On parabolic CR manifolds

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On parabolic CR manifolds

Given:

- a complex flag manifold $F = G/Q$

  (G complex semisimple Lie group, Q parabolic subgroup)
- a real form G_0 of G

we consider the natural action of G_0 on F.

Definition

A **parabolic** *(CR) manifold is a G_0-orbit in F*

We recall that (having fixed F and G_0):

- there exists a finite number of G_0-orbits in F
- only one of them is closed (compact)
- open orbits are simply connected

(see J.A.Wolf, 1969).

- homogeneous CR structure on a given parabolic manifold,
- equivariant fibrations with complex manifolds as fibers,
- topological results, in particular regarding the fundamental group.

CR manifolds and maps

Definition (CR manifold)

- *$M = M^{2n+k}$, real manifold*
- *$H = H^{0,1} \subset T^{\mathbb{C}}M$, complex subbundle (of rank n) s.t.*

$$\begin{cases} H \cap \overline{H} = 0, \\ [\Gamma(H), \Gamma(H)] \subset \Gamma(H). \end{cases}$$

The numbers n, k are the CR dimension and CR codimension, respectively.

Remark (Homogeneous CR manifolds)

An orbit M for a group of biholomorphisms of a complex manifold X is a CR manifold with CR structure

$$H = \bigcup_{p \in X} H_p, \quad H_p = T_p^{0,1}X \cap T_p^{\mathbb{C}}M, \quad p \in M.$$

In particular a parabolic manifold $M \subset F = G/Q$ is a CR manifold.

Definition (CR maps and fibrations)

A CR map between two CR manifolds (M, H) *and* (M', H') *is an*

$$f : M \to M' \quad \text{such that} \quad df^{\mathbb{C}}(H) \subset H'.$$

A CR map f *is a CR fibration if* f *is submersion and* $df^{\mathbb{C}}(H) = H'$.

Weakly nondegenerate CR manifolds

Definition

A CR manifold (M, H) is said to be weakly nondegenerate (briefly WND) in $p \in M$ if

$$\forall Z \in \Gamma(H), Z_p \neq 0, \exists Z_1 \ldots, Z_\ell \in \Gamma(H) : [Z_1, \ldots, [Z_\ell, \overline{Z}] \ldots]_p \notin H + \overline{H} \, .$$

(The case $k = 1$ corresponds to a nondegenerate Levi-form)

Criterion

Let M be a homogeneous CR manifold. Then M is WD if and only if there exists a local CR fibration $\pi : M \to M'$, with nontrivial complex fibers.

Example

The Grassmannian $Gr_{\mathbb{C}}(2,4)$ of 2-spaces of $\mathbb{C}^4$ is a flag manifold $F = SL(4,\mathbb{C})/Q$ where $Q = \{Z \in SL(4,\mathbb{C}) \,|\, Z(\langle e_1, e_2\rangle_{\mathbb{C}}) \subseteq \langle e_1, e_2\rangle_{\mathbb{C}}\}$.

We consider a Hermitian symmetric form on $\mathbb{C}^4$ associated to the matrix

$$K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $G_0 = \{Z \in SL(4,\mathbb{C}) \,|\, KZ + Z^*K = 0\} \simeq SL(3,1)$. The parabolic manifolds are given by the sets of 2-spaces with signature $(2,0)$, $(1,1)$ and $(1,0)$.

The compact orbit is $G_0 \cdot (\langle e_1, e_2\rangle_{\mathbb{C}})$. It is a CR manifold of CR-dimension 3 and CR-codimension 1, with degenerate Levi form but WND.

CR algebras

To a parabolic CR manifold immersed in a flag manifold

$$M = G_0/I_0 \hookrightarrow F = G/Q$$

we associate a pair of Lie algebras

$$(\mathfrak{g}_0, \mathfrak{q}) = (\mathrm{Lie}(G_0), \mathrm{Lie}(Q))$$

This is called a CR algebra. Note that $\mathfrak{q} \subset \mathfrak{g}$ is a complex parabolic subalgebra. Such CR algebra is called a parabolic CR algebra.

To given a parabolic CR algebra $(\mathfrak{g}_0, \mathfrak{q})$, we associate a parabolic CR manifold $M = M(\mathfrak{g}_0, \mathfrak{q})$ in the following way:

- G is a connected Lie group with Lie algebra $\mathfrak{g}$,
- Q and G_0 are analytic subgroups of G with Lie algebras $\mathfrak{q}$ and $\mathfrak{g}_0$,
- $F = G/Q$,

$M = M(\mathfrak{g}_0, \mathfrak{q})$ is the orbit of G_0 in $F = G/Q$ through the point $o = eQ$.

The isotropy subgroup of $M(\mathfrak{g}_0, \mathfrak{q})$ and its isotropy subalgebra are

$$I_0 = G_0 \cap Q, \qquad \mathfrak{i}_0 = \mathfrak{g}_0 \cap \mathfrak{q}.$$

Moreover,

$$\dim_{CR} M(\mathfrak{g}_0, \mathfrak{q}) = \dim_{\mathbb{C}} \left(\mathfrak{q}/(\mathfrak{q} \cap \bar{\mathfrak{q}})\right).$$

Let $M = M(\mathfrak{g}_0, \mathfrak{q})$ and $M' = M(\mathfrak{g}_0, \mathfrak{q}')$ be parabolic CR manifolds with $\mathfrak{i}_0 \subseteq \mathfrak{i}_0{}' := \mathfrak{g}_0 \cap \mathfrak{q}$. We have a G_0-equivariant fibration F between them:

$$M = G_0/I_0 \xrightarrow{F} M' = G_0/I_0{}'$$

Then:

- F a CR map $\iff \mathfrak{q} \subset \mathfrak{q}'$
- F a CR fibration $\iff \mathfrak{q}' = \mathfrak{q} + (\mathfrak{q}' \cap \bar{\mathfrak{q}}')$.

Standard parabolic subalgebras

It is possible to choose

- $\mathfrak{h} \subset \mathfrak{q}$, CSA of $\mathfrak{g}$
- $\mathcal{B} = \{\alpha_1, \dots, \alpha_\ell\} \subset \mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$, basis of simple roots

such that

$$\mathfrak{q} = \mathfrak{q}_{\mathcal{S}} \quad \text{for a subset } \mathcal{S} \subseteq \mathcal{B},$$

where

$$\begin{aligned} \mathfrak{q}_{\mathcal{S}} &= \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha + \mathfrak{h} + \sum_{\substack{\alpha \in \mathcal{R}^- \\ supp(\alpha) \cap \mathcal{S} = \emptyset}} \mathfrak{g}_\alpha \\ &= \underbrace{\sum_{\substack{\alpha \in \mathcal{R}^+ \\ supp(\alpha) \cap \mathcal{S} \neq \emptyset}} \mathfrak{g}_\alpha}_{\mathfrak{q}^n} + \underbrace{\mathfrak{h} + \sum_{supp(\alpha) \cap \mathcal{S} = \emptyset} \mathfrak{g}_\alpha}_{\mathfrak{q}^r} \end{aligned}$$

and $\mathrm{supp}(\alpha) := \{\alpha_j \in \mathcal{B} \,|\, \alpha = \sum_j n_j \alpha_j,\ n_j \neq 0\}$.

Remark

$$\exists \mathfrak{h}_0, \textit{CSA of } \mathfrak{g}_0 : \quad \mathfrak{h}_0 \subset \mathfrak{i}_0 := \mathfrak{g}_0 \cap \mathfrak{q}.$$

With compact parabolic manifolds, we can choose $\mathfrak{h}_0$ *maximally non-compact. This is generally not possible.*

Remark

Different choices of a root basis $\mathcal{B}$ *(corresponding to a Borel subalgebra* $\mathfrak{b} \subset \mathfrak{q}$*) are given by different choices of Weyl chambers* C*.*

Weakening of the CR structure

Let

$$M = M(\mathfrak{g}_0, \mathfrak{q}) \simeq G_0/I_0 \quad \hookrightarrow \quad F = G/Q$$

Recall that:

- $\mathfrak{i}_0 = \mathfrak{g}_0 \cap \mathfrak{q} = \mathfrak{g}_0 \cap (\mathfrak{q} \cap \bar{\mathfrak{q}})$,
- $\dim_{CR} M = \dim \mathfrak{q} - \dim(\mathfrak{q} \cap \bar{\mathfrak{q}})$.

Definition

Denote by $\mathfrak{q}_w \subset \mathfrak{q}$, the minimal parabolic subalgebra such that:

$$\mathfrak{q}_w \cap \bar{\mathfrak{q}}_w = \mathfrak{q} \cap \bar{\mathfrak{q}} \,.$$

The parabolic manifold $M_w = M(\mathfrak{g}_0, \mathfrak{q}_w)$ is called the CR-weakening of $M(\mathfrak{g}_0, \mathfrak{q})$.

Note that:

$$M_w = M(\mathfrak{g}_0, \mathfrak{q}_w) \simeq G_0/I_0 \quad \hookrightarrow \quad F' = G/Q_w. \tag{1}$$

Then:

- M_w is diffeomorphic to M as real manifold
- M_w has a different (minimal) CR structure given by the immersion (1).

(Weakening of the CR structure)

As $\mathfrak{q}_w \subset \mathfrak{q}$, the natural fibration

$$f : M_w = M(\mathfrak{g}_0, \mathfrak{q}_w) \xrightarrow[CR]{\simeq} M = M(\mathfrak{g}_0, \mathfrak{q})$$

is a CR map and a diffeomorphism.

Proposition

We have:

$$\mathfrak{q}_w = \mathfrak{q}^n + \mathfrak{q} \cap \bar{\mathfrak{q}}$$

For a suitable choice of a basis of simple roots $\mathcal{B} \subset \mathcal{R}$ (corresponding to an S-fit Weyl chamber), we obtain:

$$\mathfrak{q} = \mathfrak{q}_{\mathcal{S}}, \quad \mathcal{S} \subseteq \mathcal{B} \quad \Longrightarrow \quad \mathfrak{q}_w = \mathfrak{q}_{\mathcal{S}^*}, \quad \mathcal{S}^* = \mathcal{S} \cup \{\alpha \in \mathcal{B} \mid \bar{\alpha} > 0, \operatorname{supp}(\bar{\alpha}) \cap \mathcal{S} \neq \emptyset\}.$$

Lemma

M_w is either weakly degenerate or real (i.e. with a trivial CR structure).

Reduction to WND manifolds

Theorem (WND reduction)

Let $M = M(\mathfrak{g}_0, \mathfrak{q})$ be a parabolic CR manifold. Then there exists a G_0-equivariant CR fibration

$$\pi : M \longrightarrow M'$$

with

- *a WND base $M' = M(\mathfrak{g}_0, \mathfrak{q}')$*
- *simply-connected complex fibers (eventually disconnected).*

The fibers are non trivial $\iff$ M is WD.

Definition

The CR fibration $\pi : M \longrightarrow M'$ is said WND reduction of M.

(Reduction to WND manifolds)

Criterion

A parabolic CR manifold $M(\mathfrak{g}_0, \mathfrak{q})$ is WD if and only if there is a complex subalgebra $\mathfrak{q}'$ of $\mathfrak{g}$ such that $\mathfrak{q} \subsetneq \mathfrak{q}' \subset \mathfrak{q} + \bar{\mathfrak{q}}$.

Proposition

For a suitable choice of a basis of simple roots $\mathcal{B} \subset \mathcal{R}$ (corresponding to a V-fit Weyl chamber), we obtain:

$$\mathfrak{q} = \mathfrak{q}_{\mathcal{S}}, \quad \mathcal{S} \subseteq \mathcal{B} \quad \Longrightarrow \quad \mathfrak{q}' = \mathfrak{q}_{\mathcal{S}'}, \quad \mathcal{S}' = \{\alpha \in \mathcal{S} \,|\, \bar{\alpha} > 0\}.$$

Structure theorem

Theorem (Structure theorem)

Let $M = M(\mathfrak{g}_0, \mathfrak{q})$ *be a parabolic CR manifold.*
Then there exists a G_0*-equivariant fibration*

$$M \xrightarrow{\Psi} M_{\mathfrak{c}}$$

with

- *a real flag manifold* $M_{\mathfrak{c}} = M(\mathfrak{g}_0, \mathfrak{c})$ *as base space,*
- *simply-connected, complex fibers.*

Definition

The parabolic manifold $M_{\mathfrak{c}} = M(\mathfrak{g}_0, \mathfrak{c})$ *is called the real core of* M.

Construction

$$
\begin{tikzcd}
M \arrow[d, "\pi_{(0)}"'] & & & & \\
M^{(0)} \arrow[r, "f_{(0)}^{-1}"] & M_w^{(0)} \arrow[d, "\pi_{(1)}"] & & & \\
& M^{(1)} \arrow[r, "f_{(1)}^{-1}"] & M_w^{(1)} \arrow[d, "\pi_{(2)}"] & & \\
& & M^{(2)} & \cdots \arrow[r, "f_{(r)}^{-1}"] & M_{\mathfrak{c}}
\end{tikzcd}
$$

where

- the vertical maps are WND reductions (CR maps)
- the horizontal maps give the weakening of the CR structure (diffeomorphisms)

Each manifold $M_w^{(j)}$ is either WD or real.

$$\Psi = f_{(r)}^{-1} \circ \pi_{(r)} \circ \cdots \circ f_{(0)}^{-1} \circ \pi_{(0)}$$

Example

$$\mathcal{F}^7_{d_1,\dots,d_r} := \{(\ell_1,\dots,\ell_r) \,|\, \ell_1 \subsetneq \ell_2 \subsetneq \cdots \subsetneq \ell_r \text{ subspaces of } \mathbb{C}^7, \dim \ell_j = d_j\}.$$

Let $(e_1,\dots,e_7)$ be the standard basis of $\mathbb{C}^7$ and

$\epsilon_1 = e_1 + ie_7$, $\epsilon_2 = e_2$, $\epsilon_3 = e_3 + ie_6$, $\epsilon_4 = e_4$, $\epsilon_5 = e_5$, $\epsilon_6 = e_3 - ie_6$, $\epsilon_7 = e_1 - ie_7$

Let $G_0 = SL(7,\mathbb{R})$ and consider the parabolic manifold $M = G_0 \cdot \gamma \subset \mathcal{F}^7_{1,2,3,4,5,6,7}$ where

$$\gamma = (\langle\epsilon_1\rangle, \dots, \langle\epsilon_1,\dots,\epsilon_7\rangle) \in \mathcal{F}^7_{1,2,3,4,5,6,7}$$

The WND reduction of M is the G_0-orbit $M^{(0)} = G_0 \cdot \gamma_0 \subset \mathcal{F}^7_{2,4}$ through the flag

$$\gamma_0 = (\langle\epsilon_1,\epsilon_2\rangle, \langle\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4\rangle) \in \mathcal{F}^7_{2,4}$$

Example (continue)

Continuing the construction given by the structure theorem, we obtain that:

$M^{(1)} = G_0 \cdot \gamma_1 \subset \mathcal{F}^7_{1,3,5,7}$ is the G_0-orbit through the flag

$$\gamma_1 = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4, \epsilon_3, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_1, \epsilon_4, \epsilon_3, \epsilon_7, \epsilon_6 \rangle) \in \mathcal{F}^7_{1,3,5,6}$$

$M^{(2)} = G_0 \cdot \gamma_2 \subset \mathcal{F}^7_{1,2,4,6}$ is the G_0-orbit through the flag

$$\gamma_2 = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_4 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7, \epsilon_3, \epsilon_6 \rangle) \in \mathcal{F}^7_{1,2,4,6}$$

The manifold $M^{(2)}$ has a trivial CR structure, then $M^{(2)} = M_{\mathfrak{c}}$, the real core of M.

Real core and algebraic arc components

Definition (J.A. Wolf)

Let M be a G_0-orbit in a flag manifold $F = G/Q$.
Let $(\theta, \mathfrak{h}_0)$ an adapted Cartan pair and $\mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{q})$. Define:

$$\begin{aligned} \delta &= \textstyle\sum_{\alpha \in \mathcal{Q}^n \cap \bar{\mathcal{Q}}^n} \alpha, \\ \mathcal{Q}_a &= \{\alpha \in \mathcal{R} \,|\, (\delta|\alpha) \geq 0\}, \\ \mathfrak{q}_a &= \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{Q}_a} \mathfrak{g}^\alpha. \end{aligned}$$

The G_0-orbit $M_a = M(\mathfrak{g}_0, \mathfrak{q}_a)$ corresponding to the parabolic CR algebra $(\mathfrak{g}_0, \mathfrak{q}_a)$ is called the space of algebraic arc components of M.

The fibers of $M \longrightarrow M_a$ are the algebraic arc components.

Proposition

Let M be a compact G_0-orbit in a flag manifold. Then $M_a = M_{\mathfrak{c}}$.

In general (for non-closed orbits M) the notions of "algebraic arc component" M_a and "real core" $M_{\mathfrak{c}}$ don't coincide as shown by the previous example:

Example

$$\mathcal{F}^7_{d_1,\dots,d_r} := \{(\ell_1,\dots,\ell_r) \,|\, \ell_1 \subsetneq \ell_2 \subsetneq \cdots \subsetneq \ell_r \text{ subspaces of } \mathbb{C}^7, \dim \ell_j = d_j\}.$$

Let $(e_1,\dots,e_7)$ be the standard basis of $\mathbb{C}^7$ and

$\epsilon_1 = e_1 + ie_7,\ \epsilon_2 = e_2,\ \epsilon_3 = e_3 + ie_6,\ \epsilon_4 = e_4,\ \epsilon_5 = e_5,\ \epsilon_6 = e_3 - ie_6,\ \epsilon_7 = e_1 - ie_7$

Let $G_0 = SL(7,\mathbb{R})$ and consider the parabolic manifold $M = G_0 \cdot \gamma \subset \mathcal{F}^7_{1,2,3,4,5,6,7}$ where

$$\gamma = (\langle \epsilon_1 \rangle, \dots, \langle \epsilon_1, \dots, \epsilon_7 \rangle) \in \mathcal{F}^7_{1,2,3,4,5,6,7}.$$

The space of algebraic arc components M_a of M is the G_0-orbit $M_a = G_0 \cdot \gamma_a \subset \mathcal{F}^7_{1,4,6}$ through the flag

$$\gamma_a = (\langle \epsilon_2 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7 \rangle, \langle \epsilon_2, \epsilon_4, \epsilon_1, \epsilon_7, \epsilon_3, \epsilon_6 \rangle) \in \mathcal{F}^7_{1,2,6}$$

This shows that $M_{\mathfrak{c}}$ is different from M_a.

The fundamental group of a parabolic manifold

Theorem

$M = M(\mathfrak{g}_0, \mathfrak{q})$ *parabolic manifold*

$M_{\mathfrak{c}} = M(\mathfrak{g}_0, \mathfrak{c})$ *real core of* M

Then there is an exact sequence:

$$\begin{tikzcd} 1 = \pi_1(F) \arrow[r] & \pi_1(M) \arrow[r] & \pi_1(M_{\mathfrak{c}}) \arrow[r] & \pi_0(F) \arrow[r] \arrow[d, "\simeq"] & \pi_0(M) = 1 \\ & & & \frac{W(L_{\mathfrak{c}}, \mathfrak{h}_0)}{W(S_{\mathfrak{c}}, \mathfrak{h}_0 \cap \mathfrak{s}_{\mathfrak{c}})} & \end{tikzcd}$$

where:

$L_{\mathfrak{c}}$ *maximal reductive factor of* $I_{\mathfrak{c}} = G_0 \cap Q_{\mathfrak{c}}$

$S_{\mathfrak{c}}$ *maximal analytic semisimple subgroup of* $L_{\mathfrak{c}}$

$\mathfrak{h}_0$ *maximally non-compact CSA in* $\mathfrak{i}_0 = \mathfrak{g}_0 \cap \mathfrak{q}$

$W(L_{\mathfrak{c}}, \mathfrak{h}_0) = N_{L_{\mathfrak{c}}}(\mathfrak{h}_0)/Z_{L_{\mathfrak{c}}}(\mathfrak{h}_0)$ *and*
$W(S_{\mathfrak{c}}, \mathfrak{h}_0 \cap \mathfrak{s}_{\mathfrak{c}}) = N_{S_{\mathfrak{c}}}(\mathfrak{h}_0 \cap \mathfrak{s}_{\mathfrak{c}})/Z_{S_{\mathfrak{c}}}(\mathfrak{h}_0 \cap \mathfrak{s}_{\mathfrak{c}})$.

The fundamental group $\pi_1(M_{\mathfrak{c}})$ of the real core can be described in terms of generators and relations:

$$\Gamma = \{\xi_\alpha \,|\, \alpha = \bar{\alpha} \in \mathcal{B}, \text{ has multiplicity } 1\}$$

$$\xi_\alpha = 1 \text{ if } \alpha \in \mathcal{S}_{\mathfrak{c}}, \quad \xi_\alpha \xi_\beta = \xi_\beta \xi_\alpha^{(\alpha|\check{\beta})} \quad \forall \xi_\alpha, \xi_\beta \in \Gamma$$

(see Wiggerman, 1998)

Corollary

Let $M = M(\mathfrak{g}_0, \mathfrak{q})$ be a parabolic manifold.
Assume $\mathfrak{g}_0 = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ is a sum of simple ideals $\mathfrak{g}_j$ of the following type:

- *complex type*
- *compact type*
- *AII, AIIIa, AIV, BII, CII, DII, DIIIb, EIII, EIV, FII.*

Then M is simply-connected. (Each G_0-orbit in $F = G/Q$ is simply-connected).
Moreover, if $\mathfrak{g}_0$ has simple factors of the type given above or of type AIIIb and DIIIa, then

$$\pi_1(M) \simeq \pi_1(M_{\mathfrak{c}}).$$