Pseudoparallel Submanifolds of Kenmotsu Manifolds

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Introduction

Given an isometric immersion f:M $\rightarrow \widetilde{M}$,let σ be the second fundamental form and of M. Then J. Deprez defined the following immersion in 1985,

$$
\begin{aligned}
(\overline{R}(X,Y)\cdot\sigma)(U,V) = R^{\perp}(X,Y)\sigma(U,V) \\
-\sigma(R(X,Y)U,V) \\
-\sigma(U,R(X,Y)V)
\end{aligned}
\qquad (1)
$$

holds for all vector fields X,Y,U,V tangent to M. If $\overline{R}\cdot\sigma=0$ then M is said to be ***semiparallel*** **[Deprez, J.].**

The semiparallel hypersurfaces of Euclidean space were classified by J. Deprez in 1986 by the following theorem:

Theorem : [Deprez, J.] Let M^n be a semiparallel hypersurface of E^{n+1}. Then there are three possibilities :

(1) M^n is flat,

(2) M^n is parallel,

(3) M^n is a round cone, or a product of a round cone and a linear subspace.

Later, in 1990 Ü. Lumiste considered that a semiparallel submanifold is the second order envelope of the family of parallel submanifolds and proved the following theorem :

Theorem : [Lumiste, Ü] A submanifold M^m in $N^n(c)$ is semisymmetric if and only if M^m is the second-order envelope of the family of symmetric submanifolds in $N^n(c)$.

Also, in the case of hypersurfaces in the sphere and the hyperbolic space, F. Dillen proved in 1991 that :

Theorem : [Dillen, F.] Let M^n be a semi-parallel hypersurface of a real space form $\widetilde{M}^{n+1}(c)$ with $c \neq 0$.

Then there are three possibilities:

(1) n = 2 and M^2 is flat,

(2) M^n is parallel,

(3) There exists a totally geodesic $\widetilde{M}^2(c)$, and a vector u in the linear subspace $\mathbb{R}^3$ of $\mathbb{R}^{n+2}$, containing $\widetilde{M}^2(c)$, such that M^n is a rotation hypersurface whose profile curve is a u-helix lying in $\widetilde{M}^2(c)$, and whose axis is $u^{\perp}$. Moreover, M^n is intrinsically isometric to a cone.

Also M. Kon studied semiparallellity condition on invariant submanifolds of Sasakian manifolds but he did not call it as semiparallel. He showed by the next theorem that :

Theorem : [Kon, M.] Let M be an invariant submanifold of a Sasakian manifold. Then the following conditions are equivalent :

(i) M is totally geodesic. (ii) $(\widetilde{\nabla}_X \widetilde{\nabla}_Y \sigma)(\xi, \xi) = 0$

(iii) $\widetilde{R}(X, \xi) \cdot \sigma = 0$ (iv) $\widetilde{R}(X, Y) \cdot \sigma = 0$

X and Y being arbitrary vector fields on M.

For a (0,k)-tensor field T, k≥1 and a (0,2) -tensor field A on (M,g) we define Q(A,T) by

$$Q(A,T)(X_1,\ldots,X_k;X,Y)=-T((X\wedge_A Y)X_1,X_2,\ldots,X_k)$$
$$-\ldots-T(X_1,\ldots,X_{k-1},(X\wedge_A Y)X_k).$$

Putting into the above formula T=σ and A=g, or A=S, respectively, we obtain Q(g,σ) and Q(S,σ), where $X\wedge_A Y$ is an endomorphism defined by

$$(X\wedge_A Y)Z=A(Y,Z)X - A(X,Z)Y.$$

In the case A=g we write shortly $X\wedge_g Y=X\wedge Y$.

R. Deszcz, L. Verstraelen and Ş. Yaprak obtained some results in 1994 on hypersurfaces in 4-dimensional space form $N^4(c)$ satisfying the curvature condition **[Deszcz, R., Verstraelen, L. and Yaprak, S.]**

$$\overline{R} \cdot \sigma = L_{\sigma} Q(g, \sigma) \qquad (2)$$

A. C. Asperti, G. A. Lobos and F. Mercuri called this type submanifolds in 1999 as ***pseudoparallel ,*** where the equation (2) is defined by

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$=-L_{\sigma}[\sigma((X\wedge Y)U,V)+\sigma(U,(X\wedge Y)V)]$$

for all vector fields X, Y, U and V tangent to M.

Also, in **[Asperti, A. C., Lobos, G. A. and Mercuri, F.]** in 1999 and 2002 it was shown that a pseudoparallel hypersurface of a space form is either quasi-umbilical or a cyclic of Dupin.

On the other hand, C. Murathan, K. Arslan and R. Ezentaş defined submanifolds in 2005 satisfying the condition

$$\overline{R} \cdot \sigma = L_S Q(S, \sigma) \qquad (3)$$

This kind of submanifolds are called ***generalized Ricci-pseudoparallel***, where the equation (3) is given by

$$R^{\perp}(X,Y)\sigma(U,V)-\sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)$$
$$=-L_S[\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)]$$

for all vector fields X, Y, U and V tangent to M.

Also $\overline{R}\cdot\overline{\nabla}\sigma$ is defined by

$$\begin{aligned}(\overline{R}(X,Y)\cdot\overline{\nabla}\sigma)(U,V,W) = {} & R^{\perp}(X,Y)(\overline{\nabla}\sigma)(U,V,W) \\ & -(\overline{\nabla}\sigma)(R(X,Y)U,V,W) \\ & -(\overline{\nabla}\sigma)(U,R(X,Y)V,W) \\ & -(\overline{\nabla}\sigma)(U,V,R(X,Y)W)\end{aligned}$$

for all vector fields X,Y,U,V,W holds on M.

If $\overline{R} \cdot \overline{\nabla}\sigma = 0$ then the submanifold is said to be ***2-semiparallel*** **[Arslan, K., Lumiste, Ü., Murathan, C. and Özgür, C.].** The submanifolds satisfying the condition

$$\overline{R} \cdot \overline{\nabla}\sigma = L_{\overline{\nabla}\sigma} Q(g, \overline{\nabla}\sigma)$$

are called to be ***2-pseudoparallel*** **[Özgür, C. and Murathan, C.]**.

In 1988 H. Endo studied semiparallelity condition for an invariant submanifold of a contact metric manifold and proved the following theorem:

Theorem : Let M be an invariant submanifold of a contact metric manifold with $2>trh^2$. Then M is semiparallel and $\sigma H_A + H_A \sigma = 0$ if and only if it is totally geodesic.

Also, in 2007 Özgür, Sular and Murathan proved the following theorems:

Theorem : Let M be an invariant submanifold of a contact metric manifold with $\sigma H_A + H_A \sigma = 0$. Then M is pseudoparallel such that $2(1 - L_\sigma) > trh^2$ if and only if it is totally geodesic.

Theorem : Let M be an invariant submanifold of a contact metric manifold with $\sigma H_A + H_A \sigma = 0$. Then M is Ricci-generalized pseudoparallel such that $trh^2(2L_S - 1) > 2(2nL_S - 1)$ if and only if it is totally geodesic.

Kenmotsu Manifolds and Their Submanifolds

S. Tanno **[Tanno, S.]** in 1969 classified (2n+1)-dimensional almost contact metric manifolds M with an almost contact metric structure (φ,ξ,η,g), whose automorphism group possess the maximum dimension $(n+1)^2$.

For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c:

(1) If c>0, M is a homogeneous Sasakian manifold of constant φ-sectional curvature.

(2) If c=0, M is global Riemannian product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature.

(3) If c<0, M is a warped product space $\mathbb{R} \times_f \mathbb{C}$.

K. Kenmotsu **[Kenmotsu, K.]** in 1972 characterized the differential geometric properties of manifold of class (3); the structure so obtained is now known as Kenmotsu structure.

Definition

Let $\tilde{M}$ be a (2n+1)-dimensional an ***almost contact metric manifold*** with structure (φ,ξ,η,g) where φ is a tensor field of type (1,1), ξ is a vector field, η is a 1-form and g is the Riemannian metric on $\tilde{M}$ satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1, \qquad \eta \circ \varphi = 0,$$

$$g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\,\eta(Y),$$

$$\eta(X) = g(X,\xi), \qquad g(\varphi X, Y) + g(X, \varphi Y) = 0,$$

for all vector fields X,Y on $\tilde{M}$ **[Blair, D.]** .

An almost contact metric manifold $\tilde{M}$ is said to be a ***Kenmotsu manifold*** **[Kenmotsu, K.1972]** if the relation

$$(\tilde{\nabla}_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \qquad (1)$$

holds on $\tilde{M}$, where $\tilde{\nabla}$ is the Levi-Civita connection of g.

From the equation (1), for a Kenmotsu manifold we also have

$$\tilde{\nabla}_X \xi = X - \eta(X)\xi \qquad (2)$$

and

$$(\tilde{\nabla}_X \eta)Y = g(X,Y)\xi - \eta(X)\eta(Y). \qquad (3)$$

Moreover, the curvature tensor $\tilde{R}$ and the Ricci tensor $\tilde{S}$ of $\tilde{M}$ satisfy

$$\tilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X \qquad (4)$$

and

$$\tilde{S}(X,\xi) = -2n\eta(X). \qquad (5)$$

Now assume that M is an n-dimensional submanifold of a Kenmotsu manifold such that ξ is tangent to $\tilde{M}$. So from the Gauss formula

$$\tilde{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi),$$

which implies from **[Kobayashi, M.1986]** that

$$\nabla_X \xi = X - \eta(X)\xi \qquad \text{and} \qquad \sigma(X,\xi)=0$$

for each vector field X tangent to M.

It is also easy to see that for a submanifold M of a Kenmotsu manifold $\tilde{M}$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X,$$

for any vector fields X and Y tangent to M. From the above equation

$$R(\xi, X)\xi = X - \eta(X)\xi$$

for a submanifold M of a Kenmotsu manifold $\tilde{M}$.Moreover, the Ricci tensor S of a submanifold M satisfy

$$S(X, \xi) = -(n-1)\eta(X).$$

Motivated by the studies of the above authors, in this study, we consider pseudoparallel, generalized Ricci-pseudoparallel and 2-pseudoparallel submanifolds of Kenmotsu manifolds. We show that these type submanifolds are totally geodesic under certain conditions.

Pseudoparallel Submanifolds of Kenmotsu Manifolds

In this section, we give the main results of the study. Now we begin with the following:

Theorem : Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ ξ is tangent to M. Then M is pseudoparallel such that $L_\sigma \neq -1$ if and only if M is totally geodesic.

As a result of the previous theorem we can give the following corollary :

Corollary : Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$ tangent to ξ. Then M is semiparallel if and only if M is totally geodesic.

Theorem : Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$, ξ is tangent to M. Then M is generalized Ricci-pseudoparallel such that $L_S \neq \frac{1}{n-1}$ if and only if M is totally geodesic.

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Theorem : Let M be a submanifold of a Kenmotsu manifold $\tilde{M}$, ξ is tangent to M. Then M is 2 -pseudoparallel such that $L_{\bar{\nabla}\sigma} \neq -1$ if and only if M is totally geodesic.

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Example : Let M be a conformally flat invariant submanifold of a Kenmotsu manifold $\tilde{M}$. It is known that M is also a Kenmotsu manifold. In **[Kenmotsu, K.]** it is proved that a conformally flat Kenmotsu manifold is a manifold of constant negative curvature -1.

Again it is known **[O'Neill]** that a manifold of constant negative curvature -1 is locally isometric with the hyperbolic space $H^n(-1)$. Hence M satisfies the conditions

$$\bar{R} \cdot \sigma = -Q(g,\sigma),$$

$$\bar{R} \cdot \sigma = \frac{1}{n-1} Q(S,\sigma)$$

and

$$\bar{R} \cdot \bar{\nabla}\sigma = -Q(g, \bar{\nabla}\sigma).$$

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