On the geometry of *CR*-submanifolds of product type

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Outline

... from the beginning

$(M, g) \underset{iso}{\hookrightarrow} (\widetilde{M}, \widetilde{g}, J)$ – Kähler manifold

$T(M)$ its tangent bundle; $T(M)^{\perp}$ its normal bundle

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- $T_x(M)$ is invariant under the action of J:

$$J(T_x(M)) = T_x(M) \text{ for all } x \in M$$

M is called *complex* submanifold or **holomorphic** submanifold

... from the beginning

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Two important situations occur:

- $T_x(M)$ is anti-invariant under the action of J:

$$J(T_x(M)) \subset T(M)_x^{\perp} \text{ for all } x \in M$$

M is know as a **totally real** submanifold

... from the beginning

In **1978** A. Bejancu

- *CR-submanifolds of a Kähler manifold. I*,
  Proc. Amer. Math. Soc., 69 (1978), 135-142
- *CR- submanifolds of a Kähler manifold. II*,
  Trans. Amer. Math. Soc., 250 (1979), 333-345

started a study of the geometry of a class of submanifolds situated between the two classes mentioned above.

Such submanifolds were named *CR–submanifolds*:

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Such submanifolds were named *CR–submanifolds*:

M is a **CR-submanifold** *of a Kähler manifold $(\widetilde{M}, \widetilde{g}, J)$ if there exists a holomorphic distribution $\mathcal{D}$ on M, i.e. $J\mathcal{D}_x = \mathcal{D}_x$, $\forall x \in M$ and such that its orthogonal complement $\mathcal{D}^\perp$ is anti-invariant, namely $J\mathcal{D}_x^\perp \subset T(M)_x^\perp$, $\forall x \in M$.*

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4. $s, q \neq 0$: M is called a proper *CR*-submanifold.

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4. $s, q \neq 0$: M is called a proper *CR*-submanifold.

An example of proper generic *CR*-submanifold is furnished by any hypersurface in $\widetilde{M}$.

Notations

For any X tangent to M:

$PX = \tan(JX)$ and $FX = \mathrm{nor}(JX)$

For any N normal to M:

$tN = \tan(JN)$ and $fN = \mathrm{nor}(JN)$

Here tan and nor denotes the tangential and respectively the normal component.

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$$T(M)^\perp = J\mathcal{D}^\perp \oplus \nu \qquad J\mathcal{D}^\perp \perp \nu$$

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Denote by I and $I^\perp$ the projections on $\mathcal{D}$ and $\mathcal{D}^\perp$ respectively.

Submanifold formulas

Gauss and Weingarten formulae
(G) $\widetilde{\nabla}_X Y = \nabla_X Y + B(X, Y)$
(W) $\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N$
for any $X, Y \in \chi(M)$, and $N \in \Gamma^{\infty}(T(M)^{\perp})$.

∇ is the induced connection
$\nabla^{\perp}$ is the normal connection
B is the second fundamental form
A_N is the Weingarten operator

$$g(A_N X, Y) = \widetilde{g}(N, B(X, Y))$$

Integrability

Proposition (Bejancu - 1979, Blair & Chen - 1979)

The totally real distribution $\mathcal{D}^\perp$ of a *CR*-submanifold in a Kähler manifold is always integrable.

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The totally real distribution $\mathcal{D}^{\perp}$ of a CR-submanifold in a Kähler manifold is always integrable.

Proposition (Blair & Chen - 1979)

The distribution $\mathcal{D}$ is integrable if and only if

$$\widetilde{g}(B(X, JY), JZ) = \widetilde{g}(B(JX, Y), JZ)$$

for any vectors X, Y in $\mathcal{D}$ and Z in $\mathcal{D}^{\perp}$.

Integrability

Proposition (Bejancu, Kon & Yano - 1981)

For a CR-submanifold M in a Kähler manifold, the leaf $N^{\perp}$ of $\mathcal{D}^{\perp}$ is totally geodesic in M if and only if

$$\widetilde{g}(B(\mathcal{D},\mathcal{D}^{\perp}),J\mathcal{D}^{\perp})=0.$$

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$$\widetilde{g}(B(\mathcal{D}, \mathcal{D}^\perp), J\mathcal{D}^\perp) = 0.$$

Proposition (Chen - 1981)

If the previous result holds and if the distribution $\mathcal{D}$ is integrable, then

$$A_N JX = -JA_N X$$

for all $N \in J\mathcal{D}^\perp$.

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Definition (Chen - 1981)

A *CR*-submanifold of a Kähler manifold $\widetilde{M}$ is called ***CR*-product** if it is locally a Riemannian product of a holomorphic submanifold $N^{\top}$ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$.

Theorems of characterization

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A *CR*-submanifold of a Kähler manifold is a *CR*-product if and only if P is parallel.

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Theorem (Chen - 1981)

A *CR*-submanifold of a Kähler manifold is a *CR*-product if and only if

$$A_{J\mathcal{D}^\perp}\mathcal{D} = 0.$$

... and curvature

Lemma

Let M be a CR-product of a Kähler manifold $\widetilde{M}$. Then for any unit vectors $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$ we have

$$\widetilde{H}_B(X,Z) = 2||B(X,Z)||^2$$

where $\widetilde{H}_B(X,Z) = \widetilde{g}(Z, \widetilde{R}_{X,JX}JZ)$ is the holomorphic bisectional curvature of the plane $X \wedge Z$.

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Theorem (Chen - 1981)

Let $\widetilde{M}$ be a Kähler manifold with negative holomorphic bisectional curvature. Then every CR-product in $\widetilde{M}$ is either a holomorphic submanifold or a totally real submanifold. In particular, there exists no proper CR-product in any complex hyperbolic space $\widetilde{M}(c)$, $(c < 0)$.

CR-products in $\mathbb{C}^m$

Theorem (Chen - 1981)

Every CR-product M in $\mathbb{C}^m$ is locally the Riemannian product of a holomorphic submanifold in a linear complex subspace C^k and a totally real submanifold of a C^{m-k}, i.e.

$$M = N^{\top} \times N^{\perp} \subset \mathbb{C}^k \times \mathbb{C}^{m-k}.$$

CR-products in $\mathbb{C}P^m$

Segre embedding:

$$S_{sq} : \mathbb{C}P^s \times \mathbb{C}P^q \longrightarrow \mathbb{C}P^{s+q+sq}$$

$$(z_0, \ldots, z_s; w_0, \ldots, w_q) \mapsto (z_0 w_0, \ldots, z_i w_j, \ldots, z_s w_q)$$

$N^\perp$ = q-dimensional totally real submanifold in $\mathbb{C}P^q$

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Remark (Chen - 1981)

$m = s + q + sq$ is **the smallest dimension** of $\mathbb{C}P^m$ for admitting a *CR*-product.

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Proof.

$\{X_1, \dots, X_{2s}\}$; $\{Z_1, \dots, Z_q\}$ - orthonormal basis in $\mathcal{D}$, respectively $\mathcal{D}^\perp$
Then $\{B(X_i, Z_\alpha)\}_{i=1,\dots,2s;\alpha=1,\dots,q}$ are orthonormal vectors in ν:
recall $T(M)^\perp = J\mathcal{D}^\perp \oplus \nu$ □

Length of the second fundamental form

Theorem (Chen - 1981)

Let M be a CR-product in $\mathbb{C}P^m$. Then we have

$$||B||^2 \geq 4sq.$$

If the equality sign holds, then $N^\top$ and $N^\perp$ are both totally geodesic in $\mathbb{C}P^m$. Moreover, the immersion is rigid*. In this case $N^\top$ is a complex space form of constant holomorphic sectional curvature 4, and $N^\perp$ is a real space form of constant sectional curvature 1.

* the Riemannian structure on the submanifold M is completely determined as well as the second fundamental form and the normal connection

Length of the second fundamental form

If $\mathbb{R}P^q$ is a totally geodesic, totally real submanifold of $\mathbb{C}P^q$, then the composition of the immersions

$$\mathbb{C}P^s \times \mathbb{R}P^q \longrightarrow \mathbb{C}P^s \times \mathbb{C}P^q \xrightarrow{S_{s,q}} \mathbb{C}P^{s+q+sq} \longrightarrow \mathbb{C}P^m$$

gives the only *CR*-product in $\mathbb{C}P^m$ satisfying the equality.

Warped Products $N^{\perp} \times_f N^{\top}$

(B, g_B), (F, g_F) Riemannian manifolds, $f > 0$ smooth function on B
$M = B \times_f F$, $g = g_B + f^2 g_F$

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Theorem (Chen - 2001)

If $M = N^\perp \times_f N^\top$ is a warped product CR-submanifold of a Kähler manifold $\widetilde{M}$ such that $N^\perp$ is a totaly real submanifold and $N^\top$ is a holomorphic submanifold of $\widetilde{M}$, then M is a CR-product.

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Proof.

f should be a constant and $A_{J\mathcal{D}^\perp}\mathcal{D} = 0$ is verified. □

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Proof.

f should be a constant and $A_{J\mathcal{D}^\perp}\mathcal{D} = 0$ is verified. □

Remark (Chen - 2001)

There do not exist warped product CR-submanifolds in the for $N^\perp \times_f N^\top$ other than CR-products.

Warped Products $N^\top \times_f N^\perp$

By contrast, there exist many warped product *CR*-submanifolds $N^\top \times_f N^\perp$ which are **not** *CR*-products.

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CR-warped products

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$$\downarrow$$

CR-warped products

Theorem (Chen - 2001)

A proper CR-submanifold M of a Kähler manifold $\widetilde{M}$ is locally a CR-warped product if and only if

$$A_{JZ}X = ((JX)\mu)Z, \quad X \in \mathcal{D},\ Z \in \mathcal{D}^\perp$$

for some function μ on M satisfying $W\mu = 0$, for all $W \in \mathcal{D}^\perp$.

Sketch

Proof.

"$\Rightarrow$" is easy to prove

"$\Leftarrow$" First $\mathcal{D}$ is integrable and its leaves are totally geodesic in M.

Second, each leaf of $\mathcal{D}^\perp$ is an extrinsic sphere, i.e.
a totally umbilical submanifold with parallel mean curvature vector

By a result of S. Hiepko, Math. Ann. - 1979 one gets the warped product

$$M = N^\top \times_f N^\perp$$

where $N^\top$ is a leaf of $\mathcal{D}$ and $N^\perp$ is a leaf of $\mathcal{D}^\perp$.

□

A general Inequality for *CR*-warped products

Theorem (Chen - 2001)

Let $M = N^{\top} \times_f N^{\perp}$ be a *CR*-warped product in a Kähler manifold $\widetilde{M}$. Then

1. $||B||^2 \geq 2q||\nabla(\log f)||^2$, where $\nabla(\log f)$ is the gradient of $\log f$

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1. $||B||^2 \geq 2q||\nabla(\log f)||^2$, where $\nabla(\log f)$ is the gradient of $\log f$
2. If the equality sign holds identically, then $N^\top$ is a totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$. Moreover, M is a minimal submanifold in $\widetilde{M}$

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2. If the equality sign holds identically, then $N^\top$ is a totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$. Moreover, M is a minimal submanifold in $\widetilde{M}$
3. When M is generic and $q > 1$, the equality sign holds if and only if $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$

A general Inequality for *CR*-warped products

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Let $M = N^{\top} \times_f N^{\perp}$ be a *CR*-warped product in a Kähler manifold $\widetilde{M}$. Then

1. $||B||^2 \geq 2q||\nabla(\log f)||^2$, where $\nabla(\log f)$ is the gradient of $\log f$
2. If the equality sign holds identically, then $N^{\top}$ is a totally geodesic and $N^{\perp}$ is a totally umbilical submanifold of $\widetilde{M}$. Moreover, M is a minimal submanifold in $\widetilde{M}$
3. When M is generic and $q > 1$, the equality sign holds if and only if $N^{\perp}$ is a totally umbilical submanifold of $\widetilde{M}$
4. When M is generic and $q = 1$, then the equality sign holds if and only if the characteristic vector of M is a principal vector field with zero as its principal curvature.
(In this case M is a real hypersurface in $\widetilde{M}$.)

Equality sign when $\widetilde{M} = \widetilde{M}(c)$

For CR-warped products in complex space forms:

Theorem (Chen - 2001)

Let $M = N^{\top} \times_f N^{\perp}$ be a non-trivial CR-warped product in a complex space form $\widetilde{M}(c)$, satisfying $||B||^2 = 2q||\nabla(\log f)||^2$. Then

1. $N^{\top}$ is a totally geodesic holomorphic submanifold of $\widetilde{M}(c)$. Hence $N^{\top}$ is a complex space form $N^s(c)$ of constant holomorphic sectional curvature c

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1. $N^\top$ is a totally geodesic holomorphic submanifold of $\widetilde{M}(c)$. Hence $N^\top$ is a complex space form $N^s(c)$ of constant holomorphic sectional curvature c
2. $N^\perp$ is a totally umbilical totally real submanifold of $\widetilde{M}(c)$. Hence, $N^\perp$ is a real space form of constant sectional curvature, say $\epsilon > c/4$

Equality sign when $\widetilde{M} = \mathbb{C}^m$

Theorem (Chen - 2001)

A *CR*-warped product $M = N^\top \times_f N^\perp$ in a complex Euclidean m-space $\mathbb{C}^m$ satisfies the equality if and only if

1. $N^\top$ is an open portion of a complex Euclidean s space $\mathbb{C}^s$

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2. $N^\perp$ is an open portion of the unit q-sphere S^q

Equality sign when $\widetilde{M} = \mathbb{C}^m$

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A CR-warped product $M = N^\top \times_f N^\perp$ in a complex Euclidean m-space $\mathbb{C}^m$ satisfies the equality if and only if

1. $N^\top$ is an open portion of a complex Euclidean s space $\mathbb{C}^s$
2. $N^\perp$ is an open portion of the unit q-sphere S^q
3. up to a rigid motion of $\mathbb{C}^m$, the immersion of $M \subset \mathbb{C}^s \times_f S^q$ into $\mathbb{C}^m$ is

$$r(z,w) = \Big(z_1 + (w_0 - 1)a_1 \sum_{j=1}^{n} a_j z_j, \dots, z_s + (w_0 - 1)a_s \sum_{j=1}^{n} a_j z_j,$$

$$w_1 \sum_{j=1}^{n} a_j z_j, \dots, w_q \sum_{j=1}^{n} a_j z_j, 0, \dots, 0\Big)$$

$$z = (z_1, \dots, z_s) \in \mathbb{C}^s,\ w = (w_0, \dots, w_q) \in S^q \in \mathbb{E}^{q+1}$$

$$f = \sqrt{< a, z >^2 + < ia, z >^2},\ \text{for some point } a = (a_1, \dots, a_s) \in S^{s-1} \in \mathbb{E}^s.$$

Twisted product $N^\perp \times_f N^\top$

(B, g_B), (F, g_F) Riemannian manifolds, $f > 0$ smooth function on $B \times F$
$M = B \times_f F$, $g = g_B + f^2 g_F$

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$M = B \times_f F$, $g = g_B + f^2 g_F$

Theorem (Chen - 2000)

If $M = N^\perp \times_f N^\top$ is a twisted product *CR*-submanifold of a Kähler manifold $\widetilde{M}$ such that $N^\perp$ is a totaly real submanifold and $N^\top$ is a holomorphic submanifold of $\widetilde{M}$, then M is a *CR*-product.

Twisted product $N^{\perp} \times_f N^{\top}$

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$M = B \times_f F$, $g = g_B + f^2 g_F$

Theorem (Chen - 2000)

If $M = N^{\perp} \times_f N^{\top}$ is a twisted product CR-submanifold of a Kähler manifold $\widetilde{M}$ such that $N^{\perp}$ is a totaly real submanifold and $N^{\top}$ is a holomorphic submanifold of $\widetilde{M}$, then M is a CR-product.

Proof.

Similar to warped product case, f should be a constant and $A_{J\mathcal{D}^{\perp}}\mathcal{D} = 0$ is verified. □

Twisted product $N^\top \times_f N^\perp$

CR-submanifolds of the form $N^\top \times_f N^\perp$ = CR-twisted products

Theorem (Chen - 2000)

Let $M = N^\top \times_f N^\perp$ be a CR-twisted product in a Kähler manifold $\widetilde{M}$. Then

1. $||B||^2 \geq 2q||\nabla^\top(\log f)||^2$, where $\nabla^\top(\log f)$ is the $N^\top$-component of the gradient of $\log f$
2. If the equality sign holds identically, then $N^\top$ is a totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$.
3. If M is generic and $q > 1$, the equality sign holds if and only if $N^\top$ is totally geodesic and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$

A non-existence result

(B, g_B), (F, g_F) Riemannian manifolds, $b, f > 0$ smooth on B, resp. F

$M = {}_fB \times_b F$, $g = f^2 g_B + b^2 g_F \Longrightarrow$ doubly warped product

Similar one defines doubly twisted product

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Theorem (Şahin - 2007)

There do not exist doubly warped (resp. twisted) product *CR*-submanifolds which are not (singly) *CR*-warped (resp. *CR*-twisted) products of the form ${}_fN^\top \times_b N^\perp$ such that $N^\top$ is a holomorphic submanifold and $N^\perp$ is a totally real submanifold of $\widetilde{M}$.

Locally conformal Kähler manifolds

$(\widetilde{M}, J, \widetilde{g})$ Hermitian manifold; $\Omega = \tilde{g}(X, JY)$ Kähler 2-form
$\widetilde{M}$ is **l.c.K.** if there is a closed 1-form ω, globally defined on $\tilde{M}$, such that

$$d\Omega = \omega \wedge \Omega$$

ω is called the *Lee form* of the l.c.K. manifold $\widetilde{M}$.
Lee vector field: $\widetilde{g}(X, B) = \omega(X)$,
$\widetilde{\nabla}$: the Levi Civita connection of $(\widetilde{M}, \widetilde{g})$

$$(\tilde{\nabla}_X J)Y = \frac{1}{2}\left(\theta(Y)X - \omega(Y)JX - \tilde{g}(X, Y)A - \Omega(X, Y)B\right)$$

$\theta = \omega \circ J$: *anti-Lee form*
$A = -JB$: *anti-Lee vector field*

Integrability

Proposition (Blair & Chen - 1979)

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The holomorphic distribution $\mathcal{D}$ is integrable if and only if

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Proposition (Blair & Dragomir - 2002)

A leaf $N^\perp$ of $\mathcal{D}^\perp$ is totally geodesic in M if and only if

$$\widetilde{g}(B(X,W),JZ) = \frac{1}{2}\theta(X)\widetilde{g}(Z,W),\ X \in \mathcal{D},\ Z,W \in \mathcal{D}^\perp.$$

Ambient Kähler vs. ambient l.c.K.

New phenomena occur if the ambient is l.c.K. but not Kähler.

In general, given a submanifold $M \subset \mathbb{C}^k$ and $N \subset \mathbb{C}^{n-k}$, a conformal change $g_0 \mapsto fg_0$, $f > 0$ violates the Riemannian product property:

The induced metric on $M \times N \subset (\mathbb{C}^n, fg_0)$ is the product on the induced metrics on M and N, respectively, if and only if $f(z, w) = f_1(z)f_2(w)$, for some smooth $f_1 > 0$ and $f_2 > 0$, where $z \in \mathbb{C}^k$ and $w \in \mathbb{C}^{n-k}$.

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In view of Chen's characterization of *CR*-products in Kähler manifolds, it is natural to ask :
which *CR*-submanifolds of a l.c.K. manifold have a parallel f-structure P?

CR-submanifolds with $\nabla P = 0$

Theorem (Blair & Dragomir - 2002)

Let M be a proper CR-submanifold of a l.c.K. manifold $\widetilde{M}$. The following statements are equivalent:

- The structure P is parallel;
- M is locally a Riemannian product $N^\top \times N^\perp$, where $N^\top$ (resp. $N^\perp$) is a complex (resp. anti-invariant) submanifold of $\widetilde{M}$ of complex dimension s (resp. of real dimension q), and
  - either M is normal to the Lee field of $\widetilde{M}$
  - or $\tan(B) \neq 0$ and then $\tan(B) \in \mathcal{D}$ and $s = 1$, i.e. $N^\top$ is a complex curve in $\widetilde{M}$.

CR-warped product of the form $N^\perp \times_f N^\top$

A rather different situation occurs in l.c.K. geometry

$\{U_i\}$ open covering of $\widetilde{M}$
$\{f_i : U_i \longrightarrow \mathbb{R}\}$ such that $\tilde{g}_i = \exp(-f_i)\widetilde{g}_{|U_i}$ is Kähler metric on U_i
$M_i = M \cap U_i$, $g_i = \tilde{g}_{i\,|M_i}$

Theorem (Blair & Dragomir - 2002)

$M = N^\perp \times_f N^\top$ warped product CR-submanifold of a l.c.K. manifold $\widetilde{M}$. Then

1. $N^\top$ is totally umbilical in M of mean curvature $||\nabla \log f||$ and $d \log f = \frac{1}{2}\omega$ on $\mathcal{D}^\perp$.

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1. $N^\top$ is totally umbilical in M of mean curvature $||\nabla \log f||$ and $d \log f = \frac{1}{2}\omega$ on $\mathcal{D}^\perp$.
2. Each local CR-submanifold M_i is a warped product $N_i^\perp \times_{\alpha_i \exp(f_i)} N_i^\top$, $\alpha_i > 0$ and $g_i = \exp(-f_i)g_\perp + \alpha_i g_\top$, i.e. (M_i, g_i) is a Riemannian product.

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3. If M is normal to the Lee vector field B or $\tan(B) \in \mathcal{D}$ then M is a *CR*-product and each f_i is constant on $N_i^{\perp} = N^{\perp} \cap U_i$.

Other results

Proposition (Bonanzinga & K.Matsumoto - 2004)

If $M = N^\top \times_f N^\perp$ is a proper *CR*-warped product in a l.c.K. manifold $\widetilde{M}$, then the Lee vector field is orthogonal to $\mathcal{D}^\perp$.

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Bonanzinga and K.Matsumoto (2004) give also Chen's type inequalities for the length of the second fundamental form for both kind of *CR*-warped products in l.c.K. manifolds.

A general inequality for doubly warped product *CR*-submanifolds

Theorem (M. - 2007)

$M = {}_fN^\top \times {}_bN^\perp$ doubly warped product *CR*-submanifold in a l.c.K. manifold $\tilde{M}$. Then

$$||B||^2 \geq \frac{s}{2}\,||\mathcal{B}^{J\mathcal{D}^\perp}||^2 + \frac{p}{f^2}\left[||\nabla^{N^\top}(\ln b)||^2_{N^\top} + \frac{f^2}{4}\,||\mathcal{B}^{\mathcal{D}}||^2 - \omega(\nabla^{N^\top}(\ln b))\right].$$

If the equality sign holds identically, then $N^\top$ and $N^\perp$ are both totally umbilical submanifolds in $\tilde{M}$.

Proof.

$$||B||^2 = ||B(\mathcal{D},\mathcal{D})||^2 + 2||B(\mathcal{D},\mathcal{D}^\perp)||^2 + ||B(\mathcal{D}^\perp,\mathcal{D}^\perp)||^2$$

$$||B(U,V)||^2 = ||B_{J\mathcal{D}^\perp}(U,V)||^2 + ||B_\nu(U,V)||^2$$

$||B_{J\mathcal{D}^\perp}(\mathcal{D},\mathcal{D})||^2 = \frac{s}{2}\,||\mathcal{B}^{J\mathcal{D}^\perp}||^2.$

$||B_{J\mathcal{D}^\perp}(\mathcal{D},\mathcal{D}^\perp)||^2 = \frac{p}{f^2}\left(||\nabla^{N^\top}(\ln b)||^2_{N^\top} + \frac{f^2}{4}\,||\mathcal{B}^{\mathcal{D}}||^2 - \omega(\nabla^{N^\top}(\ln b))\right).$ $\square$

Equality sign in the inequality

Corollary

Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product CR-submanifold and totally geodesic in a l.c.K. manifold $\tilde{M}$. Then M is generic, i.e. $J_x\mathcal{D}_x^\perp = T(M)_x^\perp$, M is tangent to the Lee vector field and $\omega_{|_{N^\top}} = 2d\ln b$. (Moreover, both sides in the inequality vanish.)

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Theorem (M. - 2007)

Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product, generic *CR*-submanifold in a l.c.K. manifold $\tilde{M}$, such that $q = \dim N^\perp \geq 2$ and $N^\perp$ is totally umbilical in $\tilde{M}$. Then we have the equality sign.

Equality sign in the inequality

What happens when $q = 1$?

In this case M is a hypersurface in $\widetilde{M}$ and let N be a normal vector field on M, such that $Z = JN$ (which is tangent to $N^\perp$) is of unit length (w.r.t. $g_{N^\perp}$). Of course, Z generates $\mathcal{D}^\perp$.

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Theorem (M. - 2007)

Let $M = {}_fN^\top \times {}_bN^\perp$ be a doubly warped product, generic CR-submanifold of hypersurface type in a l.c.K. manifold $\tilde{M}$. Then the equality sign holds if and only if $A_N Z$ belongs to the holomorphic distribution $\mathcal{D}$.

Contact *CR*-submanifolds

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a submanifold M of an almost contact Riemannian manifold $(\widetilde{M}, (\phi, \xi, \widetilde{\eta}, \widetilde{g}))$ carrying an invariant distribution $\mathcal{D}$, i.e. $\phi_x \mathcal{D}_x \subseteq \mathcal{D}_x$, for any $x \in M$, such that the orthogonal complement $\mathcal{D}^\perp$ of $\mathcal{D}$ in $T(M)$ is anti-invariant, i.e. $\phi_x \mathcal{D}_x^\perp \subseteq T(M)_x^\perp$, for any $x \in M$.

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This notion was introduced by A.Bejancu & N.Papaghiuc in

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by using the terminology of *semi-invariant submanifold*.

It is customary to require that ξ be tangent to M rather than normal which is too restrictive (K. Yano & M. Kon): M must be anti-invariant, i.e. $\phi_x T_x(M) \subseteq T(M)_x^\perp$, $x \in M$

Contact CR-submanifolds

Given a contact CR submanifold M of a Sasakian manifold $\widetilde{M}$ either $\xi \in \mathcal{D}$, or $\xi \in \mathcal{D}^\perp$. Therefore

$$T(M) = H(M) \oplus \mathbf{R}\xi \oplus E(M)$$

$H(M)$ is the maximally complex, distribution of M; $\phi E(M) \subseteq T(M)^\perp$.

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Therefore, to formulate a contact analog of the notion of warped CR product one assumes that

$$\mathcal{D} = H(M) \oplus \mathbf{R}\xi$$

Notations and basic results

For any X tangent to M: $PX = \tan(\phi X)$ and $FX = \mathrm{nor}(\phi X)$
For any N normal to M: $tN = \tan(\phi N)$ and $fN = \mathrm{nor}(\phi N)$

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Proposition (Yano & Kon - 1983)

In order for a submanifold M, tangent to the structure field ξ of a Sasakian manifold $\widetilde{M}$ to be a contact CR-submanifold, it is necessary and sufficient that $FP = 0$.

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The distribution $\mathcal{D}^{\perp}$ is always completely integrable.

(Normal) Semi-invariant products

$(\widetilde{M}^{2m+1}, \phi, \xi, \eta, \widetilde{g})$ Sasakian manifold: $\phi \in \mathcal{T}_1^1(\widetilde{M})$, $\xi \in \chi(\widetilde{M})$, $\eta \in \Lambda^1(\widetilde{M})$:

$\phi^2 = -I + \eta \otimes \xi$, $\phi\xi = 0$, $\eta \circ \phi = 0$, $\eta(\xi) = 1$

$d\eta(X, Y) = \widetilde{g}(X, \phi Y)$ (the contact condition)

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the distribution $H(M) \oplus \{\xi\}$ is integrable and
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normality tensor: $S(X, Y) = N_\varphi(X, Y) - 2tdF(X, Y) + 2d\eta(X, Y)$
where $dF(X, Y) := \nabla^\perp_X FY - \nabla^\perp_Y FX - F[X, Y]$

(Normal) Semi-invariant products

Theorem (Bejancu & Papaghiuc - 1983)

A semi-invariant submanifold M of a Sasakian manifold $\tilde{M}$ is normal iff

$$A_{FZ}(PX) = PA_{FZ}X$$

for all $X \in H(M) \oplus \{\xi\}$ and $Z \in \mathcal{D}^{\perp}$.

Theorem (Bejancu & Papaghiuc - 1983)

A normal semi-invariant submanifold of a Sasakian manifold is a semi-invariant product if and only if the distribution $H(M) \oplus \{\xi\}$ is integrable.

Contact CR-products

A contact CR submanifold M of a Sasakian manifold $\widetilde{M}$ is called *contact CR product* if it is locally a Riemannian product of a ϕ-invariant submanifold $N^\top$ tangent to ξ and a totally real submanifold $N^\perp$ of $\widetilde{M}$, i.e. $N^\perp$ is ϕ anti-invariant submanifold of $\widetilde{M}$.

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Theorem (M. - 2005)

Let M be a contact *CR* submanifold of a Sasakian manifold $\widetilde{M}$, $\xi \in \mathcal{D}$. Then M is a contact *CR* product if and only if P satisfies

$$(\nabla_U P)V = -g(U_{\mathcal{D}}, V)\xi + \eta(V)U_{\mathcal{D}}$$

for all U, V tangent to M where $U_{\mathcal{D}}$ is the $\mathcal{D}$-component of U.

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Equivalently: $A_{\phi Z}X = \eta(X)Z,\ X \in \mathcal{D},\ Z \in \mathcal{D}^\perp$ (M. - 2005)

Geometric description of contact CR products in Sasakian space forms

Theorem (M. - 2005)

Let M be a complete, generic, simply connected contact CR submanifold of a complete, simply connected Sasakian space form $\widetilde{M}^{2m+1}(c)$.

If M is a contact CR product then
1. either $c \neq -3$ and M is a ϕ anti-invariant submanifold of $\widetilde{M}$ case in which M is locally a Riemannian product of an integral curve of ξ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$,
2. or $c = -3$ and M is locally a Riemannian product of $\mathbf{R}^{2s+1}$ and $N^{\perp}$ where $\mathbf{R}^{2s+1}$ is endowed with the usual Sasakian structure and $N^{\perp}$ is a totally real submanifold of $\mathbf{R}^{2m+1}$ (with the usual Sasakian structure).

ϕ-holomorphic bisectional curvature

$$\widetilde{H}_B(U,V) = \widetilde{R}(\phi U, U, \phi V, V) \quad \text{for} \quad U, V \in T(\widetilde{M})$$

Lemma (Papaghiuc - 1984)

M = contact CR-product of a Sasakian manifold $\widetilde{M}^{2m+1}$.
Then, $\widetilde{H}_B(X,Z) = 2\ \big(||B(X,Z)||^2 - 1\big)$, $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$ *unitary*.

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Theorem (M. - 2005)

Let $\widetilde{M}$ be a Sasakian manifold with $H_B < -2$. Then every contact CR product M in $\widetilde{M}$ is either an invariant submanifold or an anti-invariant submanifold, case in which M is (locally) a Riemannian product of an integral curve of ξ and a ϕ-anti-invariant submanifold of $\widetilde{M}$.

ϕ-holomorphic bisectional curvature

$$\widetilde{H}_B(U,V) = \widetilde{R}(\phi U, U, \phi V, V) \quad \text{for} \quad U, V \in T(\widetilde{M})$$

Lemma (Papaghiuc - 1984)

M = contact CR-product of a Sasakian manifold $\widetilde{M}^{2m+1}$.
Then, $\widetilde{H}_B(X,Z) = 2\left(||B(X,Z)||^2 - 1\right)$, $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$ *unitary*.

Theorem (M. - 2005)

Let $\widetilde{M}$ be a Sasakian manifold with $H_B < -2$. Then every contact *CR* product M in $\widetilde{M}$ is either an invariant submanifold or an anti-invariant submanifold, case in which M is (locally) a Riemannian product of an integral curve of ξ and a ϕ-anti-invariant submanifold of $\widetilde{M}$.

Corollary

Let $\widetilde{M}^{2m+1}(c)$, $c < -3$ be a Sasakian space form. Then there exists no strictly proper contact CR product in $\widetilde{M}$.

Some inequalities

Theorem (Papaghiuc - 1984, M. - 2005)

Let $\widetilde{M}^{2m+1}(c)$ be a Sasakian space form and let $M = N^\top \times N^\perp$ be a contact CR product in $\widetilde{M}$. Then the norm of the second fundamental form of M satisfies the inequality

$$||B||^2 \geq q\ \left((c+3)s+2\right).$$

"=" holds if and only if both $N^\top$ and $N^\perp$ are totally geodesic in $\widetilde{M}$.

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$$r : S^{2s+1} \times S^{2q+1} \longrightarrow S^{2m+1} \qquad \mathbf{m = sq + s + q}$$

$$(x_0,\ y_0, \ldots, x_s,\ y_s;\ u_0,\ v_0, \ldots, u_q,\ v_q) \longmapsto (\ldots, x_j u_\alpha - y_j v_\alpha,\ x_j v_\alpha + y_j u_\alpha, \ldots)$$

$$M = S^{2s+1} \times S^p \longrightarrow S^{2s+1} \times S^{2q+1} \xrightarrow{r} S^{2m+1}$$

contact CR product in S^{2m+1} for which the equality holds.

Some inequalities

Theorem (Papaghiuc - 1984, M. - 2005)

Let M be a strictly proper contact CR product in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, with $c \neq -3$. Then

$$m \geq sq + s + q.$$

Proof.

$\{B(X_j, Z_\alpha)\}_{i=1,\dots 2s, \alpha=1,\dots,q}$ is a linearly independent system in ν
$B(\xi, Z_\alpha) = \phi Z_\alpha \in \phi\mathcal{D}^\perp$. □

Equality sign holds

Theorem (Papaghiuc - 1984, M. - 2005)

Let $M = N^T \times N^\perp$ be a contact *CR* product in a Sasakian space form $\widetilde{M}^{2m+1}(c)$, $c \neq -3$. Let $\dim N^T = 2s + 1$, $\dim N^\perp = p$ and suppose that $m = sp + s + p$. Then N^T is a totally geodesic submanifold in $\widetilde{M}$.

Corollary

Let $M = N^T \times N^\perp$ be a strictly proper contact CR product in S^7. Then M is a Riemannian product between the sphere S^3 and a curve. Moreover, if the norm of the second fundamental form of M satisfies the equality case in the inequality we have that M is the Riemannian product between S^3 and S^1.

Interesting result in S^7

Theorem (M. - 2005)

Let $M = N^T \times N^\perp$ be a strictly proper contact *CR* product in S^7 whose second fundamental form has the norm $\sqrt{6}$. Then M is the Riemannian product between S^3 and S^1 and, up to a rigid transformation of $\mathbf{R}^8$ the embedding is given by

$$r : S^3 \times S^1 \longrightarrow S^7$$
$$r(x_1, y_1, x_2, y_2, u, v) = (x_1u, y_1u, -y_1v, x_1v, x_2u, y_2u, -y_2v, x_2v).$$

Characterization theorem

Theorem (M. - 2005)

Let $\widetilde{M}$ be a Sasakian manifold and let $M = N^{\perp} \times_f N^{\top}$ be a warped product CR submanifold such that $N^{\perp}$ is a totally real submanifold and $N^{\top}$ is ϕ holomorphic (invariant) of $\widetilde{M}$. Then M is a CR product.

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A contact CR submanifold M of a Sasakian manifold $\widetilde{M}$, tangent to ξ is called *a contact CR warped product* if it is the warped product $N^T \times_f N^{\perp}$ of an invariant submanifold N^T, tangent to ξ and a totally real submanifold $N^{\perp}$ of $\widetilde{M}$.

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Theorem (M. - 2005)

A strictly proper CR submanifold M of a Sasakian manifold $\widetilde{M}$, tangent to ξ, is locally a contact CR warped product if and only if there exists $\mu \in C^{\infty}(M)$ satisfying $W\mu = 0$ for all $W \in \mathcal{D}^{\perp}$.

$$A_{\phi Z}X = (\eta(X) - (\phi X)(\mu))\, Z\,, \quad X \in \mathcal{D}\,,\; Z \in \mathcal{D}^{\perp}.$$

A good geometric inequality

Theorem (I. Mihai - 2004, M. - 2005)

Let $M = N^{\top} \times_f N^{\perp}$ be a contact CR warped product of a Sasakian space form $\widetilde{M}^{2m+1}(c)$. Then

$$||B||^2 \geq 2q\left[||\nabla \ln f||^2 - \Delta \ln f + \frac{c+3}{2}s + 1\right].$$

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Proof.

$$||B(\mathcal{D}, \mathcal{D}^{\perp})||^2 = \sum_{j=1}^{2s+1} \sum_{\alpha=1}^{q} ||B(X_j, Z_\alpha)||^2$$

$$||B_{\phi\mathcal{D}^{\perp}}(\mathcal{D}, \mathcal{D}^{\perp})||^2 = \sum_{\alpha=1}^{q} ||\nabla \ln f||^2 + \sum_{\alpha=1}^{q} ||\phi Z_\alpha||^2$$

$$2 \sum_{j=1}^{s} \sum_{\alpha=1}^{q} \left\{ ||B_\nu(e_j, Z_\alpha)||^2 + ||B_\nu(\phi e_j, Z_\alpha)||^2 \right\} = (c+3)sq - 2q\Delta(\ln f) .$$

□

A general inequality

(the ambient $\widetilde{M}$ is not necessary a Sasakian space form)

Theorem (Hasegawa & I. Mihai - 2003, M. - 2005)

Let $M = N^{\top} \times_f N^{\perp}$ be a contact CR warped product in $\widetilde{M}$. We have

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Theorem (Hasegawa & I. Mihai - 2003, M. - 2005)

Let $M = N^\top \times_f N^\perp$ be a contact CR warped product in $\widetilde{M}$. We have

$(1)\ ||B||^2 \geq 2q\left(||\nabla \ln f||^2 + 1\right)$

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Let $M = N^\top \times_f N^\perp$ be a contact CR warped product in $\widetilde{M}$. We have
(1) $||B||^2 \geq 2q\left(||\nabla \ln f||^2 + 1\right)$
(2) If the equality sign holds, then $N^\top$ is a totally geodesic submanifold and $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$. The product manifold M is a minimal submanifold in $\widetilde{M}$.

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(3) The case $TM^\perp = \phi\mathcal{D}^\perp$. If $q > 1$ then the equality sign holds identically if and only if $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$.

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(3) $\underline{\text{The case } TM^\perp = \phi\mathcal{D}^\perp.}$ If $q > 1$ then the equality sign holds identically if and only if $N^\perp$ is a totally umbilical submanifold of $\widetilde{M}$.

(4) If $q = 1$ then the equality sign holds identically if and only if the characteristic vector field $\phi\mu$ of M satisfies $A_\mu\phi\mu = -\phi\nabla \ln f - \xi$.

(Notice that M is a hypersurface in $\widetilde{M}$ with the unitary normal vector μ).

An example of contact CR-warped product in $\mathbf{R}^{2m+1}$ satisfying the "good" equality which does not satisfy $\|B\|^2 = 2q\left(\|\nabla(\ln f)\|^2 + 1\right)$

Let $\mathbf{R}^{2s+1}$ be the Sasakian space form of ϕ sectional curvature -3. Let $S^q \subset \mathbf{R}^{q+1}$ be the unit sphere immersed in the Euclidian space $\mathbf{R}^{q+1}$. Let $\mathbf{R}^{2m+1}$ be also the Sasakian space form where $m = qh + s$ with h a positive integer, $h \leq s$.

Consider the map $r : \mathbf{R}^{2s+1} \times S^q \longrightarrow \mathbf{R}^{2m+1}$ defined by

$$r(x_1, y_1, \ldots, x_s, y_s, z, w^0, w^1, \ldots, w^q) =$$

$$(w^0 x_1, w^0 y_1, \ldots, w^q x_1, w^q y_1, \ldots, w^0 x_h, w^0 y_h, \ldots, w^q x_h, w^q y_h, x_{h+1}, y_{h+1}, \ldots, x_s, y_s, z)$$

where $(w^0)^2 + (w^1)^2 + \ldots + (w^q)^2 = 1$.

On $\mathbf{R}^{2m+1}$ we consider the (local) coordinates

$$\{X_j^\alpha, Y_j^\alpha, X_a, Y_a, Z\} \quad , \quad \alpha = 0, \ldots, q \ , \ j = 1, \ldots, h \ , \ a = h+1, \ldots, s.$$

With this notation the equations of the map r are given by

$$r : \begin{cases} X_i^\alpha = w^\alpha x_i \quad , \quad Y_i^\alpha = w^\alpha y_i \ , \\ X_a = x_a \ , \ Y_a = y_a \ , \ Z = z \ . \end{cases}$$

Proposition (M. - 2005)

We have

(1) r is an isometric immersion between the warped product $\mathbf{R}^{2s+1} \times_f S^q$ and $\mathbf{R}^{2m+1}$. The warped function is $f = \frac{1}{2}\sqrt{\sum_{i=1}^{h}(x_i^2 + y_i^2)}$.

(2) $\mathbf{R}^{2s+1}$ is a $\widetilde{\phi}$ invariant in $\mathbf{R}^{2m+1}$, i.e. $\widetilde{\phi}(r_* T(\mathbf{R}^{2s+1})) \subset r_* T(\mathbf{R}^{2s+1})$

(3) S^q is a $\widetilde{\phi}$ anti-invariant in $\mathbf{R}^{2m+1}$, i.e. $\widetilde{\phi}(r_* T(S^q)) \subset (r_* T(S^q))^{\perp}$.

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Proposition (M. - 2005)

The second fundamental form of $\mathbf{R}^{2s+1} \times_f S^q$ in $\mathbf{R}^{2m+1}$ satisfies

$$||B||^2 = 2q\left\{||\nabla \ln f||^2 - \Delta \ln f + 1\right\}.$$

Analogous results

Arslan, Ezentas, I. Mihai, Murathan – 2005

... give estimates for the norm of the second fundamental form for contact CR-warped products isometrically immersed in Kenmotsu manifolds

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Corollary (M. - 2007)

Let $\widetilde{M}$ be 1. either an α-Sasakian manifold, *2.* or a β-Kenmotsu manifold, *3.* or a cosymplectic manifold. *There is no proper doubly warped product contact CR-submanifolds in $\widetilde{M}$. More precisely we have,*

✓ *if $\xi \in \mathcal{D}$: $M = \widetilde{N}^\top \times {}_f N^\perp$, ξ is tangent to $N^\top$ and $f \in C^\infty(N^\top)$. Moreover, in case 2, β is a smooth function on $N^\top$.*

✓ *if $\xi \in \mathcal{D}^\perp$: 1. M is a ϕ-anti-invariant submanifold in $\widetilde{M}$ $(\dim \mathcal{D} = 0)$;*

2-3. $M = \widetilde{N}^\perp \times {}_f N^\top$, ξ is tangent to $N^\perp$ and $f \in C^\infty(N^\perp)$. Moreover, in case 2, β is a smooth function on $N^\perp$.

link

Non-existence result

An **a.c.m.** structure $(\phi, \xi, \eta, \widetilde{g})$ on $\widetilde{M}$ is a *trans-Sasakian structure* if $(\widetilde{M} \times \mathbf{R}, J, G)$ belongs to the class $\mathcal{W}_4$ of the Gray-Hervella classification of almost Hermitian manifolds

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

G is the product metric on $\widetilde{M} \times \mathbf{R}$.

$$(\widetilde{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)\ , \ \alpha, \beta \in C^\infty$$

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Theorem (M. - 2007)

There is no proper doubly warped product contact CR-submanifolds in trans-Sasakian manifolds.

back

$N^\perp \times {}_f N^\top$ in Kenmotsu manifold: ξ tangent to $N^\perp$

$\mathbf{C}^m$ the complex space with the usual Kähler structure
real global coordinates $(x^1, y^1, \ldots, x^m, y^m)$.

$N^{\perp} \times {}_f N^{\top}$ in Kenmotsu manifold: ξ tangent to $N^{\perp}$

$\mathbf{C}^m$ the complex space with the usual Kähler structure
real global coordinates $(x^1, y^1, \dots, x^m, y^m)$.
$\widetilde{M} = \mathbf{R} \times {}_f \mathbf{C}^m$ the warped product between the real line $\mathbf{R}$ and $\mathbf{C}^m$
$f = e^z$, z being the global coordinate on $\mathbf{R}$.

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$\mathcal{D} = \mathrm{span}\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s}\right\}$

$\mathcal{D}^\perp = \mathrm{span}\left\{\frac{\partial}{\partial z}, \frac{\partial}{\partial x^{s+1}}, \ldots, \frac{\partial}{\partial x^m}\right\}$

are integrable and denote by $N^\top$ and $N^\perp$ their integral submanifolds

$$g_{N^\top} = \sum_{i=1}^{s}\left((dx^i)^2 + (dy^i)^2\right) \ , \ g_{N^\perp} = dz^2 + e^{2z}\sum_{a=s+1}^{m}(dx^a)^2$$

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$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial x^s}, \frac{\partial}{\partial y^s}\right\}$$

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are integrable and denote by $N^\top$ and $N^\perp$ their integral submanifolds

$$g_{N^\top} = \sum_{i=1}^{s} \left((dx^i)^2 + (dy^i)^2\right) \ , \ g_{N^\perp} = dz^2 + e^{2z} \sum_{a=s+1}^{m} (dx^a)^2$$

Theorem (M. - 2007)

Then, $M = N^\perp \times {}_f N^\top$ is a contact CR-submanifold, isometrically immersed in $\widetilde{M}$.

Other Chen's type inequality

M. Djorić, L. Vrancken
Three-dimensional minimal CR submanifolds in S^6 satisfying Chen's equality
J. Geom. Phys. 56 (2006), no. 11, 2279–2288.

M. Antić, M. Djorić, L. Vrancken
4-dimensional minimal CR submanifolds of the sphere S^6 satisfying Chen's equality
Differential Geom. Appl. 25 (2007), no. 3, 290–298.

1947 – 2008

Thank you for attention!