CR AND SASAKIAN GEOMETRY
LUXEMBOURG - MARCH 24-26, 2008

Beniamino Cappelletti Montano
University of Bari (Italy)

The foliated structure of contact metric (κ,μ)-spaces

Preliminaries on Contact geometry

A contact manifold is a smooth manifold M endowed with a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \qquad (1)$$

everywhere on M.

Preliminaries on Contact geometry

A contact manifold is a smooth manifold M endowed with a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \qquad (1)$$

everywhere on M.

- M is necessarily of odd dimension, $2n+1$

Preliminaries on Contact geometry

A contact manifold is a smooth manifold M endowed with a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \qquad (1)$$

everywhere on M.

- M is necessarily of odd dimension, $2n+1$
- There exists a unique vector field ξ, called Reeb vector field, such that

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, \cdot) = 0.$$

Preliminaries on Contact geometry

A contact manifold is a smooth manifold M endowed with a 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0 \tag{1}$$

everywhere on M.

- M is necessarily of odd dimension, $2n+1$
- There exists a unique vector field ξ, called Reeb vector field, such that

$$\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, \cdot) = 0.$$

Remark

(1) implies that the contact distribution $\mathcal{D} := \ker(\eta)$ can not be integrable. In fact the maximal dimension of an integrable subbundle of $\mathcal{D}$ is n.

Legendre foliations

Let (M^{2n+1},η) be a contact manifold. Then a distribution L on M^{2n+1} is said a Legendre distribution if

- $\dim(L) = n$
- $L_x \subset \mathcal{D}_x$ for all $x \in M$
- $d\eta(X,X') = 0$ for all $X,X' \in \Gamma(L)$

A Legendre foliation of (M^{2n+1},η) is an n-dimensional foliation $\mathcal{F}$ on M^{2n+1} such that $L := T\mathcal{F}$ is a Legendre distribution.

The theory of Legendre foliations was developed by Pang, Libermann et al. in '90s

– M.Y. Pang, *The structure of Legendre foliations*, Trans. Amer. Math. Soc. 1990
– P. Libermann, *Legendre foliations on contact manifolds*, Different. Geom. Appl. 1991

Legendre foliations

A bi-Legendrian manifold is a contact manifold (M,η) foliated by two transversal Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, so that

$$TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus \mathbb{R}\xi$$

$(\mathcal{F}_1,\mathcal{F}_2)$ is called a bi-Legendrian structure on (M,η).

Legendre foliations

A bi-Legendrian manifold is a contact manifold (M,η) foliated by two transversal Legendre foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, so that

$$TM = T\mathcal{F}_1 \oplus T\mathcal{F}_2 \oplus \mathbb{R}\xi$$

$(\mathcal{F}_1,\mathcal{F}_2)$ is called a bi-Legendrian structure on (M,η).

Theorem (C. M., 2005)
Let $(M,\eta,\mathcal{F}_1,\mathcal{F}_2)$ be a bi-Legendrian manifold. There exists a unique connection ∇^{bl}, called the bi-Legendrian connection associated to $(\mathcal{F}_1,\mathcal{F}_2)$, such that

(i) $\nabla^{bl}\mathcal{F}_1 \subset \mathcal{F}_1$, $\nabla^{bl}\mathcal{F}_2 \subset \mathcal{F}_2$, $\nabla^{bl}(\mathbb{R}\xi) \subset \mathbb{R}\xi$

(ii) $\nabla^{bl}\eta = 0$, $\nabla^{bl}d\eta = 0$

(iii) $T^{bl}(X,Y) = 2d\eta(X,Y)\xi$ for all $X\in\Gamma(T\mathcal{F}_1)$, $Y\in\Gamma(T\mathcal{F}_2)$
$T^{bl}(Z,\xi) = [\xi, Z_{\mathcal{F}_1}]_{\mathcal{F}_2} + [\xi, Z_{\mathcal{F}_2}]_{\mathcal{F}_1}$ for all $Z\in\Gamma(TM)$.

Contact Riemannian Manifolds

Let (M,η) be a contact manifold. A Riemannian metric g on M is called an associated metric if the following two conditions hold:

(i) $g(X,\xi) = \eta(X)$ for all $X \in \Gamma(TM)$, that is ξ is orthogonal to $\mathcal{D}$ and has unit length,

(ii) there exists a (1,1)-tensor φ such that

$$\varphi^2 = -I + \eta \otimes \xi$$

and

$$d\eta(X,Y) = g(X,\varphi Y)$$

We refer to (φ,ξ,η,g) as a contact metric structure and to (M,φ,ξ,η,g) as a contact metric manifold.

Contact Riemannian Manifolds

A Sasakian manifold is a contact metric manifold such that

$$N := [\varphi,\varphi] + 2d\eta\otimes\xi \equiv 0.$$

Contact Riemannian Manifolds

A Sasakian manifold is a contact metric manifold such that

$$N := [\varphi,\varphi] + 2d\eta\otimes\xi \equiv 0.$$

- A contact metric manifold is Sasakian if and only if

$$(\nabla^g{}_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

Contact Riemannian Manifolds

A Sasakian manifold is a contact metric manifold such that

$$N := [\varphi,\varphi] + 2d\eta\otimes\xi \equiv 0.$$

- A contact metric manifold is Sasakian if and only if

$$(\nabla^g{}_X\varphi)Y = g(X,Y)\xi - \eta(Y)X.$$

- A contact metric manifold is Sasakian if and only if

$$R^g(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

Contact metric (κ,μ)-spaces

A contact metric (κ,μ)-space is a contact metric manifold (M,φ,ξ,η,g) such that

$$R^g(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \qquad (*)$$

for some $\kappa,\mu \in \mathbb{R}$, where

$$h := \frac{1}{2}\mathcal{L}_\xi \varphi.$$

$(*)$ is called "(κ,μ)-nullity condition". If in $(*)$ $\kappa = 1$, then $h \equiv 0$ and the manifold is Sasakian.

D.E. Blair, T. Koufogiorgos, B.J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math. 1995

Motivations

1. The condition

$$R^g(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

determines the curvature completely. In fact, for all $X,Y,Z \in \Gamma(TM)$

Motivations

1. The condition

$$R^g(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

determines the curvature completely. In fact, for all $X,Y,Z \in \Gamma(TM)$

$$\begin{aligned} R^g(X,Y)Z = {} & (1 - \mu/2)(g(Y,Z)X - g(X,Z)Y) \\ & + \frac{1-\mu/2}{1-\kappa}(g(hY,Z)hX - g(hX,Z)hY)) \\ & + g(Y,Z)hX - g(X,Z)hY - g(hX,Z)Y + g(hY,Z)X \\ & - \mu/2(g(\varphi Y,Z)\varphi X - g(\varphi X,Z)\varphi Y) \\ & + \frac{\kappa-\mu/2}{1-\kappa}(g(\varphi hY,Z)\varphi hX - g(\varphi hX,Z)\varphi hY) \\ & + \eta(X)((\kappa - 1 + \mu/2)g(Y,Z) + (\mu - 1)g(hY,Z))\xi \\ & - \eta(Y)((\kappa - 1 + \mu/2)g(X,Z) + (\mu - 1)g(hX,Z))\xi \\ & - \eta(X)\eta(Z)((\kappa - 1 + \mu/2)Y + (\mu - 1)hY) + \mu g(\varphi X,Y)\varphi Z \\ & + \eta(Y)\eta(Z)((\kappa - 1 + \mu/2)X + (\mu - 1)hX). \end{aligned}$$

Motivations

2. There are non-trivial examples of contact metric (κ,μ)-spaces, the most important being the unit tangent bundle of a Riemannian manifold of constant sectional curvature c ($\kappa = c(2 - c)$, $\mu = -2c$).

Motivations

2. There are non-trivial examples of contact metric (κ,μ)-spaces, the most important being the unit tangent bundle of a Riemannian manifold of constant sectional curvature c ($\kappa = c(2 - c)$, $\mu = -2c$).

3. In the non-Sasakian case ($\kappa \neq 1$) the contact metric structure is not "projectable". Thus contact metric (κ,μ)-spaces have no analogues in even dimension.

Motivations

2. There are non-trivial examples of contact metric (κ,μ)-spaces, the most important being the unit tangent bundle of a Riemannian manifold of constant sectional curvature c ($\kappa = c(2 - c)$, $\mu = -2c$).

3. In the non-Sasakian case ($\kappa \neq 1$) the contact metric structure is not "projectable". Thus contact metric (κ,μ)-spaces have no analogues in even dimension.

4. A complete classification is known, based on the invariant

$$I_M := \frac{1 - \mu/2}{\sqrt{1-\kappa}}$$

E. Boeckx, *A full classification of contact* (κ,μ)*-spaces*, Illinois J. Math. 2000

The bi-Legendrian structure of a contact (κ,μ)-space

Theorem (Blair, Koufogiorgos, Papantoniou)

Let (M,φ,ξ,η,g) be a contact metric (κ,μ)-space. Then $\kappa \leq 1$ and $\kappa = 1$ if and only if (M,φ,ξ,η,g) is Sasakian. Furthermore, in the case $\kappa < 1$, the operator h admits 3 eigenvalues λ, $-\lambda$, 0 and the tangent bundle of M splits as the orthogonal sum

$$TM = \mathcal{D}(\lambda) \oplus \mathcal{D}(-\lambda) \oplus \mathcal{D}(0),$$

where

$$\lambda = \sqrt{1-\kappa}$$

and

$\mathcal{D}(\lambda)$, $\mathcal{D}(-\lambda)$, $\mathcal{D}(0)$ are the eigendistributions corresponding to the eigenvalues λ, $-\lambda$, 0, respectively.

The bi-Legendrian structure of a contact (κ,μ)-space

Remarks

1. $\mathcal{D}(0) = \mathbb{R}\xi$
2. $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are integrable distributions
3. $\dim(\mathcal{D}(\lambda)) = \dim(\mathcal{D}(-\lambda)) = n$
4. For all $X,X' \in \Gamma(\mathcal{D}(\lambda))$ and for all $Y,Y' \in \Gamma(\mathcal{D}(-\lambda))$

$$d\eta(X,X') = 0, \quad d\eta(Y,Y') = 0.$$

Hence $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ define two transversal (orthogonal) Legendre foliations on M.

The bi-Legendrian structure of a contact (κ,μ)-space

Remarks

1. $\mathcal{D}(0) = \mathbb{R}\xi$
2. $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are integrable distributions
3. $\dim(\mathcal{D}(\lambda)) = \dim(\mathcal{D}(-\lambda)) = n$
4. For all $X,X' \in \Gamma(\mathcal{D}(\lambda))$ and for all $Y,Y' \in \Gamma(\mathcal{D}(-\lambda))$

$$d\eta(X,X') = 0, \quad d\eta(Y,Y') = 0.$$

Hence $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ define two transversal (orthogonal) Legendre foliations on M.

Consequently,

Any contact metric (κ,μ)-space is a "bi-Legendrian manifold"

Contact (κ,μ)-spaces as bi-Legendrian manifolds

Theorem (C.M. - Di Terlizzi, 2008)
A non-Sasakian contact metric manifold (M,φ,ξ,η,g) is a contact metric (κ,μ)-space if and only if it admits

- two mutually orthogonal Legendre distributions L_1 and L_2
- a (unique) linear connection ∇

satisfying the following properties:

(i) $\nabla L_1 \subset L_1, \quad \nabla L_2 \subset L_2,$

(ii) $\nabla \eta = 0, \quad \nabla d\eta = 0, \quad \nabla g = 0, \quad \nabla \varphi = 0, \quad \nabla h = 0,$

(iii) $T(X,Y) = 2d\eta(X,Y)\xi$ for all $X \in \Gamma(L_1)$, $Y \in \Gamma(L_2)$
$T(Z,\xi) = [\xi, Z_{L_1}]_{L_2} + [\xi, Z_{L_2}]_{L_1}$ for all $Z \in \Gamma(TM)$.

Furthermore, L_1 and L_2 are integrable and coincide with the eigen-distributions of h, and ∇ coincides with the bi-Legendrian connection ∇^{bl} associated to (L_1,L_2).

Contact (κ,μ)-spaces as bi-Legendrian manifolds

Corollary (C.M. - Di Terlizzi - Tripathi, 2008)
Every invariant submanifold of a non-Sasakian contact metric (κ,μ)-space is totally geodesic.

A submanifold M' of a contact metric manifold (M,φ,ξ,η,g) is said to be *invariant* if $\varphi(T_x M') \subset T_x M'$.

Remark
The same result does not hold, in general, for Sasakian manifolds, for which one has the following well-known result:

An invariant submanifold of a Sasakian manifold is totally geodesic *provided that the second fundamental form of the immersion is covariantly constant* (Kon, 1973).

Classification of contact metric (κ,μ)-spaces

Classification of contact metric (κ,μ)-spaces

Pang provided a classification of Legendre foliations based on the bilinear symmetric form

$$\Pi_{\mathcal{F}}(X,X') := -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi).$$

Classification of contact metric (κ,μ)-spaces

Pang provided a classification of Legendre foliations based on the bilinear symmetric form

$$\Pi_{\mathcal{F}}(X,X') := -(\mathcal{L}_X \mathcal{L}_{X'} \eta)(\xi).$$

A Legendre foliation $\mathcal{F}$ is called

- flat is $\Pi_{\mathcal{F}} \equiv 0$ or, equivalently, $[\xi,X] \in \Gamma(T\mathcal{F})$ for all $X \in \Gamma(T\mathcal{F})$
- degenerate if $\Pi_{\mathcal{F}}$ is degenerate
- non-degenerate if $\Pi_{\mathcal{F}}$ is non-degenerate
- positive (negative) definite if $\Pi_{\mathcal{F}}$ is positive (negative) definite

Classification of contact metric (κ,μ)-spaces

Pang provided a classification of Legendre foliations based on the bilinear symmetric form

$$\Pi_{\mathcal{F}}(X,X') := -(\mathcal{L}_X\mathcal{L}_{X'}\eta)(\xi).$$

A Legendre foliation $\mathcal{F}$ is called

- flat is $\Pi_{\mathcal{F}} \equiv 0$ or, equivalently, $[\xi,X]\in\Gamma(T\mathcal{F})$ for all $X\in\Gamma(T\mathcal{F})$
- degenerate if $\Pi_{\mathcal{F}}$ is degenerate
- non-degenerate if $\Pi_{\mathcal{F}}$ is non-degenerate
- positive (negative) definite if $\Pi_{\mathcal{F}}$ is positive (negative) definite

When $\mathcal{F}$ is non-degenerate, Libermann defined $\Lambda_{\mathcal{F}} : TM \to T\mathcal{F}$ such that

$$\Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z,X) = d\eta(Z,X)$$

for all $Z\in\Gamma(TM)$, $X\in\Gamma(T\mathcal{F})$. Then we extend $\Pi_{\mathcal{F}}$ to a symmetric bilinear form on TM by setting

$$\overline{\Pi}_{\mathcal{F}} := \begin{cases} \Pi_{\mathcal{F}}(Z,Z') & \text{if } Z,Z'\in\Gamma(T\mathcal{F}) \\ \Pi_{\mathcal{F}}(\Lambda_{\mathcal{F}}Z,\Lambda_{\mathcal{F}}Z') & \text{otherwise} \end{cases}$$

Classification of contact metric (κ,μ)-spaces

What about the canonical Legendre foliations of contact (κ,μ)-spaces?

Classification of contact metric (κ,μ)-spaces

What about the canonical Legendre foliations of contact (κ,μ)-spaces?

Theorem (C.M. - Di Terlizzi, 2008)
For the canonical bi-Legendrian structure $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ of a contact metric (κ,μ)-space one has

$$\Pi_{\mathcal{D}(\lambda)} = (2\sqrt{1-\kappa} - \mu + 2)g|_{\mathcal{D}(\lambda)\times\mathcal{D}(\lambda)}$$
$$\Pi_{\mathcal{D}(-\lambda)} = (-2\sqrt{1-\kappa} - \mu + 2)g|_{\mathcal{D}(-\lambda)\times\mathcal{D}(-\lambda)}$$

Hence $\nabla\Pi_{\mathcal{D}(\lambda)}=\nabla\Pi_{\mathcal{D}(-\lambda)}=0$, and only one among the following cases occurs:

(I)	Both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are positive definite	$I_M > 1$
(II)	$\mathcal{D}(\lambda)$ is positive def. and $\mathcal{D}(-\lambda)$ is negative def.	$-1 < I_M < 1$
(III)	Both $\mathcal{D}(\lambda)$ and $\mathcal{D}(-\lambda)$ are negative definite	$I_M < -1$
(IV)	$\mathcal{D}(\lambda)$ is positive definite and $\mathcal{D}(-\lambda)$ is flat	$I_M = 1$
(V)	$\mathcal{D}(\lambda)$ is flat and $\mathcal{D}(-\lambda)$ is negative definite	$I_M = -1$

Existence of contact metric (κ,μ)-structures

A well-known problem in contact geometry ...

Existence of contact metric (κ,μ)-structures

A well-known problem in contact geometry ...

... given a contact manifold (M,η), to find a Sasakian structure compatible with η

Existence of contact metric (κ,μ)-structures

A well-known problem in contact geometry ...

... given a contact manifold (M,η), to find a Sasakian structure compatible with η

... or to find conditions ensuring the existence of a Sasakian structure compatible with η

Existence of contact metric (κ,μ)-structures

A well-known problem in contact geometry ...

... given a contact manifold (M,η), to find a Sasakian structure compatible with η

... or to find conditions ensuring the existence of a Sasakian structure compatible with η

We generalize ...

Existence of contact metric (κ,μ)-structures

A well-known problem in contact geometry ...

... given a contact manifold (M,η), to find a Sasakian structure compatible with η

... or to find conditions ensuring the existence of a Sasakian structure compatible with η

We generalize …

Question

Let (M,η) be a contact manifold. Does M admit a contact metric (κ,μ)-structure compatible with η?

Which assumptions ensure the existence of a compatible contact metric (κ,μ)-structure?

Existence of contact metric (κ,μ)-structures

Theorem (C.M.)
Let (M,η) be a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1,\mathcal{F}_2)$ such that $\nabla^{bl}\Pi_{\mathcal{F}_1}=\nabla^{bl}\Pi_{\mathcal{F}_2}=0$. Assume that one of the following conditions holds:

(I) $\mathcal{F}_1$ and $\mathcal{F}_2$ are positive definite and there exist $a,b > 0$ such that $\overline{\Pi}_{\mathcal{F}_1}= ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}= ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$;

(II) $\mathcal{F}_1$ is positive definite, $\mathcal{F}_2$ negative definite and there exist $a>0$, $b<0$ such that $\overline{\Pi}_{\mathcal{F}_1}= ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}= ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$;

(III) $\mathcal{F}_1$ and $\mathcal{F}_2$ are negative definite and there exist $a,b < 0$ such that $\overline{\Pi}_{\mathcal{F}_1}= ab\overline{\Pi}_{\mathcal{F}_2}$ on $T\mathcal{F}_1$ and $\overline{\Pi}_{\mathcal{F}_2}= ab\overline{\Pi}_{\mathcal{F}_1}$ on $T\mathcal{F}_2$.

Then (M,η) admits a compatible contact metric structure (φ,ξ,η,g) such that

(i) if $a = b$ then (M,φ,ξ,η,g) is Sasakian,

(ii) if $a \neq b$ then (M,φ,ξ,η,g) is a contact metric (κ,μ)-space with

$$\kappa = 1 - \frac{1}{16}(a - b)^2, \qquad \mu = 2 - \frac{1}{2}(a + b).$$

Existence of contact metric (κ,μ)-structures

Theorem (C.M.)

Let (M,η) be a contact manifold endowed with a bi-Legendrian structure $(\mathcal{F}_1,\mathcal{F}_2)$. Assume that one of the following conditions holds:

(IV) $\mathcal{F}_1$ is positive definite, $\nabla^{bl}\Pi_{\mathcal{F}_1} = 0$ and $\mathcal{F}_2$ is flat;

(V) $\mathcal{F}_1$ is flat, $\mathcal{F}_2$ is negative definite and $\nabla^{bl}\Pi_{\mathcal{F}_2} = 0$.

Then

- for each $0 < c \leq 4$ in the case (IV)
- for each $-4 \leq c < 0$ in the case (V)

(M,η) admits a compatible contact metric (κ,μ)-structure, where

$$\kappa = 1 - c^2/16, \quad \mu = 2(1 - c/4).$$

Existence of contact metric (κ,μ)-structures

Corollary 1
Any contact metric (κ,μ)-space such that $|I_M|>1$ admits a compatible Sasakian structure.

Consequently the Betti numbers b_i, $1 \leq i \leq 2n$, of a compact contact metric (κ,μ)-space with $|I_M|>1$ are even.

Existence of contact metric (κ,μ)-structures

Corollary 1

Any contact metric (κ,μ)-space such that $|I_M|>1$ admits a compatible Sasakian structure.

Consequently the Betti numbers b_i, $1 \leq i \leq 2n$, of a compact contact metric (κ,μ)-space with $|I_M|>1$ are even.

Corollary 2

Any contact metric (κ,μ)-space such that $|I_M|<1$ admits a compatible Tanaka-Webster parallel structure.

A Tanaka-Webster parallel structure is a contact metric structure (φ,ξ,η,g) such that the Tanaka-Webster connection $\hat{\nabla}$ satisfies

$$\hat{\nabla}\hat{T} = 0, \quad \hat{\nabla}\hat{R} = 0.$$

E. Boeckx, J.T. Cho, *Pseudo-Hermitian symmetries*, Israel J. Math. 2008

Contact (κ,μ)-spaces and paracontact geometry

Let (M,η) be a contact manifold. Then if there exist

- a (1,1)-tensor $\overline{\varphi}$
- a semi-Riemannian metric $\overline{g}$

such that

- $\overline{\varphi}^2 = I - \eta\otimes\xi$,
- $\overline{g}(\overline{\varphi}X,\overline{\varphi}Y) = -\overline{g}(X,Y) + \eta(X)\eta(Y)$, $\quad d\eta(X,Y) = \overline{g}(X,\overline{\varphi}Y)$,
- $\overline{\varphi}$ induces an almost paracomplex structure on $\mathcal{D} = \ker(\eta)$,

  An *almost paracomplex structure* on a $2n$-dim. manifold is a (1,1) tensor $J\neq I$ such that
  $$J^2 = I, \quad \dim(T^+) = n = \dim(T^-),$$
  where T^+, T^- are the eigendistributions corresponding to 1 and -1

we say that $(\overline{\varphi},\xi,\eta,\overline{g})$ is a paracontact metric structure.

Contact (κ,μ)-spaces and paracontact geometry

A paracontact metric structure is called:

- integrable if, for each $X,Y\in\Gamma(\mathcal{D})$, $N_{\bar\varphi}(X,Y)\in\Gamma(\mathbb{R}\xi)$, where

$$N_{\bar\varphi}(X,Y) := \bar\varphi^2[X,Y] + [\bar\varphi X,\bar\varphi Y] - \bar\varphi[\bar\varphi X,Y] - \bar\varphi[X,\bar\varphi Y] - 2d\eta(X,Y)\xi$$

- K-paracontact if ξ is Killing (or, equivalently, the tensor field $\bar h := \frac{1}{2}\mathcal{L}_\xi\bar\varphi$ vanishes identically)

- para-Sasakian manifolds if $N_{\bar\varphi} \equiv 0$.

para-Sasakian $\Rightarrow$ K-paracontact

Contact (κ,μ)-spaces and paracontact geometry

Theorem (Zamkovoy, Ann. Glob. Anal. Geom. 2009)

On a paracontact metric manifold there exists a unique connection ∇^{pc}, called canonical paracontact connection, satisfying the following conditions:

(i) $\nabla^{pc}\eta = 0,\ \nabla^{pc}\xi = 0,\ \nabla^{pc}\overline{g} = 0,$

(ii) $(\nabla^{pc}_X\overline{\varphi})Y = (\nabla^{\overline{g}}_X\overline{\varphi})Y - \eta(Y)(X - \overline{h}X) + \overline{g}(X - \overline{h}X,Y)\xi,$

(iii) $T^{pc}(\xi,\overline{\varphi}Y) = -\overline{\varphi}T^{pc}(\xi,Y),$

(iv) $T^{pc}(X,Y) = 2d\eta(X,Y)\xi$ on $\mathcal{D} = \ker(\eta)$.

Moreover, the paracontact metric manifold is integrable if and only if $\nabla^{pc}\overline{\varphi} = 0$.

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Theorem (C.M. 2009)
There is a biunivocal correspondence

$$\Psi : \{\text{almost bi-Legendrian str.}\} \rightarrow \{\text{paracontact metric str.}\}$$

between almost bi-Legendrian structures and paracontact metric structures on the same contact manifold (M,η).

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Theorem (C.M. 2009)
There is a biunivocal correspondence

$$\Psi : \{\text{almost bi-Legendrian str.}\} \to \{\text{paracontact metric str.}\}$$

between almost bi-Legendrian structures and paracontact metric structures on the same contact manifold (M,η).

$$\Psi(\{\text{flat almost bi-Legendrian str.}\}) = \{\text{K-paracontact str.}\}$$

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Theorem (C.M. 2009)
There is a biunivocal correspondence

$$\Psi : \{\text{almost bi-Legendrian str.}\} \rightarrow \{\text{paracontact metric str.}\}$$

between almost bi-Legendrian structures and paracontact metric structures on the same contact manifold (M,η).

$$\Psi(\{\text{flat almost bi-Legendrian str.}\}) = \{\text{K-paracontact str.}\}$$

$$\Psi(\{\text{bi-Legendrian str.}\}) = \{\text{integrable paracontact metric str.}\}$$

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Theorem (C.M. 2009)
There is a biunivocal correspondence

$$\Psi : \{\text{almost bi-Legendrian str.}\} \to \{\text{paracontact metric str.}\}$$

between almost bi-Legendrian structures and paracontact metric structures on the same contact manifold (M,η).

$$\Psi(\{\text{flat almost bi-Legendrian str.}\}) = \{\text{K-paracontact str.}\}$$

$$\Psi(\{\text{bi-Legendrian str.}\}) = \{\text{integrable paracontact metric str.}\}$$

$$\Psi(\{\text{flat bi-Legendrian str.}\}) = \{\text{para-Sasakian str.}\}$$

Contact (κ,μ)-spaces and paracontact geometry

What is the link between paracontact geometry and the theory of contact metric (κ,μ)-spaces?

Theorem (C.M. 2009)
There is a biunivocal correspondence

$$\Psi : \{\text{almost bi-Legendrian str.}\} \rightarrow \{\text{paracontact metric str.}\}$$

between almost bi-Legendrian structures and paracontact metric structures on the same contact manifold (M,η).

$$\Psi(\{\text{flat almost bi-Legendrian str.}\}) = \{\text{K-paracontact str.}\}$$

$$\Psi(\{\text{bi-Legendrian str.}\}) = \{\text{integrable paracontact metric str.}\}$$

$$\Psi(\{\text{flat bi-Legendrian str.}\}) = \{\text{para-Sasakian str.}\}$$

Moreover,

$$\nabla^{bl} = \nabla^{pc}.$$

Contact (κ,μ)-spaces and paracontact geometry

Let (M,φ,ξ,η,g) be a contact metric (κ,μ)-space and $(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ the bi-Legendrian structure associated to (M,φ,ξ,η,g).

Then $\Psi(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ is called the canonical paracontact metric structure associated to (M,φ,ξ,η,g).

Question

$\Psi(\mathcal{D}(\lambda),\mathcal{D}(-\lambda))$ = ????

Contact (κ,μ)-spaces and paracontact geometry

Theorem (C.M. - Di Terlizzi)
Let (M,φ,ξ,η,g) be a non-Sasakian contact metric (κ,μ)-space.
Then the canonical paracontact metric structure associated to (M,φ,ξ,η,g) is given by

$$\bar{\varphi} = \frac{1}{2\sqrt{1-\kappa}}\mathcal{L}_{\xi}\varphi, \qquad \bar{g} = d\eta(\cdot,\bar{\varphi}) + \eta\otimes\eta$$

Moreover, $(\bar{\varphi},\xi,\eta,\bar{g})$ satisfies a "$(\bar{\kappa},\bar{\mu})$-nullity condition"

$$R^{\bar{g}}(X,Y)\xi = \bar{\kappa}(\eta(Y)X - \eta(X)Y) + \bar{\mu}(\eta(Y)\bar{h}X - \eta(X)\bar{h}Y)$$

where

$$\bar{\kappa} = \kappa - 2 + (1 - \mu/2)^2, \qquad \bar{\mu} = 2.$$

Furthermore ....

Contact (κ,μ)-spaces and paracontact geometry

.... the canonical paracontact structure $(\overline{\varphi},\xi,\eta,\overline{g})$ induces on (M,η)

- If $|I_M|<1$ a (new) contact metric (κ_1,μ_1)-structure (φ_1,ξ,η,g_1)

$$\varphi_1 = \frac{1}{2\sqrt{1-\kappa-(1-\mu/2)^2}}\mathcal{L}_\xi\overline{\varphi}, \qquad g_1 = -d\eta(\cdot,\varphi_1) + \eta\otimes\eta,$$

with

$$\kappa_1 = \kappa + (1-\mu/2)^2, \qquad \mu_1 = 2.$$

Contact (κ,μ)-spaces and paracontact geometry

.... the canonical paracontact structure $(\bar{\varphi},\xi,\eta,\bar{g})$ induces on (M,η)

- If $|I_M|<1$ a (new) contact metric (κ_1,μ_1)-structure (φ_1,ξ,η,g_1)

  $$\varphi_1 = \frac{1}{2\sqrt{1-\kappa-(1-\mu/2)^2}}\mathcal{L}_\xi\bar{\varphi}, \qquad g_1 = -d\eta(\cdot,\varphi_1) + \eta\otimes\eta,$$

  with

  $$\kappa_1 = \kappa + (1-\mu/2)^2, \quad \mu_1 = 2.$$

- If $|I_M|>1$ a (new) paracontact metric structure (φ_1,ξ,η,g_1)

  $$\varphi_1 = \frac{1}{2\sqrt{(1-\mu/2)^2-(1-\kappa)}}\mathcal{L}_\xi\bar{\varphi}, \quad g_1 = d\eta(\cdot,\varphi_1) + \eta\otimes\eta,$$

  satisfying a (κ_1,μ_1)-nullity condition with

  $$\kappa_1 = \kappa - 2 + (1-\mu/2)^2, \quad \mu_1 = 2.$$

References

1. B. Cappelletti Montano, *Some remarks on the generalized Tanaka-Webster connection of a contact metric manifold*, Rocky Mountain J. Math., in press
2. B. Cappelletti Montano, *Bi-Legendrian structures and paracontact geometry*, Int. J. Geom. Meth. Mod. Phys., in press
3. B. Cappelletti Montano, L. Di Terlizzi, *Contact metric (κ,μ)-spaces as bi-Legendrian manifolds*, Bull. Austral. Math. Soc. 77 (2008), 373–386
4. B. Cappelletti Montano, L. Di Terlizzi, M.M. Tripathi, *Invariant submanifolds of contact (κ,μ)-manifolds*, Glasgow Math. J. 50 (2008), 499–507
5. B. Cappelletti Montano, *The structure of contact metric (κ,μ)-spaces - Part I*, submitted.
6. B. Cappelletti Montano, L. Di Terlizzi *The structure of contact metric (κ,μ)-spaces - Part II*, submitted.