Workshop on CR and Sasakian Geometry

Generalized pseudohermitian geometry

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M connected $\mathcal{C}^\infty$ manifold of dimension $2n+k$, $n \geq 1$, $k \geq 0$

(HM, J) partial complex structure of type (n, k)

A *generalized pseudohermitian structure* on M is defined as a pair (h, P) where:

- h hermitian fiber metric on HM:

$$h(JX, JY) = h(X, Y) \quad \forall X, Y \in \Gamma HM$$

- P smooth projector $P : TM \to TM$ such that:

$$Im(P) = HM.$$

(M, HM, J, h, P) will be called a *generalized pseudohermitian manifold*.

Remark

If the CR codimension $k = 0$, then $P = Id$ and (M, h, J) is an almost Hermitian manifold.

$f : (M, HM, J, h, P) \to (M', HM', J', h', P')$ smooth CR map

f will be called *pseudohermitian map* if:

1. $$||f_* X||_{h'} \leq ||X||_h \quad \forall X \in H_x M \tag{0.1}$$

2. $$Im(f_* \circ P_x - P'_{f(x)} \circ f_*) \subset f_*(H_x M)^\perp \subset H'_{f(x)} M' \tag{0.2}$$
   where the orthogonal complement is relative to $h'_{f(x)}$.

If equality holds in 1) f will be called *isopseudohermitian*.

In this case $\dim_{CR} M \leq \dim_{CR} M'$.

EXAMPLES of generalized pseudohermitian manifolds

- Strongly pseudoconvex CR manifolds (M, HM) of hypersurface type

  h=fixed positive definite Levi form $\mathcal{L}_\eta \;\; \eta \in \Gamma H^0 M$
  P=projection onto HM relative to the decomposition

  $$TM = HM \oplus [\xi] \quad \xi \text{ Reeb vector field}$$

  (M, HM, J, η) is a pseudohermitian manifold according to Webster (*J. Diff. Geom.* 1978)

- Strongly pseudoconvex CR manifolds (M, HM) of arbitrary CR codimension $k \geq 1$:

  h=fixed positive definite Levi form $\mathcal{L}_\eta \;\; \eta \in \Gamma H^0 M$
  P=projection onto HM relative to the decomposition

  $$TM = HM \oplus W$$

  W: rank k subbundle of TM whose fiber at $x \in M$ is

  $$W_x := \{\xi \in T_x M \,|\, d_x\eta(X, \xi) = 0 \;\; \forall X \in H_x M\}.$$

EXAMPLES of generalized pseudohermitian manifolds

- Riemannian almost CR spaces (M, HM, g)

  g=fixed positive definite metric whose restriction to HM is Hermitian
  P=projection onto HM relative to the orthogonal decomposition

  $$TM = HM \oplus HM^{\perp}$$

  In particular:

  $M \subset \bar{M}$ CR submanifolds of almost Hermitian manifolds $(\bar{M}, g)$

- Almost contact metric manifolds $(M, \varphi, \xi, \eta, g)$
- Contact CR submanifolds of Sasakian manifolds.

NEW EXAMPLES from old

(M', HM', J', h', P') generalized pseudohermitian manifold

(M, HM, J) a CR space

$f : M \to M'$ a CR immersion

Then:

Proposition

There exists a unique generalized pseudohermitian structure (h, P) on M with respect to which f is isopseudohermitian.

NEW EXAMPLES from old

(B, HB, J', h', P') generalized pseudohermitian manifold of type (n, k)

M arbitrary manifold and $\pi : M \to B$ a submersion

Fix $\mathfrak{H} \subset TM$ a complementary subbundle to the vertical subbundle
Then:

Proposition

There exists a unique generalized pseudohermitian structure (HM, J, h, P) on M having CR dimension n and such that:

a) $HM \subset \mathfrak{H}$;

b) π is isopseudohermitian.

Algebraic structure and Levi form

(M, HM, J, h, P) generalized pseudohermitian manifold
Fact: Each T_xM carries a graded Lie algebra structure of kind 2
The non trivial Lie bracket

$$[\,,\,]_x : H_xM \times H_xM \to Ker(P_x)$$

is induced from the $C^\infty(M)$-bilinear map:

$$L : \Gamma HM \times \Gamma HM \to \Gamma Ker(P)$$

$$L(X,Y) := Q[X,Y] \quad \text{Levi-Tanaka form} \quad Q := Id - P$$

There is a well-defined vector valued quadratic form (Levi form)

$$\mathcal{L} : H_xM \to T_xM$$

$$\mathcal{L}(X) = [JX, X]_x = Q([J\tilde{X}, \tilde{X}]_x) \qquad X \in H_xM$$

- If (HM, J) is partially integrable
  $(T_xM, [,]_x)$ is pseudocomplex: $[JX, JY]_x = [X,Y]_x \; X, Y \in H_xM$
- If (HM, J) is partially integrable of kind 2, i.e.
  $\Gamma TM = \Gamma HM + [\Gamma HM, \Gamma HM]$
  $(T_xM, [,]_x)$ is the Tanaka algebra $\mathfrak{m}(x)$.

An invariant operator

(M, HM, J, h, P) generalized pseudohermitian manifold
We can define an operator

$$\Gamma : \Gamma HM \times \Gamma HM \to \Gamma HM$$

as follows:

$$\Gamma_X Y := P(\nabla^g_X Y)$$

here
g : an arbitrary Riemannian metric extending h and such that $Ker(P) = HM^\perp$
∇^g Levi-Civita connection of g.

Γ does not depend on the choice of g but only on the pair (h, P).
Γ will be called the *Koszul operator* of M.

Remark

The Koszul operator of a generalized pseudohermitian manifold is invariant under equivalence.

Given $X \in \Gamma HM$ one can define $\Gamma_X(J) : \Gamma HM \to \Gamma HM$

$$\Gamma_X(J)Y := \Gamma_X(JY) - J(\Gamma_X Y).$$

Next we define the tensorial map: $\alpha : \Gamma HM \times \Gamma HM \times HM \to C^\infty(M)$

$$\alpha(X, Y, Z) := h(\Gamma_X(J)Y, Z)$$

For each point $x \in M$, $\alpha_x : H_xM \times H_xM \times H_xM \to \mathbb{R}$ belongs to the Gray-Hervella space (*Ann. Mat. Pura Appl.* 1980)

$$W = \{\alpha \in V^* \otimes V^* \otimes V^* | \alpha(X, Y, Z) = -\alpha(X, Z, Y) = \alpha(X, JY, JZ)\}$$

where $V = (H_xM, J_x, h_x)$.

Classes of gen. pseudohermitian manifolds

Pseudohermitian structures fall into sixteen classes, according to the decomposition

$$W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$$

of W into irriducible components w.r.to the natural $U(n)$ action.
Some relevant classes:
A gen. pseudohermitian manifold (M, HM, J, h, P) will be called

of Kähler type if $\alpha = 0$ $(\Gamma_X(J)Y = 0)$

of nearly Kähler type if for each $x \in M$ $\alpha_x \in W_1$ $(\Gamma_X(J)X = 0)$

of almost Kähler type if for each $x \in M$ $\alpha_x \in W_2$
$(\mathcal{S}_{XYZ} h(\Gamma_X(J)Y, Z) = 0)$

of quasi Kähler type if for each $x \in M$ $\alpha_x \in W_1 \oplus W_2$
$(\Gamma_X(J)Y + \Gamma_{JX}(J)JY = 0)$

Remark

All the pseudohermitian manifolds in the sense of Webster are of Kähler type.

Theorem

Let (M, HM, J, P, h) be a generalized pseudohermitian manifold. Then there exists a unique connection D on HM such that:

1. *D is compatible with the metric h and J is D-parallel.*
2. *For each $X \in \Gamma HM$, the operator $\Lambda(X) := D_X - \Gamma_X : \Gamma HM \to \Gamma HM$ anticommutes with J.*
3. *For each $\xi \in \Gamma Ker(P)$ the skew-symmetric part of the tensor $\tau_\xi : \Gamma HM \to \Gamma HM$ defined by*

$$\tau_\xi(X) := D_\xi X - P[\xi, X] \quad \forall X \in \Gamma HM$$

anticommutes with J.

Remark

D is invariant under equivalence.
D coincides with the Tanaka-Webster connection for the classical pseudohermitian manifolds (they have symmetric sub torsion τ_ξ)

$\mathfrak{g} = \bigoplus_{p=-\mu}^{\mu} \mathfrak{g}_p$ semisimple Levi-Tanaka algebra

$S(\mathfrak{g}) = G/G_+$ standard homogeneous CR manifold

$S(\mathfrak{g})$ carries a standard gen. pseudohermitian structure (h, P):

Fix an adapted Cartan decomposition of $\mathfrak{g}$
(Medori-Nacinovich, *Compositio Math.* 1997)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \bigoplus_{0}^{\mu} \mathfrak{k}_{|j|}, \quad \mathfrak{p} = \bigoplus_{0}^{\mu} \mathfrak{p}_{|j|} \quad \mathfrak{k}_{|j|} = \mathfrak{k} \cap (\mathfrak{g}_j \oplus \mathfrak{g}_{-j})$$

The analytic subgroup $K \subset G$ corresponding to $\mathfrak{k}$ acts transitively on S
$S = K/K_o$ reductive homogeneous space in the sense of Nomizu

$T_oS \cong \mathfrak{n} \quad \mathfrak{n} := \mathfrak{k}_{|1|} \oplus \bigoplus_{p=1}^{\mu} \mathfrak{k}_{|p|} \quad \mathfrak{n}$ reductive summand, $H_oS \cong \mathfrak{k}_{|1|}$

h:= K-invariant Hermitian metric on HM induced from the Killing form

P:= K-invariant tensor field corresponding to the linear projection
$P_o : \mathfrak{n} \to \mathfrak{n}$ onto $\mathfrak{k}_{|1|}$

$S = K/K_0 \;\; \mathfrak{k} = \mathfrak{k}_{|0|} \oplus \mathfrak{n} \;\; \mathfrak{n} := \mathfrak{k}_{|1|} \oplus \bigoplus_{p=1}^{\mu} \mathfrak{k}_{|p|}$ reductive decomposition

$P(S)$:= $U(n)$ reduction of the frame bundle $\mathcal{F}(HS)$ of HS

Wang's theorem:
$\{$K-Invariant connection on $P(S)\} \leftrightarrow \{$Equivariant $\Lambda : \mathfrak{n} \to \mathfrak{u}(n)\}$

Theorem

The canonical connection of S is th K-invariant connection on $P(S)$ corresponding to the linear map $\Lambda : \mathfrak{n} \to \mathfrak{u}(n)$ defined by

$$\Lambda(Z)(X) = [Z, X]_{|1|} \qquad X \in \mathfrak{k}_{|1|}.$$

The sub torsion $\tau_\xi = 0$ for every ξ. The pseudoholomorphic curvature H is non-negative, namely for each unit vector $X \in H_o S \cong \mathfrak{k}_{|1|}$

$$H(p) = ||[JX, X]||^2 \qquad p = \mathit{Span}(X, JX)$$

$H(p)$ defined as usual by $H(p) = h(R(X, JX)JX, X)$
R curvature of D

The equivalence problem

We shall treat gen. pseudohermitian manifolds having kind 2, i.e.

$$\Gamma TM = \Gamma HM + [\Gamma HM, \Gamma HM].$$

This means that the Tanaka form

$$L : HM \wedge HM \to Ker(P)$$

such that

$$L(X \wedge Y) = Q[\tilde{X}, \tilde{Y}] \qquad Q := Id - P$$

is a surjective bundle homomorphism covering Id_M.

Proposition

a) L induces a surjective bundle map $\mathfrak{F}(HM) \to \mathfrak{F}(Ker(P))$ covering the identity, between the frame bundles of HM and $Ker(P)$.

b) h extends canonically to a Riemannian metric g with respect to which $TM = HM \oplus Ker(P)$ is an orthogonal decomposition.

The Riemannian metric g in b) will be called the canonical metric *of (P, h).*

Theorem

The equivalence problem for generalized pseudohermitian manifolds of type (n, k) and having kind 2 reduces in a natural way to the equivalence of complete parallelisms in spaces of dimension $N = n^2 + 2n + k$.

Actually, we have a correspondence

$$(M, HM, J, h, P) \mapsto (P(M), \gamma)$$

where

$P(M)$ canonical $U(n)$ reduction of the frame bundle $\mathcal{F}(HM)$ of HM

$\gamma = \omega + \theta : TP(M) \to \mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}^k$ a parallelism such that

ω connection form of the canonical connection D

θ a kind of "solder" form

The correspondence is compatible with the respective isomorphisms.

Automorphism group

Theorem

The automorphism group $Psh(M)$ of a generalized pseudohermitian manifold of type of type (n, k) and having kind 2 is a Lie group having dimension

$$\dim(Psh(M)) \leq n^2 + 2n + k$$

If equality holds: a) $Psh(M)$ is transitive, i.e. M is homogeneous

b) M has constant pseudoholomorphic curvature.

Moreover:

If J is partially integrable, then $\dim(Psh(M)) = n^2 + 2n + k$ can hold only for $k \in \{0, 1, n^2 - 1, n^2\}$ and the Tanaka algebra $\mathfrak{m}(x)$ at an arbitrary point must be isomorphic to

$$\mathfrak{m} = \mathbb{C}^n \oplus W^* \qquad [X, Y](h) = \Im h(X, Y)$$

where $W \subset \mathfrak{H}_s(\mathbb{C}^n)$ is one of the following subspaces

$$W = \{0\}, \quad W = \langle I_n \rangle, \quad W = \mathfrak{H}_s(\mathbb{C}^n) \cap \mathfrak{sl}(n, \mathbb{C}), \quad W = \mathfrak{H}_s(\mathbb{C}^n).$$

PROBLEM: Finding a sharp estimate for dim $Psh(M)$ for each type (n, k).

Assume (HM, J) is partially integrable and strongly regular:

the Hermitian Tanaka algebras $(\mathfrak{m}(x), h_x)$, $x \in M$ are all isomorphic to a fixed one $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$

Then h induces canonically an inner product $\langle , \rangle$ on $\mathfrak{m}_{-1}$
Let

$$\mathfrak{k}_o := \{A \in Der(\mathfrak{m}) | [A, J] = 0, \ \langle AX, Y \rangle + \langle X, AY \rangle = 0 \ \ \forall X, Y \in \mathfrak{m}_{-1}\}$$

Then

Theorem

$$\dim Psh(M) \leq \dim_{\mathbb{R}} (k_o) + 2n + k$$

This inequality is sharp.

The maximum dimension is obtained for example in the following cases:

1. M = affine CR quadric = nilpotent group with $Lie(M) = \mathfrak{m}$
   h = arbitrary left invariant Hermitian metric on HM
   P left invariant projection such that $Ker(P_e) = \mathfrak{m}_{-2}$

2. M = Compact homogenous standard CR manifold K/K_o of kind 2
   (P, h)= standard pseudohermitian structure discussed previously.

Remark

Examples 1) are flat.
Examples 2) show that in our pseudohermitian geometry, there are manifolds with large automorphism group but with nonconstant pseudoholomorphic curvature.

PROBLEM: Is the above list exhaustive?

Pseudohermitian immersions (results in collaboration with G. Dileo)

$f : (M, HM, J, h, P) \to (M', HM', J', h', P')$ isopseudohermitian immersion.

There is a well-defined subbundle $HM^\perp$ of the pullback $f^*(HM')$

$H_xM^\perp$:=orthogonal complement to $f_*(H_xM)$ in $H_{f(x)}M'$

We shall drop f in the notation for semplicity, assuming $M \subset M'$

Remark

For each $X \in \mathfrak{X}(M)$: $P'(QX) \in \Gamma HM^\perp$ $\qquad Q = Id - P$

D canonical connection on HM
D' canonical connection on $HM'_{|M}$

Basic formulas for isopseudohermitian immersions

Theorem

For each $X \in \mathfrak{X}(M)$, $Y \in \Gamma HM \;\; \zeta \in \Gamma HM^{\perp}$ *we have*
Gauss like formula:

$$D'_X Y = \underbrace{D_X Y + \beta_{P'QX}(Y)}_{\text{tangent}} + \underbrace{\alpha(X, Y)}_{\text{normal}}$$

Weingarten like formula:

$$D'_X \zeta = \underbrace{-A_\zeta X}_{\text{tangent}} + \underbrace{D^{\perp}_X \zeta}_{\text{normal}}$$

Here for each $\zeta \in \Gamma HM^{\perp}$:

1. $\beta_\zeta : HM \to HM$ bundle homomorphism defined by:

$$h(\beta_\zeta Y, Z) = -\frac{1}{4} h'(P'([Y, Z] + [JY, JZ]), \zeta)$$

2. $A_\zeta : TM \to HM$ bundle homomorphism such that:
$h(A_\zeta X, Y) = h'(\alpha(X, Y), \zeta) \quad X \in \mathfrak{X}(M) \; Y \in \Gamma HM$

An interpretation of β

$$D'_X Y = D_X Y + \beta_{P'QX}(Y) + \alpha(X,Y)$$

Remark

β_ζ is skew-symmetric and commutes with J
In general, $A_\zeta : HM \to HM$ fails to be symmetric.

However:

Proposition

For an isopseudohermitian immersion $f : (M, HM, J, h, P) \to (M', HM', J', h', P')$ the following are equivalent:
a) $\beta_\zeta = 0 \qquad \forall \zeta \in HM^\perp$
b) $\mathcal{L}(X) = \mathcal{L}'(X) \qquad \forall X \in HM.$

If M' is of Quasi-Kähler type, then a) implies that $\alpha : HM \times HM \to HM^\perp$ is symmetric.

Isopseudohermitian immersions with $\beta \equiv 0$ will be called regular.

Remark: The pseudohermitian immersions introduced by Dragomir (*Amer. J. Math.* 1995) are regular.

Theorem

For a regular isopseudohermitian immersion

$$R'(X, JX, X, JX) = R(X, JX, X, JX) + 2||\alpha(X, X)||^2 \qquad X \in \ HM$$

provided that M' is of quasi Kähler type. Hence for the pseudoholomorphic curvatures:

$$H \leq H'$$

Theorem

There is no regular isopseudohermitian immersion

$$f : S(\mathfrak{g}) \to M$$

from a compact standard homogeneous pseudohermitian manifold into a gen. pseudohermitian M of quasi Kähler type having non positive pseudoholomorphic curvature.