On parabolic CR manifolds

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On parabolic CR manifolds

Given:

- a complex flag manifold $F = G/Q$
  
  ($G$ complex semisimple Lie group, $Q$ parabolic subgroup)

- a real form $G_0$ of $G$

we consider the natural action of $G_0$ on $F$.

**Definition**

A **parabolic (CR) manifold** is a $G_0$-orbit in $F$

We recall that (having fixed $F$ and $G_0$):

- there exists a finite number of $G_0$-orbits in $F$
- only one of them is closed (compact)
- open orbits are simply connected

(see J.A.Wolf, 1969).
homogeneous CR structure on a given parabolic manifold,
equivariant fibrations with complex manifolds as fibers,
topological results, in particular regarding the fundamental group.
CR manifolds and maps

**Definition (CR manifold)**

- $M = M^{2n+k}$, real manifold
- $H = H^{0,1} \subset T^C M$, complex subbundle (of rank $n$) s.t.
  \[
  \begin{cases}
  H \cap \overline{H} = 0, \\
  [\Gamma(H), \Gamma(H)] \subset \Gamma(H).
  \end{cases}
  \]

The numbers $n, k$ are the CR dimension and CR codimension, respectively.

**Remark (Homogeneous CR manifolds)**

An orbit $M$ for a group of biholomorphisms of a complex manifold $X$ is a CR manifold with CR structure

$$H = \bigcup_{p \in X} H_p, \quad H_p = T^{0,1}_p X \cap T^C_p M, \quad p \in M.$$ 

In particular a parabolic manifold $M \subset F = G/Q$ is a CR manifold.
Definition (CR maps and fibrations)

A **CR map** between two CR manifolds \((M, H)\) and \((M', H')\) is an

\[ f : M \to M' \text{ such that } df^\mathbb{C}(H) \subset H'. \]

A **CR map** \(f\) is a **CR fibration** if \(f\) is submersion and \(df^\mathbb{C}(H) = H'\).
Weakly nondegenerate CR manifolds

**Definition**

A CR manifold \((M, H)\) is said to be **weakly nondegenerate** (briefly **WND**) in \(p \in M\) if

\[
\forall Z \in \Gamma(H), \ Z_p \neq 0, \ \exists Z_1 \ldots, Z_\ell \in \Gamma(H) : [Z_1, \ldots, [Z_\ell, \overline{Z}] \ldots]_p \notin H + \overline{H}.
\]

(The case \(k = 1\) corresponds to a nondegenerate Levi-form)

**Criterion**

Let \(M\) be a homogeneous CR manifold. Then \(M\) is WD if and only if there exists a local CR fibration \(\pi : M \to M'\), with nontrivial complex fibers.
Example

The Grassmannian $Gr_{\mathbb{C}}(2, 4)$ of 2-spaces of $\mathbb{C}^4$ is a flag manifold $F = SL(4, \mathbb{C})/Q$ where $Q = \{ Z \in SL(4, \mathbb{C}) \mid Z(\langle e_1, e_2 \rangle_{\mathbb{C}}) \subseteq \langle e_1, e_2 \rangle_{\mathbb{C}} \}$. 

We consider a Hermitian symmetric form on $\mathbb{C}^4$ associated to the matrix

$$K = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.$$ 

Let $G_0 = \{ Z \in SL(4, \mathbb{C}) \mid KZ + Z^*K = 0 \} \simeq SL(3, 1)$. The parabolic manifolds are given by the sets of 2-spaces with signature $(2, 0)$, $(1, 1)$ and $(1, 0)$. 

The compact orbit is $G_0 \cdot (\langle e_1, e_2 \rangle_{\mathbb{C}})$. It is a CR manifold of CR-dimension 3 and CR-codimension 1, with degenerate Levi form but WND.
CR algebras

To a parabolic CR manifold immersed in a flag manifold

$$M = G_0/I_0 \hookrightarrow F = G/Q$$

we associate a pair of Lie algebras

$$(g_0, q) = (\text{Lie}(G_0), \text{Lie}(Q))$$

This is called a CR algebra. Note that $q \subset g$ is a complex parabolic subalgebra. Such CR algebra is called a parabolic CR algebra.

To given a parabolic CR algebra $(g_0, q)$, we associate a parabolic CR manifold $M = M(g_0, q)$ in the following way:

- $G$ is a connected Lie group with Lie algebra $g$,
- $Q$ and $G_0$ are analytic subgroups of $G$ with Lie algebras $q$ and $g_0$,
- $F = G/Q$,

$M = M(g_0, q)$ is the orbit of $G_0$ in $F = G/Q$ through the point $o = eQ$. 
The isotropy subgroup of $M(g_0, q)$ and its isotropy subalgebra are

$$I_0 = G_0 \cap Q, \quad i_0 = g_0 \cap q.$$  

Moreover,

$$\dim_{CR} M(g_0, q) = \dim \mathbb{C} (q/(q \cap \bar{q})).$$

Let $M = M(g_0, q)$ and $M' = M(g_0, q')$ be parabolic CR manifolds with $i_0 \subseteq i_0' := g_0 \cap q$. We have a $G_0$-equivariant fibration $F$ between them:

$$M = G_0/I_0 \xrightarrow{F} M' = G_0/I_0'$$

Then:

- $F$ a CR map $\iff q \subset q'$
- $F$ a CR fibration $\iff q' = q + (q' \cap \bar{q'})$. 
Standard parabolic subalgebras

It is possible to choose

- $h \subset q$, CSA of $g$
- $B = \{\alpha_1, \ldots, \alpha_\ell\} \subset \mathcal{R} = \mathcal{R}(g, h)$, basis of simple roots such that

$$q = q_S \text{ for a subset } S \subseteq B,$$

where

$$q_S = \sum_{\alpha \in \mathcal{R}^+} g_\alpha + h + \sum_{\alpha \in \mathcal{R}^- \atop \text{supp}(\alpha) \cap S = \emptyset} g_\alpha$$

$$= \sum_{\alpha \in \mathcal{R}^+ \atop \text{supp}(\alpha) \cap S \neq \emptyset} g_\alpha + h + \sum_{\text{supp}(\alpha) \cap S = \emptyset} g_\alpha$$

$$= q^n + \sum_{\text{supp}(\alpha) \cap S = \emptyset} q^r$$

and $\text{supp}(\alpha) := \{\alpha_j \in B \mid \alpha = \sum_j n_j \alpha_j, \ n_j \neq 0\}$. 
Remark

\[ \exists h_0, \text{CSA of } g_0 : \quad h_0 \subset i_0 := g_0 \cap q. \]

With compact parabolic manifolds, we can choose \( h_0 \) maximally non-compact. This is generally not possible.

Remark

Different choices of a root basis \( \mathcal{B} \) (corresponding to a Borel subalgebra \( b \subset q \)) are given by different choices of Weyl chambers \( C \).
Weakening of the CR structure

Let

\[ M = M(g_0, q) \cong G_0/I_0 \quad \hookrightarrow \quad F = G/Q \]

Recall that:

- \( i_0 = g_0 \cap q = g_0 \cap (q \cap \bar{q}) \),
- \( \dim_{CR} M = \dim q - \dim(q \cap \bar{q}) \).

**Definition**

Denote by \( q_w \subset q \), the *minimal* parabolic subalgebra such that:

\[ q_w \cap \bar{q}_w = q \cap \bar{q} . \]

The parabolic manifold \( M_w = M(g_0, q_w) \) is called the **CR-weakening** of \( M(g_0, q) \).

Note that:

\[ M_w = M(g_0, q_w) \cong G_0/I_0 \quad \hookrightarrow \quad F' = G/Q_w . \quad (1) \]

Then:

- \( M_w \) is diffeomorphic to \( M \) as real manifold
- \( M_w \) has a different (minimal) CR structure given by the immersion (1).
(Weakening of the CR structure)

As \( q_w \subset q \), the natural fibration

\[
f : M_w = M(g_0, q_w) \xrightarrow{\sim_{CR}} M = M(g_0, q)
\]

is a CR map and a diffeomorphism.

**Proposition**

We have:

\[
q_w = q^n + q \cap \bar{q}
\]

For a suitable choice of a basis of simple roots \( B \subset \mathbb{R} \) (corresponding to an S-fit Weyl chamber), we obtain:

\[
q = q_S, \quad S \subseteq B \quad \implies \quad q_w = q_{S^*}, \quad S^* = S \cup \{\alpha \in B \mid \bar{\alpha} > 0, \ \text{supp}(\bar{\alpha}) \cap S \neq \emptyset\}
\]

**Lemma**

\( M_w \) is either weakly degenerate or real (i.e. with a trivial CR structure).
Theorem (WND reduction)

Let $M = M(\mathfrak{g}_0, q)$ be a parabolic CR manifold. Then there exists a $G_0$-equivariant CR fibration

$$\pi : M \longrightarrow M'$$

with

- a WND base $M' = M(\mathfrak{g}_0, q')$
- simply-connected complex fibers (eventually disconnected).

The fibers are non trivial $\iff M$ is WD.

Definition

The CR fibration $\pi : M \longrightarrow M'$ is said WND reduction of $M$. 
**Criterion**

A parabolic CR manifold $M(\mathfrak{g}_0, \mathfrak{q})$ is WD if and only if there is a complex subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ such that $\mathfrak{q} \subsetneq \mathfrak{q}' \subset \mathfrak{q} + \overline{\mathfrak{q}}$.

**Proposition**

For a suitable choice of a basis of simple roots $\mathcal{B} \subset \mathcal{R}$ (corresponding to a V-fit Weyl chamber), we obtain:

$q = q_S, \quad S \subseteq \mathcal{B} \quad \implies \quad q' = q_{S'}, \quad S' = \{ \alpha \in S \mid \tilde{\alpha} > 0 \}$.
Structure theorem

Theorem (Structure theorem)

Let $M = M(g_0, q)$ be a parabolic CR manifold. Then there exists a $G_0$-equivariant fibration

$$M \xrightarrow{\psi} M_c$$

with

- a real flag manifold $M_c = M(g_0, c)$ as base space,
- simply-connected, complex fibers.

Definition

The parabolic manifold $M_c = M(g_0, c)$ is called the real core of $M$. 
Construction

\[ \begin{array}{ccc}
M & \xrightarrow{f_{(0)}^{-1}} & M_{w}^{(0)} \\
\pi_{(0)} & & \downarrow \\
M^{(0)} & \rightarrow & M_{w}^{(0)} \\
\pi_{(1)} & & \\
M^{(1)} & \xrightarrow{f_{(1)}^{-1}} & M_{w}^{(1)} \\
\pi_{(2)} & & \\
M^{(2)} & \rightarrow & M_{w}^{(2)} \\
& & \\
& \cdots & \xrightarrow{f_{(r)}^{-1}} M_{c} \\
\end{array} \]

where
- the vertical maps are WND reductions (CR maps)
- the horizontal maps give the weakening of the CR structure (diffeomorphisms)

Each manifold \( M_{w}^{(j)} \) is either WD or real.

\[ \Psi = f_{(r)}^{-1} \circ \pi_{(r)} \circ \cdots \circ f_{(0)}^{-1} \circ \pi_{(0)} \]
Example

\[ \mathcal{F}_{d_1, \ldots, d_r} := \{ (l_1, \ldots, l_r) \mid l_1 \subsetneq l_2 \subsetneq \cdots \subsetneq l_r \text{ subspaces of } \mathbb{C}^7, \dim l_j = d_j \}. \]

Let \((e_1, \ldots, e_7)\) be the standard basis of \(\mathbb{C}^7\) and

\[ \epsilon_1 = e_1 + ie_7, \quad \epsilon_2 = e_2, \quad \epsilon_3 = e_3 + ie_6, \quad \epsilon_4 = e_4, \quad \epsilon_5 = e_5, \quad \epsilon_6 = e_3 - ie_6, \quad \epsilon_7 = e_1 - ie_7 \]

Let \(G_0 = SL(7, \mathbb{R})\) and consider the parabolic manifold \(M = G_0 \cdot \gamma \subset \mathcal{F}_{1,2,3,4,5,6,7}\)
where

\[ \gamma = (\langle \epsilon_1 \rangle, \ldots, \langle \epsilon_1, \ldots, \epsilon_7 \rangle) \in \mathcal{F}_{1,2,3,4,5,6,7} \]

The WND reduction of \(M\) is the \(G_0\)-orbit \(M^{(0)} = G_0 \cdot \gamma_0 \subset \mathcal{F}_{2,4}\) through the flag

\[ \gamma_0 = (\langle \epsilon_1, \epsilon_2 \rangle, \langle \epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4 \rangle) \in \mathcal{F}_{2,4} \]
Example (continue)

Continuing the construction given by the structure theorem, we obtain that:

\( M^{(1)} = G_0 \cdot \gamma_1 \subset \mathcal{F}_{1,3,5,7} \) is the \( G_0 \)-orbit through the flag

\[
\gamma_1 = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4, \epsilon_3, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4, \epsilon_3, \epsilon_7, \epsilon_6 \rangle) \in \mathcal{F}_{1,3,5,6}
\]

\( M^{(2)} = G_0 \cdot \gamma_2 \subset \mathcal{F}_{1,2,4,6} \) is the \( G_0 \)-orbit through the flag

\[
\gamma_2 = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_4 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7, \epsilon_3, \epsilon_6 \rangle) \in \mathcal{F}_{1,2,4,6}
\]

The manifold \( M^{(2)} \) has a trivial CR structure, then \( M^{(2)} = M_c \), the real core of \( M \).
**Definition (J.A. Wolf)**

Let $M$ be a $G_0$-orbit in a flag manifold $F = G/Q$. Let $(\theta, h_0)$ an adapted Cartan pair and $\mathcal{R} = \mathcal{R}(\mathfrak{g}, q)$. Define:

\[
\delta = \sum_{\alpha \in Q^\vee \cap \mathfrak{q}_n} \alpha,
\]
\[
Q_a = \{ \alpha \in \mathcal{R} \mid (\delta|\alpha) \geq 0 \},
\]
\[
q_a = \mathfrak{h} \oplus \sum_{\alpha \in Q_a} \mathfrak{g}^\alpha.
\]

The $G_0$-orbit $M_a = M(\mathfrak{g}_0, q_a)$ corresponding to the parabolic CR algebra $(\mathfrak{g}_0, q_a)$ is called the **space of algebraic arc components** of $M$.

The fibers of $M \longrightarrow M_a$ are the **algebraic arc components**.

**Proposition**

Let $M$ be a **compact** $G_0$-orbit in a flag manifold. Then $M_a = M_c$. 
In general (for non-closed orbits $M$) the notions of ”algebraic arc component” $M_a$ and ”real core” $M_c$ don’t coincide as shown by the previous example:

**Example**

$$\mathcal{F}^7_{d_1,\ldots,d_r} := \{ (l_1, \ldots, l_r) \mid l_1 \subsetneq l_2 \subsetneq \cdots \subsetneq l_r \text{ subspaces of } \mathbb{C}^7, \dim l_j = d_j \}.$$  
Let $(e_1, \ldots, e_7)$ be the standard basis of $\mathbb{C}^7$ and

$$\epsilon_1 = e_1 + ie_7, \quad \epsilon_2 = e_2, \quad \epsilon_3 = e_3 + ie_6, \quad \epsilon_4 = e_4, \quad \epsilon_5 = e_5, \quad \epsilon_6 = e_3 - ie_6, \quad \epsilon_7 = e_1 - ie_7$$

Let $G_0 = SL(7, \mathbb{R})$ and consider the parabolic manifold $M = G_0 \cdot \gamma \subset \mathcal{F}^7_{1,2,3,4,5,6,7}$ where

$$\gamma = (\langle \epsilon_1 \rangle, \ldots, \langle \epsilon_1, \ldots, \epsilon_7 \rangle) \in \mathcal{F}^7_{1,2,3,4,5,6,7}.$$  

The space of algebraic arc components $M_a$ of $M$ is the $G_0$-orbit $M_a = G_0 \cdot \gamma_a \subset \mathcal{F}^7_{1,4,6}$ through the flag

$$\gamma_a = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7, \epsilon_3, \epsilon_6 \rangle) \in \mathcal{F}^7_{1,2,6}.$$  

This shows that $M_c$ is different from $M_a$. 
The fundamental group of a parabolic manifold

**Theorem**

\[ M = M(g_0, q) \text{ parabolic manifold} \]
\[ M_c = M(g_0, c) \text{ real core of } M \]

Then there is an exact sequence:

\[
1 = \pi_1(F) \longrightarrow \pi_1(M) \longrightarrow \pi_1(M_c) \longrightarrow \pi_0(F) \longrightarrow \pi_0(M) = 1
\]

where:

- \( L_c \) maximal reductive factor of \( I_c = G_0 \cap Q_c \)
- \( S_c \) maximal analytic semisimple subgroup of \( L_c \)
- \( h_0 \) maximally non-compact CSA in \( i_0 = g_0 \cap q \)
- \( W(L_c, h_0) = N_{L_c}(h_0)/Z_{L_c}(h_0) \) and
- \( W(S_c, h_0 \cap s_c) = N_{S_c}(h_0 \cap s_c)/Z_{S_c}(h_0 \cap s_c) \).
The fundamental group $\pi_1(M_c)$ of the real core can be described in terms of generators and relations:

$$
\Gamma = \{ \xi_\alpha \mid \alpha = \tilde{\alpha} \in B, \text{ has multiplicity 1} \}
$$

$$
\xi_\alpha = 1 \text{ if } \alpha \in S_c, \quad \xi_\alpha \xi_\beta = \xi_\beta \xi_{(\alpha | \beta)} \quad \forall \xi_\alpha, \xi_\beta \in \Gamma
$$

(see Wiggerman, 1998)

**Corollary**

Let $M = M(g_0, q)$ be a parabolic manifold. Assume $g_0 = g_1 \oplus \cdots \oplus g_r$ is a sum of simple ideals $g_j$ of the following type:

- complex type
- compact type
- $A_{IIIa}, A_{IV}, B_{II}, C_{II}, D_{II}, D_{IIIb}, E_{III}, E_{IV}, F_{II}$.

Then $M$ is simply-connected. (Each $G_0$-orbit in $F = G/Q$ is simply-connected). Moreover, if $g_0$ has simple factors of the type given above or of type $A_{IIIb}$ and $D_{IIIa}$, then

$$
\pi_1(M) \simeq \pi_1(M_c).
$$