Para-CR Geometry. II. Generalizations of a para-CR structure

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An ϵ-quaternionic CR structure

Summary

We define the notion of ϵ-quaternionic CR structure on $4n+3$-dimensional manifold M as a triple $\omega = (\omega_1, \omega_2, \omega_3)$ of 1-forms, which satisfy some structure equations. Here $\epsilon = \pm 1$.

It defines a decomposition $TM = VM + HM$ of the tangent bundle into a direct sum of the horizontal subbundle $HM = \mathrm{Ker}\omega$ and a complementary vertical rank 3 subbundle V and an ϵ-hypercomplex structure

$\mathbf{J} = (J_1, J_2, J_3 = J_1 J_2 = -J_2 J_1) \quad$ on HM.

It is a joint work with Y. Kamishima.

- We associate with ω a 1-parameter family of pseudo-Riemannain metrics g_t.

- We show that the metric g_1 is Einstein metric and

- that the ϵ-quaternionic CR structure is equivalent to an ϵ-3-Sasakian structure subordinated to the pseudo-Riemannian manifold (M, g_1) (which is defined as Lie algebra of Killing fields span(ξ_1, ξ_2, ξ_3) isomorphic to $sp(1, \mathbb{R})$ for $\epsilon = 1$ and $sp(1)$ for $\epsilon = -1$ with some properties.)

- The cone $C(M) = \mathbb{R}^+ \times M$, $\hat{g} = dr^2 + r^2 g_1$ over a ϵ-quaternionic CR manifold (M, ω) has a canonical ϵ-hyperKaehler structure (in particular, is Ricci-flat).

- Under assumption that the Killing vectors ξ_α are complete and define (almost) free action of the corresponding group $K = Sp(1, \mathbb{R})$ or $Sp(1)$, the orbit manifold $Q = M/K$ has a structure of ϵ-quaternionic Kähler manifold.

- Homogeneous manifolds with ϵ-quaternionic CR structure are described.

- A simple reduction construction, which associates with an ϵ-quaternionic CR manifold with a symmetry group G a new ϵ-quaternionic CR manifold $M' = \mu^{-1}(0)/G$ is presented.

Definition of ϵ-quaternionic CR structure

Let $\omega = (\omega_1, \omega_2, \omega_3)$ be a triple of 1-forms on a $4n+3$-dimensional manifold M which are linearly independent, i.e. $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$. We associate with ω a triple $\rho = (\rho_1, \rho_2, \rho_3)$ of 2-forms by

$$\rho_1 = d\omega_1 - 2\varepsilon\omega_2 \wedge \omega_3,$$

$$\rho_2 = d\omega_2 + 2\omega_3 \wedge \omega_1,$$

$$\rho_3 = d\omega_3 + 2\omega_1 \wedge \omega_2,$$

where $\varepsilon = +1$ or -1.

A triple $\mathbf{J} = (J_1, J_2, J_3)$ of anticommuting endomorphisms of a distribution HM is called an ε-hypercomplex structure if they satisfies ε-quaternionic relations

$$J_1^2 = -\varepsilon J_2^2 = -\varepsilon J_3^2 = -1, \; J_3 = J_1 J_2 = -J_2 J_1.$$

For $\varepsilon = 1$, this means that J_1 is a complex structure and $J_2, J_3 = J_1 J_2$ are para-complex structures.

Definition 1 *A triple of 1-forms $\omega = (\omega_\alpha)$ is called a ϵ-quaternionic CR structure if the associated 2-forms (ρ_α) are non degenerate on the distribution*

$$H = \mathrm{Ker}\,\omega = \mathrm{Ker}\,\omega_1 \cap \mathrm{Ker}\,\omega_2 \cap \mathrm{Ker}\,\omega_3,$$

have the same 3-dimensional kernel V and three fields of endomorphisms J_α on H, defined by

$$J_1 = -\varepsilon(\rho_3|_H)^{-1} \circ \rho_2|_H,$$

$$J_2 = (\rho_1|_H)^{-1} \circ \rho_3|_H, \; J_3 = (\rho_2|_H)^{-1} \circ \rho_1|_H.$$

form an ϵ-hypercomplex structure on HM.

Associated metric and the canonical vector fields

We define

1) one-parameter family of pseudo-Riemannian metrics g_t on a ϵ-quaternionic CR manifold M by

$$g_t = g_V^t + g_H \tag{1}$$

where

$$\begin{aligned} g_V^t &= t(\omega_1 \otimes \omega_1 - \varepsilon\omega_2 \otimes \omega_2 - \varepsilon\omega_3 \otimes \omega_3) \\ &= t \sum \varepsilon_\alpha \omega_\alpha \otimes \omega_\alpha \end{aligned} \tag{2}$$

$$g_H = \rho_1 \circ J_1 = \rho_2 \circ J_2 = -\varepsilon\rho_3 \circ J_3,$$

and $\varepsilon_1 = 1,\ \varepsilon_2 = \varepsilon_3 = -\varepsilon$.

2) Three vertical vector fields $\xi_\alpha \in VM$ dual to the 1-forms ω_α:

$$\omega_\beta(\xi_\alpha) = \delta_{\alpha\beta}.$$

Then

$$g_t \circ \xi_\alpha = t\varepsilon_\alpha \omega_\alpha, \tag{3}$$

Properties of the canonical vector fields

We will denote by $\mathcal{L}_X$ the Lie derivative in direction of X.

(1) The vector fields ξ_α preserves the decomposition $TM = V \oplus H$ and span a 3-dimensional Lie algebra $\mathfrak{a}_\varepsilon$ of Killing fields of the metric g_t for $t > 0$, which is isomorphic to $\mathfrak{sp}(1, \mathbb{R})$ for $\varepsilon = 1$ and $\mathfrak{sp}(1)$ for $\varepsilon = -1$. More precisely, the following cyclic relations hold:

$$[\xi_1, \xi_2] = 2\xi_3, \ \ [\xi_2, \xi_3] = -2\varepsilon\xi_1, \ \ [\xi_3, \xi_1] = 2\xi_2.$$

(2) The vector field ξ_α preserves the forms ω_α and ρ_α for $\alpha = 1, 2, 3$. Moreover, the following relations hold :

$$\mathcal{L}_{\xi_2}\omega_3 = -\mathcal{L}_{\xi_3}\omega_2 = \omega_1, \;\; \mathcal{L}_{\xi_3}\omega_1 = \varepsilon\mathcal{L}_{\xi_1}\omega_3 = -\varepsilon\omega_2,$$

$$\mathcal{L}_{\xi_1}\omega_2 = \varepsilon\mathcal{L}_{\xi_2}\omega_1 = \omega_3,$$

and similar relations for ρ_α.

Extension of the endomorphisms J_α

We extend endomorphisms J_α of H to endomorphisms $\bar{J}_\alpha$ of the tangent bundle TM by :

$$\begin{aligned} &\bar{J}_\alpha \xi_\alpha = 0, \quad \bar{J}_\alpha|_H = J_\alpha \\ &\bar{J}_1 \xi_2 = -\varepsilon \xi_3, \quad \bar{J}_1 \xi_3 = \varepsilon \xi_2, \\ &\bar{J}_2 \xi_3 = \xi_1, \quad \bar{J}_2 \xi_1 = \varepsilon \xi_3, \\ &\bar{J}_3 \xi_1 = \xi_2, \quad \bar{J}_3 \xi_2 = \varepsilon \xi_1. \end{aligned} \tag{4}$$

The endomorphisms $\bar{J}_\alpha$, $\alpha = 1, 2, 3$ at a point x constitute the standard basis of the Lie algebra $sp(1)_\varepsilon \subset \mathrm{End}(T_x M)$ where
$sp(1)_{-1} = sp(1), \; sp(1)_{+1} = sp(1, \mathbb{R})$.

Integrability of extended endomorphisms $\bar{J}_\alpha$

Proposition 2 *Let (M, ω) be an ϵ-quaternionic CR manifold. Then $T_\alpha :=$ Ker ω_α, $\bar{J}_\alpha)$ is a Levi-non-degenerate $(-\epsilon_\alpha)$-CR structure.*

This means that T_α is a contact distribution, and J_α is an integrable ϵ_α-complex structure, i.e. J_1 is a complex structure and J_2, J_3 are para-complex structure.
Integrability means that the Nijenhuis tensor $N(\bar{J}_\alpha, \bar{J}_\alpha)_{T_\alpha} = 0$ or, equivalently, the eigendistributions $T_\alpha^\pm$ of $\bar{J}_\alpha|_{T_\alpha}$ are involutive.

Contact metric 3-structure

Let (M, g) be a $(4n+3)$-dimensional manifold with a pseudo-Riemannian metric g of signature $(3+4p, 4q)$.

A contact metric 3-structure is $(\xi_\alpha, \phi_\alpha), \alpha = 1, 2, 3\}$ where ξ_α are three orthonormal vector fields which define contact forms $\eta_\alpha := g \circ \xi_\alpha$, and ϕ_α are skew-symmetric endomorphisms with kernel $\mathrm{Ker}\, \phi_\alpha = \mathbb{R}\xi_\alpha$ such that

(1) $\phi_\alpha^2|_{\xi_\alpha^\perp} = -\mathrm{Id}$, $\phi_\alpha(\xi_\alpha) = 0$;

(2) $\phi_\alpha = \phi_\beta \phi_\gamma - \xi_\beta \otimes \eta_\gamma = -\phi_\gamma \phi_\beta + \xi_\gamma \otimes \eta_\beta$.

K-contact structures

A contact metric 3-structure is called a K-contact 3-structure if ξ_α are Killing fields.

3-Sasakian structure

A K-contact 3-structure is called Sasakian 3-structure if it is normal, i. e. if the following tensors $N_{\eta_\alpha}(\cdot,\cdot)$, $(\alpha = 1, 2, 3)$ vanish:

$$N^{\eta_\alpha}(X,Y) := N_{\phi_\alpha}(X,Y) + (X\eta_\alpha(Y) - Y\eta_\alpha(X))\xi_\alpha \tag{5}$$

$(\forall\ X, Y \in TM)$. Here

$$N_{\phi_\alpha}(X,Y) =$$

$$[\phi_\alpha X, \phi_\alpha Y] - [X,Y] - \phi_\alpha[\phi_\alpha X, Y] - \phi_\alpha[X, \phi_\alpha Y]$$

is the usual Nijenhuis tensor of a field of endomorphisms ϕ_α.

Theorem The following three structures on a $(4n+3)$-dimensional manifold M are equivalent: contact pseudo-metric 3-structures, quaternionic CR structures and pseudo-Sasakian 3-structures.

If ω is a quaternionic CR structure, then the associated 3-Sasakian metric is

$$g = g_1 = \sum \omega_\alpha \otimes \omega_\alpha + \rho_1 \circ J_1,$$

the Killing vectors are vertical vectors ξ_α dual to 1-forms ω_α and $\phi_\alpha = \bar{J}_\alpha$.
The metric g is an Einstein metric.

ε-quaternionic Kähler manifolds

Recall that a (pseudo-Riemannian) quaternionic Kähler manifold (respectively, para-quaternionic Kähler manifold) is defined as a $4n$-dimensional pseudo-Riemannian manifold (M, g) with the holonomy group

$H \subset Sp(1)Sp(p, q)$
(respectively, $H \subset Sp(1, \mathbb{R}) \cdot Sp(n, \mathbb{R})$).

This means that the manifold M admits a parallel 3-dimensional subbundle Q (quaternionic subbundle) of the bundle of endomorphisms which is locally generated by three skew-symmetric endomorphisms J_1, J_2, J_3 which satisfy the quaternionic relations (respectively, para-quaternionic relations). To unify the notations, we will call a quaternionic Kähler manifold also a $(\varepsilon = -1)$-quaternionic Kähler manifold and a para-quaternionic Kähler manifold a $(\varepsilon = 1)$-quaternionic Kähler manifold.

Any ε-quaternionic Kähler manifold is Einstein and its curvature tensor has the form

$$R = \nu R_1 + W$$

,

ε-quaternionic Kähler manifold associated with a ε-quaternionic CR manifolds

Let (M, ω) be a ε-quaternionic CR manifold. We will assume that the Lie algebra $sp(1)_\varepsilon = \text{span}(\xi_\alpha)$ of vector fields is complete and generates a free action of the group $Sp(1)_\varepsilon$ on M. Then the orbit space $B = M/Sp(1)_\varepsilon$ is a smooth manifold and $\pi : M \to B$ is a principal bundle. Moreover, the pseudo-Riemannian metric g_1 of (M, ω) induces a pseudo-Riemannian metric g_B on B such that $\pi : M \to B$ is a Riemannian submersion with totally geodesic fibers.

Theorem 3 *The space of orbit* $N = M/Sp(1)_\varepsilon$ *has a natural structure of* ε*-quaternionic Kähler manifold.*
Conversely, The bundle of orthonormal frames over a ε*-quaternionic Kähler manifold* N *has a structure of* ε*-quaternionic CR manifold.*

Examples of homogeneous ϵ-quaternionic CR manifolds of classical Lie groups:

(C_n) $\varepsilon = +1$, $S\mathbb{H}'^{n,n} = Sp_{n+1}(\mathbb{R})/Sp_n(\mathbb{R})$;
$\varepsilon = -1$, $S_{\mathbb{H}}^{p,q} = Sp_{p+1,q}/Sp_{p,q}$

$(\ A_n)$ $\varepsilon = +1$, $SU_{p+1,q+1}/U_{p,q}$;
$\varepsilon = -1$, $SU_{p+2,q}/U_{p,q}$;

$(BD_n\ )$$\varepsilon = +1$, $SO_{p+2,q+2}/SO_{p,q}$,
$\varepsilon = -1$, $SO_{p+4,q}/SO_{p,q}$.

Momentum map of a ϵ-quaternionic CR manifold with a symmetry group

Let (M, ω) be a ϵ-quaternionic CR manifold and G be a Lie group of its authomorphisms, i.e. transformations which preserves 1-forms ω. We denote by $\mathfrak{g}^*$ the dual space of the Lie algebra $\mathfrak{g}$ of G and we will consider elements $X \in \mathfrak{g}$ as vector fields on M. We define a momentum map as

$$\mu : M \to \mathbb{R}^3 \otimes \mathfrak{g}^*,\ x \mapsto \mu_x,$$

$$\mu_x(X) = \omega(X_x) = (\omega_1(X_x), \omega_2(X_x), \omega_3(X_x)) \in \mathbb{R}^3.$$

Lemma 4 *The momentum map is G-equivariant, where G acts on $\mathbb{R}^3 \otimes \mathfrak{g}^*$ by the coadjoint representation on the second factor.*

Reduction of ϵ-quaternionic CR manifold with a symmetry group

Let $M' = \mu^{-1}(0)$ be the zero level set of the momentum map. It consists of all point $x \in M$ such that the tangent space $\mathfrak{g}x$ to the orbit Gx is horizontal: $\mathfrak{g}x \subset H_x$. In general, it is a stratified manifold.

Lemma 5 *(1)* $\dim Gx \leq \dim T_x(M') \leq 3 \dim Gx$;

(2) If the group G is one dimensional group without fixed point, then M' is a smooth regular (i.e. closed imbedded) submanifold of dimension $4n$.

Theorem 6 *Let* (M, ω_α) *be an* ϵ*-quaternionic CR manifold and* G *a connected Lie group of its authomorphisms. Assume that* G *acts properly on the manifold* $M' = \mu^{-1}(0)$*. Then the* ϵ*-quaternionic CR structure of* M *induces a* ϵ*-quaternionic CR structure* $\widehat{\omega}_\alpha$ *on the orbit space* $\widehat{M} = M'_{\mathrm{reg}}/G$*.*

ε-hyperKähler structure on the cone over an ϵ-quaternionic CR manifold

Theorem 7 *Let* (M, ω_α) *be a* ϵ*-quaternionic CR manifold and* g_t *is the natural metric. Then the cone* $N = \mathbb{R}^+ \times M$ *with the metric* $g^N = dr^2 + r^2 g_1$ *is a* ε*-hyperKähler manifold. Conversely, if the cone metric* g^N *on the cone* N *over a manifold* M *is* ε*-hyperKähler with a parallel* ε*-hypercomplex structure* J_α*, then the manifold* M *has the canonical* ϵ*-quaternionic CR structure* $\omega_\alpha = dr \circ J_\alpha$ *such that* g_1 *is the associated natural metric.*