Hypo contact and Sasakian structures on Lie groups

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$SU(2)$-structures in 5-dimensions

Definition

An $SU(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N^5 is given by a 1-form η and by three 2-forms ω_i such that

$$\begin{array}{l} \omega_i \wedge \omega_j = \delta_{ij} v, \; v \wedge \eta \neq 0, \\ i_X \omega_3 = i_Y \omega_1 \Rightarrow \omega_2(X, Y) \geq 0, \end{array}$$

where i_X denotes the contraction by X.

Remark

The pair (η, ω_3) defines a $U(2)$-structure or an almost contact metric structure on N^5, i.e. (η, ξ, φ, g) such that

$$\begin{array}{l} \eta(\xi) = 1, \quad \varphi^2 = -\mathrm{Id} + \xi \otimes \eta, \\ g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \end{array}$$

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Sasakian structures

An almost contact metric structure (η, ξ, φ, g) on N^{2n+1} is said contact metric if $2g(X, \varphi Y) = d\eta(X, Y)$. On N^5 the pair (η, ω_3) defines a contact metric structure if $d\eta = -2\omega_3$.

(η, ξ, φ, g) is called normal if

$$N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y],$$

satisfies the condition $N_\varphi = -\mathrm{d}\eta \otimes \xi$.

Definition (Sasaki)

A Sasakian structure on N^{2n+1} is a normal contact metric structure.

Theorem (Boyer, Galicki)

A Riemannian manifold (N^{2n+1}, g) has a compatible Sasakian structure if and only if the cone $N^{2n+1} \times \mathbb{R}^+$ equipped with the conic metric $\tilde{g} = dr^2 + r^2 g$ is Kähler.

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Sasaki-Einstein structures

Example

Sasaki-Einstein structure on N^5

$$d\eta = -2\omega_3, \; d\omega_1 = 3\eta \wedge \omega_2, \; d\omega_2 = -3\eta \wedge \omega_1 .$$

On $S^2 \times S^3$ there exist an infinite family of explicit Sasaki-Einstein metrics [Gauntlett, Martelli, Sparks, Waldram, ...].

Definition (Boyer, Galicki)

(N^{2n+1}, g, η) is Sasaki-Einstein if the conic metric $\tilde{g} = dr^2 + r^2 g$ on the symplectic cone $N^{2n+1} \times \mathbb{R}^+$ is Kähler and Ricci- flat (CY).

•$N^{2n+1} \times \mathbb{R}^+$ has an integrable $SU(n+1)$-structure, i.e. an Hermitian structure $(J, \tilde{g})$, with $F = d(r^2\eta)$, and a $(n+1,0)$-form $\Psi = \Psi_+ + i\Psi_-$ of lenght 1 such that $dF = d\Psi = 0 \Rightarrow \tilde{g}$ has holonomy in $SU(n+1)$.

• N^{2n+1} has a real Killing spinor, i.e. the restriction of a parallel spinor on the Riemannian cone [Friedrich, Kath].

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An $SU(2)$-structure P on N^5 induces a spin structure on N^5 and P extends to $P_{Spin(5)} = P \times_{SU(2)} Spin(5)$.

The spinor bundle is $P \times_{SU(2)} \Sigma$, where $\Sigma \cong \mathbb{C}^4$ and $Spin(5)$ acts transitively on the sphere in Σ with stabilizer $SU(2)$ in a fixed unit spinor $u_0 \in \Sigma$.

Then the $SU(2)$-structures are in one-to-one correspondence with the pairs $(P_{Spin(5)}, \psi)$, with ψ a unit spinor such that $\psi = [u, u_0]$ for any local section u of P, i.e. $\psi \in P \times_{Spin(5)} (Spin(5)u_0)$.

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An $SU(2)$-structure on N^5 is hypo if

$$d\omega_3 = 0, \ d(\eta \wedge \omega_1) = 0, \ d(\eta \wedge \omega_2) = 0.$$

Proposition (Conti, Salamon)

An $SU(2)$-structure P on N^5 is hypo if and only if the spinor ψ (defined by P) is generalized Killing (in the sense of Bär, Gauduchon, Moroianu) , i.e.

$$\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi,$$

where O is a section of $\mathrm{Sym}(TN^5)$ and $\cdot$ is the Clifford multiplication.

If N^5 is simply connected and Sasaki-Einstein, then $O = \pm \mathrm{Id}$ [Friedrich, Kath].

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• Any oriented hypersurface N^5 of (M^6, F, Ψ) with an integrable $SU(3)$-structure (F, Ψ) has in a natural way a hypo structure. The generalized Killing spinor ψ on N^5 is the restriction of the parallel spinor on M^6 and O is just given by the Weingarten operator. If ψ is the restriction of a parallel spinor over the Riemannian cone then O is a constant multiple of the identity.

• Nilmanifolds cannot admit Sasaki-Einstein structures but they can admit hypo structures.

Theorem (Conti, Salamon)

The nilpotent Lie algebras admitting a hypo structure are

$$\begin{array}{ll} (0,0,12,13,14), & (0,0,0,12,13+24), \\ (0,0,0,12,13), & (0,0,0,0,12+34), \\ (0,0,0,0,12), & (0,0,0,0,0). \end{array}$$

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• Any oriented hypersurface N^5 of (M^6, F, Ψ) with an integrable $SU(3)$-structure (F, Ψ) has in a natural way a hypo structure.

The generalized Killing spinor ψ on N^5 is the restriction of the parallel spinor on M^6 and O is just given by the Weingarten operator. If ψ is the restriction of a parallel spinor over the Riemannian cone then O is a constant multiple of the identity.

• Nilmanifolds cannot admit Sasaki-Einstein structures but they can admit hypo structures.

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Hypo evolution equations

Theorem (Conti, Salamon)

A real analytic hypo structure (η, ω_i) *on* N^5 *determines an integrable* $SU(3)$*-structure on* $N^5 \times I$*, with* I *some open interval, if* (η, ω_i) *belongs to a one-parameter family of hypo structures* $(\eta(t), \omega_i(t))$ *which satisfy the evolution equations*

$$\begin{cases} \partial_t \omega_3(t) = -\hat{d}\eta(t), \\ \partial_t(\omega_2(t) \wedge \eta(t)) = \hat{d}\omega_1(t), \\ \partial_t(\omega_1(t) \wedge \eta(t)) = -\hat{d}\omega_2(t). \end{cases}$$

The $SU(3)$-structure on $N^5 \times I$ is given by

$$F = \omega_3(t) + \eta(t) \wedge dt,$$
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η-Einstein structures

Definition

An almost contact metric manifold $(N^{2n+1}, \eta, \xi, \varphi, g)$ is η-Einstein if there exist $a, b \in \mathcal{C}^\infty(N^{2n+1})$ such that

$$\mathrm{Ric}_g(X, Y) = a\, g(X, Y) + b\, \eta(X)\eta(Y),$$

where $\mathrm{scal}_g = a(2n+1) + b$ and $\mathrm{Ric}_g(\xi, \xi) = a + b$.

If $b = 0$, a Sasaki η-Einstein is Sasaki- Einstein.

Theorem (Conti, Salamon)

A hypo structure on N^5 is η-Einstein $\Leftrightarrow$ it is Sasakian.

For a Sasaki η-Einstein structure on N^5 we have

$$d\eta = -2\omega_3, \; d\omega_1 = \lambda\omega_2 \wedge \eta, \; d\omega_2 = -\lambda\omega_1 \wedge \eta$$

and for the associated generalized Killing spinor

$$O = a\,\mathrm{Id} + b\,\eta \otimes \xi,$$

with a and b constants [Friedrich, Kim].

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Hypo-contact structures

In general, for a hypo structure the 1-form η is not a contact form.

A hypo structure is contact if and only if $d\eta = -2\omega_3$.

Problem

Find examples of manifolds N^5 with a hypo-contact structure.

Examples

- Sasaki η-Einstein manifolds.

An example is given by the nilmanifold associated to

$$(0,0,0,0,12+34) \cong \mathfrak{h}_5.$$

- Contact Calabi-Yau structures, defined by the equations $d\eta = -2\omega_3$, $d\omega_1 = d\omega_2 = 0$ [Tomassini, Vezzoni].

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Classification in the hypo-contact case

Theorem (De Andres, Fernandez, –, Ugarte)

A 5-dimensional solvable Lie algebra $\mathfrak{g}$ has a hypo-contact structure $\Leftrightarrow$ $\mathfrak{g}$ is isomorphic to one of the following:

$$\begin{aligned}
\mathfrak{g}_1 &: \quad [e_1, e_4] = [e_2, e_3] = e_5 \ (\textit{nilpotent and } \eta\textit{-Einstein});\\
\mathfrak{g}_2 &: \quad \tfrac{1}{2}[e_1, e_5] = [e_2, e_3] = e_1,\ [e_2, e_5] = e_2,\\
&\quad\ \ [e_3, e_5] = e_3,\ [e_4, e_5] = -3e_4;\\
\mathfrak{g}_3 &: \quad \tfrac{1}{2}[e_1, e_4] = [e_2, e_3] = e_1,\ [e_2, e_4] = [e_3, e_5] = e_2,\\
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\mathfrak{g}_5 &: \quad [e_1, e_5] = [e_2, e_4] = e_1,\\
&\quad\ \ [e_3, e_4] = e_2,\ [e_3, e_5] = -e_3,\ [e_4, e_5] = e_4.
\end{aligned}$$

$\Rightarrow$ Description of the 5-dimensional solvable Lie algebras which admit a hypo-contact structure.

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Consequences

• All the 5-dimensional solvable Lie algebras with a hypo-contact structure are irreducible.

• $\mathfrak{g}_1 \cong \mathfrak{h}_5$ is the unique nilpotent Lie algebra with a hypo-contact structure.

• The Lie algebras of the classification cannot be Einstein since they are contact [Diatta].

• The unique 5-dimensional solvable Lie algebras with a η-Einstein hypo-contact structure are $\mathfrak{g}_1$ and $\mathfrak{g}_3$.

• If $\mathfrak{g}$ is such that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ and admits a contact Calabi-Yau structure then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_1$.

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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups G_i $(1 \leq i \leq 5)$ with Lie algebra $\mathfrak{g}_i$:

Theorem (De Andres, Fernandez, –, Ugarte)

Any left-invariant hypo-contact structure on any G_i $(1 \leq i \leq 5)$ determines a Riemannian metric with holonomy $SU(3)$ on $G_i \times I$, for some open interval I.

For the nilpotent Lie group G_1 we get the metric found by Gibbons, Lü, Pope and Stelle.

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New metrics with holonomy $SU(3)$

Studying the Conti-Salamon evolution equations for the left-invariant hypo-contact structures on the simply-connected solvable Lie groups G_i $(1 \leq i \leq 5)$ with Lie algebra $\mathfrak{g}_i$:

Theorem (De Andres, Fernandez, –, Ugarte)

Any left-invariant hypo-contact structure on any G_i $(1 \leq i \leq 5)$ determines a Riemannian metric with holonomy $SU(3)$ on $G_i \times I$, for some open interval I.

For the nilpotent Lie group G_1 we get the metric found by Gibbons, Lü, Pope and Stelle.

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A Sasakian manifold $(N^{2n+1}, \eta, \xi, \varphi, g)$ is called homogeneous Sasakian if (η, ξ, φ, g) are invariant under the group of isometries acting transitively on the manifold.

Theorem (Perrone)

A homogeneous 3-dimensional Sasakian manifold has to be a Lie group endowed with a left-invariant Sasakian structure.

Theorem (Geiges, Cho-Chung)

Any 3-dimensional Sasakian Lie algebra is isomorphic to one of the following: $\mathfrak{su}(2)$, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \times \mathbb{R}$, $\mathfrak{h}_3$, where $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of $\mathbb{R}$.

Problem

Classify 5-dimensional Lie groups with a left-invariant Sasakian structure.

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Classify 5-dimensional Lie groups with a left-invariant Sasakian structure.

General results

Proposition (Andrada,–, Vezzoni)

Let $(\mathfrak{g}, \eta, \xi)$ be a contact Lie algebra. Then $\dim \mathfrak{z}(\mathfrak{g}) \leq 1$.

Proposition (Andrada,–, Vezzoni)

Let $(\mathfrak{g}, \eta, \xi, \varphi, g)$ be a Sasakian Lie algebra.
• If $\dim \mathfrak{z}(\mathfrak{g}) = 1$, then $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}\,\xi$ and $(\ker \eta, \theta, \varphi, g)$ is a Kähler Lie algebra, where θ is the component of the Lie bracket of $\mathfrak{g}$ on $\ker \eta$.
• If $\mathfrak{z}(\mathfrak{g}) = \{0\}$, then $\mathrm{ad}_\xi \varphi = \varphi\, \mathrm{ad}_\xi$, and one has the orthogonal decomposition

$$\mathfrak{g} = \ker \mathrm{ad}_\xi \oplus (\mathrm{Im}\, \mathrm{ad}_\xi).$$

If $\mathfrak{g}$ is a $(2n+1)$-dimensional Sasakian nilpotent Lie algebra, then $\mathfrak{g} \cong \mathfrak{h}_{2n+1}$.

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Sasakian 5-dimensional Lie algebras

Theorem (Andrada, –, Vezzoni)

Let $\mathfrak{g}$ be a 5-dimensional Sasakian Lie algebra. Then

1. *if $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, $\mathfrak{g}$ is solvable with $\dim \mathfrak{z}(\mathfrak{g}) = 1$ and the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ carries an induced Kähler structure;*
2. *if $\mathfrak{z}(\mathfrak{g}) = \{0\}$, $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, or $\mathfrak{su}(2) \times \mathfrak{aff}(\mathbb{R})$, or $\mathfrak{g}_3 \cong \mathbb{R}^2 \ltimes \mathfrak{h}_3$.*

• $\mathfrak{g}$ is either solvable or a direct sum.

• A 5-dimensional Sasakian solvmanifold is either a compact quotient of H_5 or of $\mathbb{R} \ltimes (H_3 \times \mathbb{R})$ with structure equations

$$(0, -13, 12, 0, 14 + 23).$$

• A 5-dimensional Sasakian η-Einstein Lie algebra is isomorphic either to $\mathfrak{g}_1 \cong \mathfrak{h}_5$, or $\mathfrak{g}_3$ or to $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$.

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$SU(n)$-structures in $(2n+1)$-dimensions

Definition

An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is determined by the forms

$$\eta = e^{2n+1}, \quad \phi = e^1 \wedge e^2 + \ldots + e^{2n-1} \wedge e^{2n},$$
$$\Omega = (e^1 + ie^2) \wedge \ldots \wedge (e^{2n-1} + ie^{2n}).$$

As for the case of $SU(2)$-structures in dimensions 5 we have that an $SU(n)$-structure P_{SU} on N^{2n+1} induces a spin structure P_{Spin} and if we fix a unit element $u_0 \in \Sigma = (\mathbb{C}^2)^{\otimes 2n}$ we have that

$$P_{SU} = \{u \in P_{Spin} \,|\, [u, u_0] = \psi\}.$$

The pair (η, ϕ) defines a $U(n)$-structure or an almost contact metric structure on N^{2n+1}.

The $U(n)$-structure is a contact metric structure if $d\eta = -2\phi$.

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Example

$N^{2n+1} \hookrightarrow M^{2n+2}$ (with holonomy in $SU(n+1)$).
Then the restriction of the parallel spinor defines an $SU(n)$-structure (η, ϕ, Ω) where the forms ϕ and $\Omega \wedge \eta$ are the pull-back of the Kähler form and the complex volume form on the CY manifold M^{2n+2}.

Proposition (Conti, –)

Let N^{2n+1} be a real analytic manifold with a real analytc $SU(n)$-structure P_{SU} defined by (η, ϕ, Ω). The following are equivalent:

1. The spinor ψ associated to P_{SU} is a generalized Killing spinor, i.e. $\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi$.
2. $d\phi = 0$ and $d(\eta \wedge \Omega) = 0$.
3. A neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$ has a CY structure which restricts to P_{SU}.

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Generalized Killing spinors

Example

$N^{2n+1} \hookrightarrow M^{2n+2}$ (with holonomy in $SU(n+1)$).
Then the restriction of the parallel spinor defines an $SU(n)$-structure (η, ϕ, Ω) where the forms ϕ and $\Omega \wedge \eta$ are the pull-back of the Kähler form and the complex volume form on the CY manifold M^{2n+2}.

Proposition (Conti, –)

Let N^{2n+1} be a real analytic manifold with a real analytc $SU(n)$-structure P_{SU} defined by (η, ϕ, Ω). The following are equivalent:

1. The spinor ψ associated to P_{SU} is a generalized Killing spinor, i.e. $\nabla_X \psi = \frac{1}{2} O(X) \cdot \psi$.
2. $d\phi = 0$ and $d(\eta \wedge \Omega) = 0$.
3. A neighbourhood of $M \times \{0\}$ in $M \times \mathbb{R}$ has a CY structure which restricts to P_{SU}.

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The assumption of real analycity is certainly necessary to prove that (1) or (2) implies (3), but the fact that (1) implies (2) does not require this hypothesis.

(2) $\Rightarrow$ (3) can be described in terms of evolution equations in the sense of Hitchin. Indeed, suppose that there is a family $(\eta(t), \phi(t), \Omega(t))$ of $SU(n)$-structures on N^{2n+1}, with t in some interval I, then the forms

$$\eta(t) \wedge dt + \phi(t), \quad (\eta(t) + idt) \wedge \Omega(t)$$

define a CY structure on $N^{2n+1} \times I$ if and only if (2) holds for $t = 0$ and the evolution equations

$$\frac{\partial}{\partial t}\phi(t) = -\hat{d}\eta(t), \quad \frac{\partial}{\partial t}(\eta(t) \wedge \Omega(t)) = i\hat{d}\Omega(t)$$

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Contact $SU(n)$-structures

Definition

An $SU(n)$-structure (η, ϕ, Ω) on N^{2n+1} is contact if $d\eta = -2\phi$.

In this case N^{2n+1} is contact metric with contact form η and we may consider the symplectic cone over (N^{2n+1}, η) as the symplectic manifold $(N^{2n+1} \times \mathbb{R}^+, -\frac{1}{2}d(r^2\eta))$.

If N^{2n+1} is Sasaki-Einstein, we know that the symplectic cone is CY with the cone metric $r^2g + dr^2$ and the Kähler form equal to the conical symplectic form.

Problem

If one thinks the form ϕ as the pullback to $N^{2n+1} \cong N^{2n+1} \times \{1\}$ of the conical symplectic form, which types of contact $SU(n)$-structures give rise to a CY symplectic cone but not necessarily with respect to the cone metric?

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Proposition (Conti, –)

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Examples

• 5-dimensional hypo-contact solvable Lie groups [De Andres, Fernandez, –, Ugarte].

• The $(2n+1)$-dimensional real Heisenberg Lie group H_{2n+1}

$$\begin{array}{l} de^i = 0, \quad i = 1, \dots, 2n, \\ de^{2n+1} = e^1 \wedge e^2 + \dots + e^{2n-1} \wedge e^{2n}. \end{array}$$

• A two-parameter family of examples in the sphere bundle in $T\mathbb{CP}^2$ [Conti].

• A 7-dimensional compact example, quotient of the Lie group $SU(2) \ltimes \mathbb{R}^4$, which has a weakly integrable generalized G_2-structure [–, Tomassini].

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Proposition (Conti, –)

H: compact Lie group
ρ a representation of H on V.
Then $H \ltimes_\rho V$ has a left-invariant contact structure if and only if $H \ltimes_\rho V$ is either $SU(2) \ltimes \mathbb{R}^4$ or $U(1) \ltimes \mathbb{C}$.

Then, if H is compact, the example $SU(2) \ltimes \mathbb{R}^4$ is unique in dimensions > 3.

If H is solvable we have

Proposition (Conti, –)

H : 3-dimensional solvable Lie group.
There exists $H \ltimes \mathbb{R}^4$ admitting a contact $SU(3)$-structure whose associated spinor is generalized Killing if and only if the Lie algebra of H is isomorphic to one of the following

$$(0,0,0), \quad (0,\pm 13, 12),$$
$$(0,12,13), \quad (0,0,13).$$

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Let N^{2n+1} be a $(2n+1)$-dimensional manifold endowed with a contact metric structure (η, ϕ, g) and a spin structure compatible with the metric g and the orientation.

We say that a spinor ψ on N^{2n+1} is compatible if

$$\eta \cdot \psi = i^{2n+1}\psi, \quad \phi \cdot \psi = -ni\psi.$$

Suppose that S^1 acts on N^{2n+1} preserving both metric and contact form, so that the fundamental vector field X satisfies

$$\mathcal{L}_X \eta = 0 = \mathcal{L}_X \phi.$$

and denote by t its norm.

The moment map is given by $\mu = \eta(X)$.

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Assume that 0 is a regular value of μ and consider the hypersurface $\iota : \mu^{-1}(0) \to N^{2n+1}$.

Then the contact reduction is given by $N^{2n+1}//S^1 = \mu^{-1}(0)/S^1$ [Geiges, Willett].

• The contact $U(n)$-structure on N^{2n+1} induces a contact $U(n-1)$-structure on $N^{2n+1}//S^1$.

Let ν be the unit normal vector field, dual to the 1-form $i_{t^{-1}X}\phi$.

• The choice of an invariant compatible spinor ψ on N^{2n+1} determines a spinor

$$\psi^{\pi} = \iota^* \psi + i\nu \cdot \iota^* \psi.$$

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Then the contact reduction is given by $N^{2n+1}//S^1 = \mu^{-1}(0)/S^1$ [Geiges, Willett].

• The contact $U(n)$-structure on N^{2n+1} induces a contact $U(n-1)$-structure on $N^{2n+1}//S^1$.

Let ν be the unit normal vector field, dual to the 1-form $i_{t^{-1}X}\phi$.

• The choice of an invariant compatible spinor ψ on N^{2n+1} determines a spinor

$$\psi^\pi = \iota^*\psi + i\nu \cdot \iota^*\psi.$$

on $N^{2n+1}//S^1$.

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Theorem (Conti, –)

N^{2n+1} with a contact $U(n)$-structure (g, η, ϕ) and a compatible generalized Killing spinor ψ.
Suppose that S^1 acts on N^{2n+1} preserving both structure and spinor and acts freely on $\mu^{-1}(0)$ with 0 regular value. Then the induced spinor ψ^π on $N^{2n+1}//S^1$ is generalized Killing if and only if at each point of $\mu^{-1}(0)$ we have

$$dt \in span < i_X\phi, \eta >,$$

where X is the fundamental vector field associated to the S^1-action, and t is the norm of X.

Example

If we apply the previous theorem to $SU(2) \ltimes \mathbb{R}^4$ we get a new hypo-contact structure on $S^2 \times \mathbb{T}^3$.

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From a result by Grantcharov and Ornea the contact reduction of a η-Einstein-Sasaki structure is Sasaki.

Corollary (Conti, –)

N^{2n+1} with an η-Einstein-Sasaki structure (g, η, ϕ, ψ), and let S^1 act on M preserving the structure in such a way that 0 is a regular value for the moment map μ and S^1 acts freely on $\mu^{-1}(0)$. Then the Sasaki quotient $M//S^1$ is also η-Einstein if and only if

$$dt \in span < i_X\phi, \eta > .$$

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