-dimensional manifold M is a pair (HM, K), where $HM \subset TM$ is a rank $2n$ distribution and $K \in \mathrm{End}(HM)$ is a field of endomorphisms such that $K^2 = \mathrm{id}$ and $K \neq \pm\mathrm{id}$. Note that K is defined by eigenspace decomposition $HM = H_- + H_+$.

Para-CR structure

An almost para-CR structure is said to be a para-CR structure, if the eigenspace subdistributions $H_{\pm}M \subset HM$ are integrable or equivalently if the following integrability conditions hold:

$$[KX, KY] + [X, Y] \in \Gamma(HM)\,, \qquad (1)$$

$$[X, Y] + [KX, KY] - K([X, KY] + [KX, Y]) = 0$$

for all X, $Y \in \Gamma(HM)$.
If the eigenspace distributions $H_{\pm}$ have the same rank, we say that (HM, K) is an (almost) para-CR structure in the strong sense.

Codimension 1 para-CR structure

Let $(HM, K))$ be a codimension 1 para-CR structure. Locally $HM = \mathrm{Ker}\,\theta$ where 1-form θ is defined up to a scaling.

The symmetric form

$g^H = d\theta \circ K$ on HM

is called the Levi-form. A para-CR manifold is called Levi non-degenerate if g^H is non-degenerate or, equivalently, if HM is a contact distribution.

Then the contact form θ defines a pseudo-Riemannian metric on M

$$g = g^\theta := d\theta^2 + g^H.$$

Note that $g^H(H_\pm, H_\pm) = 0$ where $H_\pm$ are eigen-distributions of K.

Classification of homogeneous compact Levi non-degenerate CR manifolds (-, A.Spiro).

Let $(M = G/L, HM, J)$ be a simply connected homogeneous compact Levi-non-degenerate CR manifold. Then it is either

a) a standard CR homogeneous manifold which is homogeneous S^1-bundle over a flag manifold $F = G/K$, with CR structure induced by an invariant complex structure on F; or

b) the Morimoto-Nagano spaces , i.e. sphere bundles $S(N) \subset TN$ of a compact rank one symmetric space $N = G/H$, with the CR structure induced by the natural complex structure of $TN = G^{\mathbb{C}}/H^{\mathbb{C}}$; or one of the manifolds

c) $SU_n/T^1 \cdot SU_{n-2}$, $SU_p \times SU_q/T^1 \cdot U_{p-2} \cdot U_{q-2}$, $SU_n/T^1 \cdot SU_2 \cdot SU_2 \cdot SU_{n-4}$, $SO_{10}/T^1 \cdot SO_6$, $E_6/T^1 \cdot SO_8$.

These manifolds admit canonical holomorphic fibration over a flag manifold (F, J_F) with typical fiber $S(S^k)$, where $k = 2, 3, 5, 7$ or 9, respectively;

the CR structure is determined by the invariant complex structure J_F on F and an invariant CR structure on the typical fiber, depending on one complex parameter.

We describe a class of homogeneous Levi non-degenerate para-CR manifolds of a semisimple group.

Homogeneous contact manifold

Homogeneous contact manifolds of a Lie group G correspond to coadjoint orbits of G, ($\approx$ adjoint orbits for a semisimple G) and are split into two classes:

If $N = \mathrm{Ad}_G z \subset \mathfrak{g}$ is a non conical orbit of an element $z \in \mathfrak{g}$, then the corresponding contact manifold $M = G/L$ is a homogeneous line (or circle) bundle over N;

If N is a conical orbit , then $M = \mathbb{P}N$ is the projectivization of N.

We describe homogeneous non-degenerate para-CR manifolds $(M = G/L, HM, K)$ of a semisimple Lie group G which correspond to an orbit $N = \mathrm{Ad}_G\, z$ of a semisimple non compact element $z \in \mathfrak{g}$ under additional assumption that the para-complex structure K is invariant with respect to the Reeb vector field Z, defined by

$$\theta(Z) = 1,\ d\theta(Z, .) = 0.$$

The field Z is Hamiltonian, i.e. it preserves θ. The orbit N of a semisimple element z is not conical and the associated homogeneous contact manifold $(M = G/L, HM)$ admit a global G-invariant contact form θ; the associated Reeb vector is also G-invariant.

A construction of invariant para-CR structure

Let $N = \mathrm{Ad}_G\, z = G/C_G(z) \subset \mathfrak{g}$ be the adjoint orbit of a non-compact semisimple element. The associated homogeneous contact manifold is $(M = G/L, \mathrm{Ker}\,\theta)$ where

$$\mathrm{Lie}(L) = \mathfrak{l} := C_{\mathfrak{g}(z)} \cap z^{\perp}$$

and θ is invariant 1-form on M which is the invariant extension of the 1-form $B \circ z \in \mathfrak{g}^*$ defined by z. (B is the Killing form).

The contact manifold $(M, H = \mathrm{Ker}\,\theta)$ has the canonical invariant para-CR structure

$HM = H^{-}M^{+}H^{+}M$ defined as follows.

Let $\mathfrak{h} \ni z$ be a Cartan subalgebra of $\mathfrak{g}$ and R the root system of $(\mathfrak{g}, \mathfrak{h})$. Denote by $R_z = R \cap z^\perp$ the roots which belong to the hyperplane $z^\perp$ and by $R_+, R_- = -R_+$ the roots which belong to positive and negative half-spaces $\mathfrak{h}_\pm$ defined by z. Then

$$\mathfrak{g} = (\mathfrak{h} + \mathfrak{g}R_0) + (\mathbb{R}z + \mathfrak{g}(R_-) + \mathfrak{g}(R_+)) =$$

$$\mathfrak{l} + (\mathbb{R}z + \mathfrak{m}_- + \mathfrak{m}_+)$$

where $\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_\alpha$ for $P \subset R$.
Then Ad_L-invariant decomposition $\mathfrak{m} = (\mathfrak{m}_- + \mathfrak{m}_+)$ defines an invariant Levi non degenerate para-CR structure $HM = H^- M + H^+ M$ on $M = G/L$.

Para-CR-manifolds M^3 and 2d order ODE

(P.Nurowski,G.Sparling,CQG,2003)

ODE $y'' = Q(x, y, y')$ is equivalent to para-CR structure

$$HM = \mathrm{Ker}\,\theta = H_- + H_+ = \mathrm{Ker}\,\rho + \mathrm{Ker}\,\rho',$$

on the contact manifold $M^3 = J^1(\mathbb{R})$, where $\theta = dy - pdx,\ \ \rho = dp - Qdx,\ \rho' = dx$.

Under a point transformation

$$\tilde{x} = \tilde{x}(x, y),\ \ \tilde{y} = \tilde{y}(x, y)$$

the forms are transformed by $\tilde{\theta} = a\theta$, $\tilde{\rho} = b\rho + c\theta,\ \ \tilde{\rho}' = b'\rho + c'\theta$).

This shows that the para-CR structure $H_\pm$ is invariant under point transformations.

Solutions of the ODE are integral curves of the (1-dimensional) Lagrangian distribution H_+. PN-GS considered the 8-dimensional principal bundle $\pi : P \to M$ of adapted frames for the para-CR structure $H_\pm$ (G-structure) and constructs an associated para-Fefferman bundle $F \to M$ with a canonical conformal metric of signature $(2, 2)$.

Using it, they define two fundamental invariants w_1, w_2 of the ODE (known by S.Lie and Segre) and solve the problem of equivalency of ODE under point transformations.
The duality between H_- and H_+ leads to a interesting duality between equivalence classes of ODE, which was known by E. Cartan.

Para CR structures and parabolic Monge-Ampere Equations

(-, G. Manno, F. Pugliese)

Let $HM = \mathrm{Ker}\,\theta$ be a contact distribution on a $(2n+1)$-dimensional manifold M.

In Darboux coordinates $(w^a) = (z, x^i, p_i)$,

$\theta = dz - \sum p_i dx^i$

and we can locally identify M with the manifolds $J^1(\mathbb{R}^n)$ of 1-jets of functions $z = z(x)$.

The tangent space $T_w\Sigma$ of any n-dimensional integral submanifold $\Sigma \subset M$ of HM is a Lagrangian subspace of the symplectic space (H_w, ω_w), where $\omega = d\theta|_H$.

The first prolongation of (M, HM) is the set $M^{(1)} = Lagr(TM)$ of all Lagrangian subspaces of TM. It is a bundle over M with a fiber $Lagr(T_w M) = Sp(n, \mathbb{R})/GL(n, \mathbb{R})$.

A 2d order PDE is a submanifold $\mathcal{E} \subset M^{(1)}$ and its solution is an n-dimensional integral submanifold $\Sigma \subset M$ of HM which is tangent to $\mathcal{E}$:

$T_w\Sigma \in \mathcal{E},\ w \in \Sigma$.

PDE associated to a subdistribution $D \subset HM$

We associate to an n-dimensional subdistribution $D \subset HM$ a PDE

$$\mathcal{E}(D) = \{L \in M^{(1)},\ L \cap D_w \neq 0\}.$$

A solution of $\mathcal{E}(D)$ is an n-dimensional integrale submanifold Σ of HM such that $T_w\Sigma \cap D_w \neq 0$.
Let $X_i,\ i = 1, \cdots, n$ be a local basis of the ω-orthogonal distribution $D^{\perp} \subset HM$ and
$\theta_i := X_i \cdot \theta$.
Consider n-form $\rho := \theta_1 \wedge \cdots \wedge \theta_n$.

Equation $\mathcal{E}(D)$ in coordinates

Proposition 1 *An integrale submanifold* $\Sigma \subset M$ *of* HM *is a solution of* $\mathcal{E}(D)$ *if and only if* $\rho|_\Sigma = 0$.

We may assume that $X_i = \widehat{\partial}_i + q_{ij}(x^k, p_m, z)\partial_{p_j}$ where $\widehat{\partial}_i := \widehat{\partial}_i + p_i\partial_z$.
Then $\theta_i = \omega \circ X_i = -dp_i + q_{ij}dx^j$.
If $\Sigma = \Sigma_z(x)$ is the graph of a function $z = z(x^i)$, then $p_i = z_{,i}$ and

$$\theta_i|\Sigma = (-z_{,ij} + q_{ij})dx^j.$$

The equation $\rho|\Sigma = 0$ take the form of the Monge-Ampere equation

$$\det ||z_{,ij} - q_{ij}(x^k, z_{,m}, z)|| = 0.$$

Parabolic Monge-Ampere equation associated with a Lagrangian distribution

Vector fields $X_i = \widehat{\partial}_i + q_{ij}(x^k, p_m, z)\partial_{p_j}$ generate a Lagrangian distribution D if and only if the matrix $||q_{ij}||$ is symmetric. The corresponding equation $\mathcal{E}(D)$ is called the parabolic Monge-Ampere equation (MAE).

Proposition 2 *There exist a natural 1-1 correspondence between Lagrangian distributions on (M, HM) and parabolic MAE.*

In particular, a non degenerate para-CR structure $H_{\pm}$ defines a pair of dual parabolic Monge-Ampere equations.

In the case $n = 2$, a local classification of Lagrangian distributions and associated parabolic MAE was given by R.Bryant and P.Griffiths in analytic case and R.Alonso Blanco, G. Manno and F.Pugliese in C^∞ case.

Proposition 3 *Any integrable n-dimensional subdistribution D of HM is a Lagrangian distribution, locally given by*

$$D = \mathsf{span}\{\partial_{p_1}, \cdots, \partial_{p_n}\}.$$

Theorem 4 *The equation*
$\det ||z_{,ij}|| = 0$
is contactomorphic to the trivial equation

$$z_{,11} = 0.$$

Maximally homogeneous CR structures

(-,C.Medori, A. Tomassini)

Summary

We will consider a para-CR structure (HM, K) on a manifold M as a Tanaka structure i.e. a distribution together with a principal bundle of adapted coframes.

We associate with any point $x \in M$ of a para-CR manifold a non positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}_0$ and consider its full prolongation $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}_0)^{\infty}$.

A para-CR structure is of a semisimple type if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra.

We give a classification of maximally homogeneous para-CR manifolds of semisimple type in terms of graded real semisimple Lie algebras.

Gradations of a Lie algebra

Recall that a gradation of depth k of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i = \mathfrak{g}^{-k} + \mathfrak{g}^{-k+1} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^j + \cdots$$

such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, for any $i, j \in \mathbb{Z}$ and $\mathfrak{g}^{-k} \neq \{0\}$. Note that $\mathfrak{g}^0$ is a subalgebra and $\mathfrak{g}^i$ is a $\mathfrak{g}^0$-module. An element $x \in \mathfrak{g}^j$ has degree j and we write $d(x) = j$. The gradation is determined by a derivation δ of $\mathfrak{g}$ defined by $\delta_{|\mathfrak{g}_j} = j \cdot id$.

Special types of gradations

Definition 5 *A gradation* $\mathfrak{g} = \sum \mathfrak{g}^i$ *of a Lie algebra is called*

1. *fundamental, if the negative part* $\mathfrak{m} = \sum_{i<0} \mathfrak{g}^i$ *is generated by* $\mathfrak{g}^{-1}$*;*

2. *effective or transitive, if the non-negative part*

   $$\mathfrak{g}^{\geq 0} = \mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

   contains no non-trivial ideal of $\mathfrak{g}$*;*

3. *non-degenerate, if*

   $$X \in \mathfrak{g}^{-1},\ [X, \mathfrak{g}^{-1}] = 0 \implies X = 0\,.$$

Fundamental algebra of a distribution

We associate to a distribution $\mathcal{H}$ and a point $x \in M$ a graded Lie algebra $\mathfrak{m}(x)$.

We have a filtration of the Lie algebra $\mathcal{X}(M)$ of vector fields defined inductively by

$$\begin{aligned} \Gamma(\mathcal{H})_{-1} &= \Gamma(\mathcal{H}), \\ \Gamma(\mathcal{H})_{-i} &= \Gamma(\mathcal{H})_{-i+1} + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})_{-i+1}], \text{for } i > 1. \end{aligned}$$

Evaluating vector fields at a point $x \in M$, we get a flag

$$T_x M = \mathcal{H}_{-d}(x) \supsetneq \mathcal{H}_{-d+1}(x) \supset \cdots \supset \mathcal{H}_{-2}(x) \supset \mathcal{H}_x$$

in $T_x M$, where $\mathcal{H}_{-i}(x) = \{X_{|_x} \mid X \in \Gamma(\mathcal{H})_{-i}\}$.

The commutators of vector fields induce a structure of fundamental negatively graded Lie algebra on the associated graded space

$$\mathfrak{m}(x) = \mathrm{gr}(T_x M) = \mathfrak{m}^{-d}(x) + \mathfrak{m}^{-d+1}(x) + \cdots + \mathfrak{m}^{-1}(x)\,,$$

where $\mathfrak{m}^{-j}(x) = \mathcal{H}_{-j}(x)/\mathcal{H}_{-j+1}(x)$.
A distribution $\mathcal{H}$ is called a regular of depth d and type $\mathfrak{m}$ if all graded Lie algebras $\mathfrak{m}(x)$ are isomorphic to a given fundamental Lie algebra

$$\mathfrak{m} = \mathfrak{m}^{-d} + \mathfrak{m}^{-d+1} + \cdots + \mathfrak{m}^{-1}\,.$$

A distribution $\mathcal{H}$ is called non-degenerate if the Lie algebra $\mathfrak{m}$ is non-degenerate.

Para-CR algebras

Definition 6 *A pair* $(\mathfrak{m}, K_o)$*, where*
$\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$
is a negatively graded fundamental Lie algebra and K_o *is an involutive endomorphism of* $\mathfrak{m}^{-1}$*, is called a para-CR algebra of depth* d*.*
If, moreover, the ± 1*-eigenspaces* $\mathfrak{m}^{-1}_{\pm}$ *of* K_o *on* $\mathfrak{m}^{-1}$ *are commutative subalgebras of* $\mathfrak{m}$*, then* $(\mathfrak{m}, K_o)$ *is called an integrable para-CR structure.*

Regular para-CR structures

Definition 7 *Let* $(\mathfrak{m}, K_o)$ *be a para-CR algebra of depth* d. *A almost para-*CR *structure* (HM, K) *on* M *is called regular of type* $(\mathfrak{m}, K_o)$ *and depth* d *if, for any* $x \in M$, *the pair* $(\mathfrak{m}(x), K_x)$ *is isomorphic to* $(\mathfrak{m}, K_o)$. *We say that the regular almost para-*CR *structure is non-degenerate if the graded algebra* $\mathfrak{m}$ *is non-degenerate.*

A regular almost para-CR structure of type $(\mathfrak{m}, K_0)$ is integrable if and only the Lie algebra $(\mathfrak{m}, K_0)$ is integrable.

Prolongations of negatively graded Lie algebras

The full prolongation of a negatively graded fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is defined as a maximal graded Lie algebra

$$\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) + \mathfrak{g}^{0}(\mathfrak{m}) + \mathfrak{g}^{1}(\mathfrak{m}) + \cdots$$

with the negative part

$$\mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) = \mathfrak{m}$$

such that $\forall k \geq 0$, $X \in \mathfrak{g}^{k}(\mathfrak{m})$

$$[X, \mathfrak{g}^{-1}(\mathfrak{m})] = \{0\} \;\Rightarrow X = 0$$

N.Tanaka proved that the full prolongation $\mathfrak{g}(\mathfrak{m})$ always exists and it is unique up to an isomorphism. Moreover, it can be defined inductively by

$$\mathfrak{g}^i(\mathfrak{m}) = \begin{cases} \mathfrak{m}^i \\ \{A \in \mathsf{Der}(\mathfrak{m}, \mathfrak{m}) \,:\, A(\mathfrak{m}^j) \subset \mathfrak{m}^j\,, \forall j < 0\} \\ \{A \in \mathsf{Der}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^h(\mathfrak{m})) \,:\, A(\mathfrak{m}^j) \subset \mathfrak{g}(\mathfrak{m})^{i+j}\,, \end{cases}$$

where $\mathsf{Der}(\mathfrak{m}, V)$ is the space of derivations of Lie algebra $\mathfrak{m}$ with values in the $\mathfrak{m}$-module V.

Prolongations of non-positively graded Lie algebras

The full prolongation of a non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^0$ is a graded Lie subalgebra

$$(\mathfrak{m}+\mathfrak{g}^0)^\infty = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}+\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\cdots$$

of $\mathfrak{g}(\mathfrak{m})$, defined inductively by

$$\mathfrak{g}^i = \{X \in \mathfrak{g}(\mathfrak{m})^i : [X, \mathfrak{m}^{-1}] \subset \mathfrak{g}^{i-1}\}.$$

A graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$ has finite type (resp.,semisimple type) if $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ is a finite dimensional (resp., finite dimensional semisimple) Lie algebra.

Lemma 8 *Let* $(\mathfrak{m} = \sum_{i<0} \mathfrak{m}^i, K_o)$ *be an integrable para-CR algebra and* $\mathfrak{g}^0$ *the subalgebras of* $\mathfrak{g}^0(\mathfrak{m})$ *consisting of any* $A \in \mathfrak{g}^0(\mathfrak{m})$ *such that* $A|_{\mathfrak{m}^{-1}}$ *commutes with* K_o*. Then the graded Lie algebra* $(\mathfrak{m} + \mathfrak{g}^0)$ *is of finite type if and only if* $\mathfrak{m}$ *is non-degenerate.*

A regular almost para-CR structure of type $(\mathfrak{m}, K_0)$ is of finite type or, respectively, of semisimple type, if the Lie algebra $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ is finite-dimensional or, respectively, semisimple.

Tanaka structures

Definition 9 *Let* $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ *be a negatively graded Lie algebra generated by* $\mathfrak{m}^{-1}$ *and* G^0 *a closed Lie subgroup of (grading preserving) automorphisms of* $\mathfrak{m}$*. A Tanaka structure of* type $(\mathfrak{m}, G^0)$ *on a manifold* M *is a regular distribution* $\mathcal{H} \subset TM$ *of type* $\mathfrak{m}$ *together with a principal* G^0*-bundle* $\pi : Q \to M$ *of adapted coframes of* $\mathcal{H}$*. A coframe* $\varphi : \mathcal{H}_x \to \mathfrak{m}^{-1}$ *is called* adapted *if it can be extended to an isomorphism* $\varphi : \mathfrak{m}_x \to \mathfrak{m}$ *of Lie algebra.*

We say that the Tanaka structure of type $(\mathfrak{m}, G^0)$ is of finite type (respectively semisimple type $(\mathfrak{m}, G^0)$), if the graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$ is of finite type (respectively semisimple type).
Let P be a Lie subgroup of a connected Lie group G and $\mathfrak{p}$ (respectively, $\mathfrak{g}$) the Lie algebra of P (respectively, G).

Maximally homogeneous Tanaka structures

Theorem 10 *Let* $(\pi : Q \to M, \mathcal{H})$ *be a Tanaka structure on* M *of semisimple type* $(\mathfrak{m}, G^0)$*. Then the Tanaka prolongation of* $(\pi, \mathcal{H})$ *is a* P*-principal bundle* $\mathcal{G} \to M$*, with the parabolic structure group* P*, equipped with a Cartan connection* $\kappa : T\mathcal{G} \to \mathfrak{g}$*, where* $\mathfrak{g}$ *is the full prolongation of* $\mathfrak{m} + \mathfrak{g}^0$ *and* $\mathrm{Lie}P = \mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}_i$*. Moreover,* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is a Lie group and* $\dim \mathrm{Aut}(\mathcal{H}, \pi) \leq \dim \mathfrak{g}$.

If the equality holds, the Tanaka structure is called to be maximally homogeneous.

Tanaka structures of semisimple type

Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}^0 + \mathfrak{g}^+$ be a fundamental graded Lie algebra, $\tilde{G}$ the simply connected Lie group defined by $\mathfrak{g}$ and $\tilde{P} = \tilde{G}^0 \cdot \tilde{G}^+$ the parabolic subgroup generated by $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^+$.

Then the flag manifold $F = \tilde{G}/\tilde{P}$ has invariant Tanaka structure $(\mathcal{H}, \pi : Q \to G/P)$ of type $(\mathfrak{m}, G^0)$ where $G^0 \subset GL(\mathfrak{m})$ is the adjoint representation of $\tilde{G}^0$ on $\mathfrak{m}$.

It is called the standard maximally homogeneous Tanaka structure.

Any maximally homogeneous Tanaka structure is locally isomorphic to the standard one.

Standard maximally homogeneous almost para-CR manifolds

Let $\mathfrak{g} = \sum_{-d}^{d} \mathfrak{g}^i = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$ be an effective fundamental gradation of a semisimple Lie algebra $\mathfrak{g}$ with negative part $\mathfrak{m} = \mathfrak{g}^-$ and positive part $\mathfrak{g}^+$. Let $F = \tilde{G}/\tilde{P}$ be associated the simply connected real flag manifold, where Lie$\tilde{P} = \mathfrak{p} = \mathfrak{g}_0 + \mathfrak{g}_+$.

A decomposition

$$\mathfrak{g}^{-1} = \mathfrak{g}_+^{-1} + \mathfrak{g}_-^{-1} \tag{2}$$

of $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ into two submodules determines invariant almost para-CR structure (HF, K) on $F = \tilde{G}/\tilde{P}$. It is called standard almost para-CR manifold.

Theorem 11 *Let $F = \tilde{G}/\tilde{P}$ be the simply connected flag manifold associated with a (real) semisimple effective fundamental graded Lie algebra $\mathfrak{g}$.*
A decomposition
$\mathfrak{g}^{-1} = \mathfrak{g}^{-1}_{+} + \mathfrak{g}^{-1}_{-}$
of $\mathfrak{g}^{-1}$ into complementary G^0-submodules $\mathfrak{g}^{-1}_{\pm}$ determines an invariant almost para-CR structure (HM, K) such that ± 1-eigenspaces $H_{\pm}M$ of K are subdistributions of HM associated with $\mathfrak{g}^{-1}_{\pm}$.
Conversely, any standard almost para-CR structure (HM, K) on F can be obtained in such a way.

Moreover, (HM, K) is:

1. an almost para-CR structure if $\mathfrak{g}_+^{-1}$ and $\mathfrak{g}_-^{-1}$ have the same dimensions,

2. a para-CR structure if and only if $\mathfrak{g}_+^{-1}$ and $\mathfrak{g}_-^{-1}$ are commutative subalgebras of $\mathfrak{g}$,

3. non-degenerate if and only if $\mathfrak{g}$ has no graded ideals of depth one.

The classification of maximally homogeneous almost para-CR structures of semisimple type, up to local isomorphisms (i.e. up to coverings), reduces to the
description of all gradation of semisimple Lie algebras $\mathfrak{g}$ and
to decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules.

Fundamental gradations of a semisimple Lie algebra

A $\mathbb{Z}$-gradation

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^{-1} + \mathfrak{g}^{0} + \mathfrak{g}^{1} + \cdots + \mathfrak{g}^{k} \quad [\mathfrak{g}^{i}, \mathfrak{g}^{j}] \subset \mathfrak{g}^{i+j} \tag{3}$$

of a (real or complex) semi-simple Lie algebra $\mathfrak{g}$ is called fundamental if the subalgebra

$$\mathfrak{g}^{\pm} = \mathfrak{g}^{\pm k} + \cdots + \mathfrak{g}^{\pm 1}$$

is generated by $\mathfrak{g}^{\pm 1}$.

Examples. Fundamental gradations of $sl(V)$

Let V be a (complex or real) vector space and $V = V^1 + \cdots + V^k$ a decomposition of V into a direct sum of subspaces. It defines a fundamental gradation $sl(V) = \sum_{i=-k}^{k} \mathfrak{g}^i$ of the Lie algebra $sl(V)$, where

$$\mathfrak{g}^i = \{A \in sl(V),\, AV^j \subset V^{i+j},\, j = 1, \dots, k\ \}\ .$$

Fundamental gradations of a complex semisimple Lie algebra $\mathfrak{g}$

Let
$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$
be a root space decomposition of a complex semisimple Lie algebra $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$.

We fix a system of simple roots
$\Pi = \{\alpha_1, \cdots, \alpha_\ell\} \subset R$.

Any disjoint decomposition
$\Pi = \Pi^0 \cup \Pi^1$ of Π
defines a fundamental gradation of $\mathfrak{g}$ as follows.
We define the function $d : R \to \mathbb{Z}$ by

$$d|_{\Pi^0} = 0,\ d|_{\Pi^1} = 1,\ d(\alpha) = \sum k_i d(\alpha_i),\ \forall \alpha = \sum k_i \alpha_i.$$

Then the fundamental gradation is given by

$$\mathfrak{g}^0 = \mathfrak{h} + \sum_{\alpha \in R,\ d(\alpha)=0} \mathfrak{g}_\alpha\ , \qquad \mathfrak{g}^i = \sum_{\alpha \in R,\ d(\alpha)=i} \mathfrak{g}_\alpha.$$

Any fundamental gradation of $\mathfrak{g}$ is conjugated to a unique gradation of such form.

Fundamental gradations of a real semisimple Lie algebra

Any real semisimple Lie algebra $\hat{\mathfrak{g}}$ is a real form of a complex semisimple Lie algebra $\mathfrak{g}$, that is it is the fixed point set $\hat{\mathfrak{g}} = \mathfrak{g}^{\sigma}$ of some antilinear involution σ of $\mathfrak{g}$, i.e. an antilinear involutive map $\sigma : \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a Lie algebra over $\mathbb{R}$.

We can always assume that σ preserves a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and induces an automorphism of the root system R. A root $\alpha \in R$ is called compact (or black) if $\sigma\alpha = -\alpha$. It is always possible to choose a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ such that, for any non compact root $\alpha_i \in \Pi$, the corresponding root $\sigma\alpha_i$ is a sum of one non-compact root $\alpha_j \in \Pi$ and a linear combination of compact roots from Π. The roots α_i and α_j called to be equivalent.

Theorem 12 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra $\mathfrak{g}$, $\sigma : \mathfrak{g} \to \mathfrak{g}$ an antilinear involution and $\mathfrak{g}^\sigma$ the corresponding real form. The gradation of $\mathfrak{g}$, associated with a decomposition $\Pi = \Pi^0 \cup \Pi^1$, defines a gradation $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^i)^\sigma$ of $\mathfrak{g}^\sigma$ if and only if*

Π^1 consists of non compact roots and

any two equivalent roots are either both in Π^0 or both in Π^1.

Decomposition of a $\mathfrak{g}^0$-module $\mathfrak{g}^1$ into irreducible submodules

Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a fundamental gradation of a complex semisimple Lie algebra $\mathfrak{g}$. We set

$$R^i = \{\alpha \in R \mid d(\alpha) = i\} = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{g}^i\}$$

and

$$\Pi^i = \Pi \cap R^i = \{\alpha \in \Pi \mid d(\alpha) = i\}.$$

For any simple root $\gamma \in \Pi$, we put

$$R(\gamma) = \{\gamma + (R^0 \cup \{0\})\} \cap R =$$

$$\{\alpha = \gamma + \phi^0 \in R, \ \phi^0 \in R^0 \cup \{0\}\}.$$

We associate to any set of roots $Q \subset R$ a subspace

$$\mathfrak{g}(Q) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \subset \mathfrak{g}\,.$$

Proposition 13 *The decomposition of a $\mathfrak{g}^0$-module $\mathfrak{g}^1$ into irreducible submodules is given by*

$$\mathfrak{g}^1 = \sum_{\gamma \in \Pi^1} \mathfrak{g}(R(\gamma))\,.$$

Moreover, γ is a lowest weight of the irreducible submodule $\mathfrak{g}(R(\gamma))$. In particular, the number of the irreducible components $= \#\Pi^1$.

Proposition 14 *For any simple root $\gamma \in \Pi^1$ of label one, there are two possibilities:*

i) $\sigma^*\gamma = \gamma + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$. Then $\sigma^*\gamma \in R(\gamma)$ and the $\mathfrak{g}^0$-module $\mathfrak{g}(R(\gamma))$ is σ-invariant;

ii) $\sigma^*\gamma = \gamma' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta$, where $\gamma \neq \gamma' \in \Pi^1$. Then, $\sigma^* R(\gamma) = R(\gamma')$ and the two irreducible $\mathfrak{g}^0$-modules $\mathfrak{g}(R(\gamma))$ and $\mathfrak{g}(R(\gamma'))$ determine one irreducible submodule $\mathfrak{g}^\sigma \cap (\mathfrak{g}(R(\gamma)) + \mathfrak{g}(R(\gamma')))$ of $\mathfrak{g}^\sigma$.

Corollary 15 *Let* $\mathfrak{g}^\sigma = \sum(\mathfrak{g}^\sigma)^i$ *be a graded real semisimple Lie algebra. Then irreducible submodules of the* $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *correspond to vertices* γ *with label one without curved arrow and to pairs* (γ, γ') *of equivalent vertices with label one. In particular, decompositions* $\Pi^1 = \Pi^1_- \cup \Pi^1_+$ *such that equivalent roots belong to the same component correspond to decomposition*

$$(\mathfrak{g}^\sigma)^{-1} = (\mathfrak{g}^\sigma)^{-1}_- + (\mathfrak{g}^\sigma)^{-1}_+$$

of $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *into submodules, where*

$$(\mathfrak{g}^\sigma)^{-1}_\pm = \mathfrak{g}^\sigma \cap \sum_{\gamma \in \Pi^1_\pm} \mathfrak{g}(R(-\gamma)). \tag{4}$$

Maximally homogeneous para-CR manifolds

Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra associated with a Satake diagram $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ with the fundamental gradation defined by a subset $\Pi^1 \subset \Pi'$ and $F = \tilde{G}/\tilde{P}$ the associated flag manifold.

An almost para-CR structure on $F = \tilde{G}/\tilde{P}$ associated with a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is integrable i.e. is a CR structure if and only if $(\mathfrak{g}^\sigma)^0$-submodules $(\mathfrak{g}^\sigma)^{-1}_+$ and $(\mathfrak{g}^\sigma)^{-1}_-$ are Abelian subalgebras of $\mathfrak{g}^\sigma$.

We give now a simple criterion for this. We need the following definitions.

Definition 16 *We say that a subset* $\Pi^1 \subset \Pi$ *of a system* Π *of simple roots of a root system* R *is* admissible *if*
there is no root of R *of the form*

$$2\alpha + \sum k_i\phi_i,\ \alpha \in \Pi^1, \phi_i \in \Pi_0 = \Pi \setminus \Pi^1. \quad (5)$$

Definition 17 *Let $\mathfrak{g}^\sigma$ be a real semisimple Lie algebra with a fundamentally gradation defined by a subset $\Pi^1 \subset \Pi$. We say that a decomposition $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ is alternate, if the vertices from Π^1_+ and Π^1_- appear in the Satake diagram in alternate order.*
More precisely, this means that after deleting vertices from Π^1_+ (respectively, Π^1_-), one gets a graph, each connected component of which has not more then one vertex from Π^1_- (respectively, from Π^1_+).

Proposition 18 *Let* $\mathfrak{g}^\sigma$ *be a semisimple real Lie algebra with the fundamental gradation associated to a subset* $\Pi^1 \subset \Pi$ *and* $F = \tilde{G}/\tilde{P}$ *the associated flag manifold. A decomposition*

$\Pi^1 = \Pi^1_+ \cup \Pi^1_-$

defines a para-CR structure on the flag manifold F *if and only if the subset* Π^1 *is admissible and the decomposition of* Π^1 *is alternate.*

The following Proposition describes admissible subsystems Π^1 of a system Π of simple roots for any indecomposable root system R.
We denote by $\Pi = \{\alpha_1, \cdots, \alpha_\ell\}$ the simple roots of $\mathfrak{g}$.
If $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ is a Satake diagram which defines a real form $\mathfrak{g}^\sigma$ of $\mathfrak{g}$ we denote elements of a subset $\Pi^1 \subset \Pi'$ which defines a fundamental gradation

of $\mathfrak{g}^\sigma$ by

$$\alpha_{i_1}, \cdots, \alpha_{i_k}, \;\; i_1 < i_2 < \cdots < i_k.$$

Proposition 19 *A subset $\Pi^1 \subset \Pi$ of a system Π of simple roots of a root system R is admissible in the following cases:*

- *For $R = A_\ell$, in all cases;*

- *For $R = B_\ell$ and C_ℓ, under the additional condition : if $i_k = \ell$, then $i_{k-1} = \ell - 1$;*

- *For $\mathfrak{g} = D_\ell$, under the condition: if $i_k < \ell - 1$, then $i_{k-1} = i_k - 1$;*

- *For $\mathfrak{g} = E_6$, in all cases except the following ones:*

  $$\Pi^1 = \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_3, \alpha_6\}, \{\alpha_4, \alpha_6\}, \{\alpha_1, \alpha_4$$

- *For $\mathfrak{g} = E_7$, in all cases except the following ones:*

  $$\Pi^1 = \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_1, \alpha_6\}$$

- *For* $\mathfrak{g} = E_8$, *in all cases except the following ones:*

  $$\Pi^1 = \{\alpha_1, \alpha_4\}, \{\alpha_1, \alpha_5\}, \{\alpha_1, \alpha_6\}, \{\alpha_1, \alpha_7\}$$

- *For* $\mathfrak{g} = F_4$, *in all cases except the following ones:*

  $$\Pi^1 = \{\alpha_1, \alpha_3\}, \{\alpha_1, \alpha_4\}, \{\alpha_2, \alpha_4\}, \{\alpha_3, \alpha_4\}, \{\alpha_1, \alpha_3$$

- *For* $\mathfrak{g} = G_2$, *in the case*

  $$\Pi^1 = \{\alpha_1, \alpha_2\}.$$

Theorem 20 *Let* $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ *be a Satake diagram of a simple real Lie algebra* $\mathfrak{g}^\sigma$ *and* $\Pi^1 \subset \Pi'$ *an admissible subset described in Proposition 19. Let* $\tilde{G}$ *be the simply connected Lie group with the Lie algebra* $\mathfrak{g}^\sigma$ *and* $\tilde{P}$ *the parabolic subgroup of* $\tilde{G}$ *generated by the non-negatively graded subalgebra*

$$\mathfrak{p} = \sum_{i \geq 0} (\mathfrak{g}^\sigma)^i$$

with the grading element $\vec{d}_{\Pi^1}$*. Then the alternate decomposition* $\Pi^1 = \Pi^1_+ \cup \Pi^1_-$ *defines a decomposition*

$$(\mathfrak{g}^\sigma)^1 = (\mathfrak{g}^\sigma)^1_+ + (\mathfrak{g}^\sigma)^1_-$$

of $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^1$ *into a sum of two commutative subalgebras. This decomposition determines an invariant para-CR structure on the simply connected flag manifold* $F = \tilde{G}/\tilde{P}$*. Moreover, any simply connected maximally homogeneous para-CR manifolds of semisimple type is a direct product of such manifolds.*

Maximal homogeneous para-CR manifolds of depth 2

Theorem 21 *Let M be a non degenerate maximally homogeneous weak para-CR manifold of semisimple type $(\mathfrak{m}, K_0)$ and depth 2. Then, up to coverings, M is isomorphic to a direct product of the following flag manifolds $F = G/P$ of a simple Lie group G associated with a graded Lie algebra $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ equipped with an invariant para-CR structure:*

$\mathfrak{g}$ is of type A_ℓ:

i) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{R})$ and $F = F_{p,q}(\mathbb{R}) = \mathsf{SL}_{\ell+1}(\mathbb{R})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{R}^{\ell+1}$;

ii) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$ and $F = F_{p,q}(\mathbb{C}) = \mathsf{SL}_{\ell+1}(\mathbb{C})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{C}^{\ell+1}$;

iii) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{H})$ and $F = F_{p,q}(\mathbb{H}) = \mathsf{SL}_{\ell+1}(\mathbb{H})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{H}^{\ell+1}$;

$\mathfrak{g}$ is of type D_ℓ:

i) $\mathfrak{g} = \mathfrak{so}_{2\ell}(\mathbb{C})$ and $F = \mathrm{SO}^+_{2\ell}(\mathbb{C})/P$ is the manifold of all isotropic $(1,\ell)$-flags in the complex Euclidean space $V = (\mathbb{C}^{2\ell}, <, >)$, where P is the standard $(1,\ell)$-flag $f_0 = \mathbb{C} \subset \mathbb{C}^\ell$;

ii) $\mathfrak{g} = \mathfrak{so}_{2\ell}(\mathbb{C})$ and $F = \mathrm{SO}^+_{2\ell}(\mathbb{C})/P$ is the manifold of all isotropic $(\ell-1,\ell)$-flags in the complex Euclidean space $V = (\mathbb{C}^{2\ell}, <, >)$, where P is the standard $(\ell-1,\ell)$-flag $f_0 = \mathbb{C}^{\ell-1} \subset \mathbb{C}^\ell$;

iii) $\mathfrak{g} = \mathfrak{so}_{\ell,\ell}$ (the normal form of D_ℓ) and $F = F_{1,\ell} = \mathrm{SO}_{\ell,\ell}/P$ is the manifold of isotropic $(1,\ell)$-flags in the pseudo-Euclidean space $\mathbb{R}^{\ell,\ell}$;

iv) $\mathfrak{g} = \mathfrak{so}_{\ell,\ell}$ (the normal form of D_ℓ) and $F = F_{\ell-1,\ell} = \mathrm{SO}_{\ell,\ell}/P$ is the manifold of isotropic $(\ell-1,\ell)$-flags in the pseudo-Euclidean space $\mathbb{R}^{\ell,\ell}$.

$\mathfrak{g}$ is of type E_6:

i) $\mathfrak{g} = \mathfrak{e}_6$ (see subsection 6.2 for the description of the manifold F);

ii) $\mathfrak{g} = \mathfrak{e}_6^{\mathrm{norm}} = E\,I$ (the normal form of E_6) with the maximal compact subalgebra $\mathfrak{sp}_4$ and $F = \mathrm{E}_6^{\mathrm{norm}}/P$ is the flag manifold described like in the complex case;

iii) $\mathfrak{g} = \mathfrak{e}_6(\mathfrak{f}_4){=}E\,IV$ the real form of $\mathfrak{e}_6$ with maximal compact subalgebra $\mathfrak{f}_4$ and $F = \mathrm{E}_6(\mathfrak{f}_4)/P$ is the flag manifold described like in the complex case.

Moreover, we have

	$\dim H_- F$	$\dim H_+ F$	$\operatorname{codim} HF$
$A_\ell i) - iii)$	$p(q-p)$	$(q-p)(\ell+1-q)$	$p(\ell+1-q)$
$D_\ell i), iii)$	$\frac{(\ell-1)(\ell-2)}{2}$	$\ell-1$	$\ell-1$
$ii), iv)$	$\ell-1$	$\ell-1$	$\frac{(\ell-1)(\ell-2)}{2}$
$E_6 i) - iii)$	8	8	8

where the dimensions have to be intended over $\mathbb{C}$ whenever $\mathfrak{g}$ has a complex structure).
In particular, the weak para-CR structure is a para-CR structure in cases A_ℓ for $p+q=\ell+1$, D_ℓ $ii)$ and $iv)$ and E_6.