On Special Complex Structures and Hermitian Metrics

A. Tomassini

Università di Parma

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Introduction

M^{2n} $2n$-dimensional (compact) manifold

- F non-degenerate 2-form on M^{2n}
- F is *symplectic* if $dF = 0$
- J *almost complex structure* on M^{2n}, i.e.
  $J \in \mathrm{End}(TM^{2n})$ s.t. $J^2 = -\mathrm{id}_{TM^{2n}}$
- J is *integrable* if it is induced by a complex structure.

Newlander-Nirenberg

$$J \text{ is integrable} \qquad \Longleftrightarrow \qquad N_J = 0$$

where

$$N_J = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]$$

$\forall X, Y$ vector fields on M.

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$$g_J(JX, JY) = g(X, Y)\,, \quad \forall X, Y\,.$$

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Weaker conditions

1) $dF = 0$, J non-integrable.

2) $dF \neq 0$, J integrable.

1)

- Special symplectic manifolds,
- Geometry of Lagrangian submanifolds.

2)

- *Geometry with Torsion,*
- *Generalized Kähler Geometry,*
- *Bi-Hermitian Structures,*
- *Special metrics on Complex manifolds e.g.* balanced, strong KT, astheno-Kähler

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Def. A *special symplectic Calabi-Yau manifold* (SSCY) is the datum of (M^6, F, J, ψ) where

- F is a symplectic structure
- J is a F-calibrated almost complex structure
- $g_J(\cdot,\cdot) := F(\cdot, J\cdot)$
- $\psi \in \wedge^{3,0}(M)$, $\psi \neq 0$,

s.t.

$$\begin{aligned} d\,\mathfrak{Re}\,\psi &= 0 \\ \psi \wedge \overline{\psi} &= \frac{4}{3} i\, F^3 \end{aligned}$$

Rem.

- (M^6, F, J, ψ) is in particular *Half-flat*, in the sense of Hitchin, Chiossi and Salamon.
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Theorem (P. de Bartolomeis,—, *Ann. Inst. Fouriér* '06)

There exists a compact complex manifold M such that

- *M has a symplectic structure satisfying the Hard Lefschetz Condition;*
- *M admits a SSCY structure;*
- *M has no Kähler structures.*

$$M = (\mathbb{C}^3, *)/\Gamma$$

where $*$ is defined by

$$^t(z_1, z_2, z_3) *^t (w_1, w_2, w_3) =$$

$$^t(z_1 + w_1, e^{-w_1} z_2 + w_2, e^{w_1} z_3 + w_3)$$

and Γ is a certain closed subgroup of $(\mathbb{C}^3, *)$ finitely generated.

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Def. A Hermitian metric g on (M^{2n}, J) is said to be *strong Kähler with torsion*, (SKT), if

$$\partial_J \overline{\partial}_J F = 0\,,$$

where F is the fundamental $(1,1)$-form of g.

Def. (J. Jost, Y.-T. Yau, *Acta Math. '93*). A Hermitian metric g on (M^{2n}, J) is said to be *astheno Kähler*, if

$$\partial_J \overline{\partial}_J F^{n-2} = 0\,.$$

Rem.

- If (M, J) is a complex surface, then every Hermitian metric is astheno-Kähler.
- In complex dimension 3, strong KT means astheno-Kähler.

Strong Kähler with torsion metrics

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Bismut connection

(M, J, g) Hermitian manifold.
The *Bismut connection* ∇^B is characterized by the following

$$\nabla^B g = 0\,, \quad \nabla^B J = 0\,,$$

$$g(X, T^{\nabla^B}(Y, Z))\,\text{totally skew-symmetric}$$

The *torsion form* of the Bismut connection

$$T(X, Y, Z) := g(X, T^{\nabla^B}(Y, Z))$$

is $JdF = -d^cF$.

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} T(X, Y, Z)\,,$$

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- (M, J) compact complex surface $\Rightarrow$ every conformal class of a Hermitian metric has a strong KT representative (Gauduchon, *Math. Ann.* '84).

- $\dim_{\mathbb{R}} M > 4$ compact examples of strong KT metrics on nilmanifolds (Fino, Parton, Salamon, *Comm. Math. Helv. '04*).

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Link with the Generalized Kähler structures

A *generalized Kähler structure* on M (*GK structure*) (see Gualtieri-Apostolov *Comm. Math. Phys.* '07 and Gualtieri *DPhil thesis* '03), is equivalent to assign a triple (g, J_+, J_-) where:

- g is a Riemannian metric on M
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$$d^c_+ F_+ + d^c_- F_- = 0\,, \quad dd^c_+ F_+ = 0\,, \quad dd^c_- F_- = 0\,,$$

F_+, F_- fundamental forms of (g, J_+), (g, J_-),

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Example (M, g, J) Kähler

$$J_+ = J\,, \quad J_- = \pm J$$

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Rem. If M has a GK structure, then M has a strong KT metric.

Pb. When does a compact complex manifold (M, J) admit a GK structure (g, J_+, J_-) with $J = J_+$?

Interesting case: $J_+ \neq \pm J_-$, i.e. the *GK* structure is not induced by a Kähler metric on (M, J).

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Existence results

- (M, J) compact complex surface.

Classification theorem of generalized Kähler structures

(Apostolov and Gualtieri, *Comm. Math. Phys.* '07)

- $\dim_{\mathbb{R}} M = 6$.

By [Cavalcanti and Gualtieri, *J. of Sympl. Geom.* '05]

every nilmanifold carries a GC structure

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there are no nilmanifolds (different from Tori) admitting an invariant GK structure.

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Compact example

- $\mathfrak{s}_{a,b}$ solvable Lie algebra defined by:

$$\begin{cases} de^1 = a\, e^1 \wedge e^2 \,, \\ de^2 = 0 \,, \\ de^3 = \frac{1}{2} a\, e^2 \wedge e^3 \,, \\ de^4 = \frac{1}{2} a\, e^2 \wedge e^4 \,, \\ de^5 = b\, e^2 \wedge e^6 \,, \\ de^6 = -b\, e^2 \wedge e^5 \,, \end{cases} \tag{1}$$

a, b real parameters different from zero.

- $S_{a,b}$ simply-connected Lie group whose Lie algebra is $\mathfrak{s}_{a,b}$

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a, b real parameters different from zero.

- $S_{a,b}$ simply-connected Lie group whose Lie algebra is $\mathfrak{s}_{a,b}$

$(t, x_1, x_2, x_3, x_4, x_5)$ global coordinates on $\mathbb{R}^6$.

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Link with the Generalized Kähler structures

- Product on $S_{a,b}$

$$(t, x_1, x_2, x_3, x_4, x_5) \cdot (t', x_1', x_2', x_3', x_4', x_5') =$$

$$(t + t',\ e^{-a\,t} x_1' + x_1,\ e^{\frac{a}{2}t} x_2' + x_2,\ e^{\frac{a}{2}t} x_3' + x_3,$$

$$x_4' \cos(b\,t) - x_5' \sin(b\,t) + x_4,$$

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- $S_{a,b}$ unimodular semidirect product

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Theorem (A. Fino, —, to appear in *J. of Sympl. Geom.*)

- $S_{1,\frac{\pi}{2}}$ *has a compact quotient*

$$M^6 = S_{1,\frac{\pi}{2}}/\Gamma\,.$$

- M^6 *is the total space of a* $\mathbb{T}^2$*-bundle over the Inoue surface.*
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$$\varphi^1_+ = e^1 + ie^2, \quad \varphi^2_+ = e^3 + ie^4, \quad \varphi^3_+ = e^5 + ie^6,$$

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$(\varphi^1_\pm, \varphi^2_\pm, \varphi^3_\pm)$ $(1,0)$-forms associated with $J_\pm$.

- $J_\pm$ integrable.
- $g = \sum_{\alpha=1}^{6} e^\alpha \otimes e^\alpha$ $\quad$ $J_\pm$-Hermitian.

Then

$$d^c_+ F_+ + d^c_- F_- = 0\,, \quad dd^c_+ F_+ = 0\,, \quad dd^c_- F_- = 0\,,$$

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- $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ where

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It can be checked that

i) Γ acts freely and in a properly discontinuos way on $S_{1,\frac{\pi}{2}}$

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Link with the Generalized Kähler structures

Uniform subgroup

- $S_{1,\frac{\pi}{2}}$ is isomorphic to $(\mathbb{R}^6 = \mathbb{R} \ltimes (\mathbb{R} \times \mathbb{C} \times \mathbb{C}), *)$ where

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Characterization in terms of currents

We shortly recall some known facts on positive currents on an n-dimensional complex manifold M.

- $\Omega^{p,q}(M)$ (respectively by $\mathcal{D}^{p,q}(M)$) space of (p,q)-forms (respectively (p,q)-forms with compact support) on M.

On $\mathcal{D}^{p,q}(M)$ consider the $\mathcal{C}^\infty$-topology.

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Theorem(A. Fino, —, to appear in *Adv. in Math.*)
Let M be a complex manifold of complex dimension $n \geq 2$. If $M \setminus \{p\}$ admits a strong KT metric, then there exists a strong KT metric on M.

Idea of the proof

It is sufficient to show that

If F is the fundamental 2-form of a strong KT metric on $\mathbb{B}^n(r) \setminus \{0\}$, $n \geq 2$, then there exist $0 < R \leq r$ and $\hat{F} \in \Lambda^{1,1}(\mathbb{B}^n(R))$ such that

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• The last theorem is the generalization of the *Miyaoka Extension Theorem* (*Proc. Japan Acad. '74*) for Kähler manifolds, to the strong KT case.

As a corollary, we have the following

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Let M be a complex manifold of complex dimension $n \geq 2$ and $\tilde{M}$ be the blow-up of M at a point $p \in M$.
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On the Iwasawa manifold $\mathbb{I}(3) = \Gamma \backslash H_3^{\mathbb{C}}$ the condition for a Hermitian metric to be strong KT is not stable under small deformations of the complex structure underlying the strong KT structure.

- $\mathfrak{n}_{t,s}$ family of 2-step nilpotent Lie algebras with structure equations

$$\begin{cases} de^i = 0, \quad i = 1, \dots, 4, \\ de^5 = t(e^1 \wedge e^2 + 2\, e^3 \wedge e^4) + s(e^1 \wedge e^3 - e^2 \wedge e^4), \\ de^6 = s(e^1 \wedge e^4 + e^2 \wedge e^3), \end{cases}$$

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The associated $(1,0)$-forms

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and therefore J is integrable.

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• For $t = 0$ and $s = 1$ the complex structure J coincides with the bi-invariant complex structure J_0 on the complex Heisenberg group.

• The complex structure J_0 cannot admit any compatible strong KT metric, since otherwise it has to be balanced and by Fino-Parton-Salamon (*Comm. Math. Helv.* '04) the balanced condition is complementary to the strong KT one.

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Balanced Hermitian metrics

Let (M, J) be a complex manifold of complex dimension n.

Def. *An Hermitian metric g on (M, J) is said to be* balanced *if*

$$dF^{n-1} = 0\,,$$

where $F(\cdot, \cdot) = g(\cdot, J\cdot)$ is the fundamental form of g.

Example

- Let $*$ be the product on $\mathbb{C}^3$ given by

$$^t(z_1, z_2, z_3) *^t (w_1, w_2, w_3) =$$

$$^t(z_1 + w_1, e^{-w_1} z_2 + w_2, e^{w_1} z_3 + w_3)$$

Then $(\mathbb{C}^3, *)$ has a uniform discrete subgroup Γ and

$$M = (\mathbb{C}^3, *)/\Gamma$$

is a compact complex solvmanifold.

Then

$$\varphi_1 = dz_1\,,\ \varphi_2 = e^{z_1}dz_2\,,\ \varphi_3 = e^{-z_1}dz_3\,,$$

induce invariant complex $(1,0)$-forms on M and

$$g = \frac{1}{2}\sum_{j=1}^{3} \varphi_j \otimes \overline{\varphi}_j + \overline{\varphi}_j \otimes \varphi_j$$

is a balanced metric on M.

$\mathrm{SU}(2)$-structures

- N 5-dimensional manifold $L(N)$ principal bundle of linear frames on N.

An $\mathrm{SU}(2)$-*structure* on N is an $\mathrm{SU}(2)$-reduction of $L(N)$. We have the following

Proposition(Conti, Salamon *Trans. Amer. Math. Soc.* '07)
$\mathrm{SU}(2)$-structures on a 5-manifold N are in $1:1$ correspondence with quadruplets $(\eta, \omega_1, \omega_2, \omega_3)$, where η is a 1-form and ω_i are 2-forms on N satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \quad v \wedge \eta \neq 0,$$

for some 4-form v, and $i_X\omega_3 = i_Y\omega_1 \Rightarrow \omega_2(X, Y) \geq 0$, where i_X denotes the contraction by X. Equivalently, an $\mathrm{SU}(2)$-structure on N can be viewed as the datum of (η, ω_3, Φ), where η is a 1-form, ω_3 is a 2-form and $\Phi = \omega_1 + i\omega_2$ is a complex 2-form such that

$$\eta \wedge \omega_3^2 \neq 0\,, \qquad \Phi^2 = 0$$
$$\omega_3 \wedge \Phi = 0\,, \qquad \Phi \wedge \overline{\Phi} = 2\omega_3^2$$

and Φ is of type $(2,0)$ with respect to ω_3.

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for some 4-form v, and $i_X\omega_3 = i_Y\omega_1 \Rightarrow \omega_2(X, Y) \geq 0$, where i_X denotes the contraction by X. Equivalently, an $\mathrm{SU}(2)$-structure on N can be viewed as the datum of (η, ω_3, Φ), where η is a 1-form, ω_3 is a 2-form and $\Phi = \omega_1 + i\omega_2$ is a complex 2-form such that

$$\begin{aligned} \eta \wedge \omega_3^2 \neq 0\,, &\qquad \Phi^2 = 0 \\ \omega_3 \wedge \Phi = 0\,, &\qquad \Phi \wedge \overline{\Phi} = 2\omega_3^2 \end{aligned}$$

and Φ is of type $(2,0)$ with respect to ω_3.

As a corollary of the last Proposition, we obtain the useful local characterization of $\mathrm{SU}(2)$-structures (see Conti-Salamon):

Corollary

If $(\eta, \omega_1, \omega_2, \omega_3)$ is an $\mathrm{SU}(2)$-structure on a 5-dimensional manifold N, then locally, there exists a basis of 1-forms $\{e^1, \ldots, e^5\}$ such that

$$\eta = e^1, \;\; \omega_1 = e^{24} + e^{53}, \;\; \omega_2 = e^{25} + e^{34}, \;\; \omega_3 = e^{23} + e^{45}\,.$$

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SU(2)-structures

$\mathrm{SU}(2)$-structures naturally arise on hypersurfaces of 6-manifolds with an $\mathrm{SU}(3)$-structure.

- $f : N \longrightarrow M$ oriented hypersurface in a 6-manifold M endowed with an $\mathrm{SU}(3)$-structure (F, Ψ_+, Ψ_-), $\mathbb{U}$ the unit normal vector field. Then

$$\eta = -i_{\mathbb{U}} F, \quad \omega_1 = i_{\mathbb{U}} \Psi_-, \quad \omega_2 = -i_{\mathbb{U}} \Psi_+, \quad \omega_3 = f^* F. \tag{3}$$

defines an $\mathrm{SU}(2)$-structure on N.

- Conversely, an $\mathrm{SU}(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on N induces an $\mathrm{SU}(3)$-structure (F, Ψ_+, Ψ_-) on $N \times \mathbb{R}$ given by

$$F = \omega_3 + \eta \wedge dt, \qquad \Psi = \Psi_+ + i\Psi_- = (\omega_1 + i\omega_2) \wedge (\eta + i dt), \tag{4}$$

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Definition

An $\mathrm{SU}(2)$-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on a 5-dimensional manifold N is called *balanced* if it satisfies

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In Conti-Salamon (*Trans. Amer. Math. Soc.* '07), an $\mathrm{SU}(2)$-structure is said to be *hypo* if

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Proposition

Any 5-dimensional compact nilmanifold has an invariant balanced $\mathrm{SU}(2)$*-structure.*

Proof. It is easy to check that the $\mathrm{SU}(2)$-structure given by

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There exist also 5-dimensional solvable non-nilpotent Lie algebras with no invariant hypo structure, but having a balanced $\mathrm{SU}(2)$-structure.

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Balanced $\mathrm{SU}(2)$-structures

- The solvable non-nilpotent Lie algebra

$$(0, 0, 13, -14, 34)$$

has a balanced $\mathrm{SU}(2)$-structure, but it has no hypo structure and the corresponding solvable Lie group G has a compact quotient $N = G/\Gamma$.

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Link with holomorphic symplectic geometry

(X, J) complex surface.

A *holomorphic symplectic structure* on X is the datum of a d-closed and non-degenerate $(2, 0)$-form ω on X.
Let g be a J-Hermitian metric on X and ω_3 be the fundamental form of (g, J). Then, up to a conformal change, we may assume that

$$\omega_1^2 = \omega_2^2 = \omega_3^2 \,.$$

Then we have the following

Proposition

Let (X, J) be a complex surface equipped with a holomorphic symplectic structure $\omega = \omega_1 + i\omega_2$ and let ω_3 be the Kähler form of a J-Hermitian metric g as above. Then, for any integral closed 2-form Ω on X annihilating $\cos\theta\,\omega_1 + \sin\theta\,\omega_2$ and $-\sin\theta\,\omega_1 + \cos\theta\,\omega_2$ for some θ, there is a principal circle bundle $\pi\colon N \longrightarrow X$ with connection form ρ such that Ω is the curvature of ρ and such that the $\mathrm{SU}(2)$-structure $(\eta, \omega_1^\theta, \omega_2^\theta, \omega_3^\theta)$ on N given by

$$\begin{aligned}
\eta &= \rho,\\
\omega_1^\theta &= \pi^*(\cos\theta\,\omega_1 + \sin\theta\,\omega_2),\\
\omega_2^\theta &= \pi^*(-\sin\theta\,\omega_1 + \cos\theta\,\omega_2),\\
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\end{aligned}$$

is a balanced $\mathrm{SU}(2)$-structure.

Example

Let $X = \Gamma \backslash G$ be the *Kodaira-Thurston* manifold, where the Lie algebra $\mathfrak{g}$ of G has the following structure equations

$$de^1 = 0\,, \quad de^2 = 0\,, \quad de^3 = 0\,, \quad de^4 = -e^{23}\,.$$

Then

$$\varphi^1 = e^1 + ie^4\,, \quad \varphi^2 = e^2 + ie^3$$

$$\omega = \left(e^{12} + e^{34}\right) + i\left(e^{13} - e^{24}\right) = \omega_1 + i\omega_2$$

define a complex structure and holomorphic symplectic structure on X respectively. If $g = \sum_{i=1}^{4} e^i \otimes e^i$, then

$$\omega_3 = e^{14} + e^{23}\,.$$

Therefore, the previous Proposition applies.

Differential equations

We establish the evolution equations that allow the construction of new balanced structures in dimension six from balanced $\mathrm{SU}(2)$-structures in dimension five.

Proposition (M. Fernández, —, L. Ugarte, R. Villacampa, *J. Math. Phys.* '09)
Let $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ be a family of $\mathrm{SU}(2)$-structures on a 5-manifold N, for $t \in I = (a,b)$. Then, the $\mathrm{SU}(3)$-structure on $M = N \times I$ given by

$$F = \omega_3(t) + \eta(t) \wedge dt, \qquad \Psi = (\omega_1(t) + i\omega_2(t)) \wedge (\eta(t) + i dt), \tag{7}$$

is balanced if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is a balanced $\mathrm{SU}(2)$-structure for any t in the open interval I, and the following evolution equations

$$\begin{cases} \partial_t(\omega_1 \wedge \eta) = -d\omega_2 \\ \partial_t(\omega_2 \wedge \eta) = d\omega_1 \\ \partial_t(\omega_3 \wedge \omega_3) = -2\, d(\omega_3 \wedge \eta) \end{cases} \tag{8}$$

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Proof. A direct calculation shows that the $\mathrm{SU}(3)$-structure given by (7) satisfies

$$dF^2 = d(\omega_3 \wedge \omega_3) + (\partial_t(\omega_3 \wedge \omega_3) + 2\, d(\omega_3 \wedge \eta)) \wedge dt,$$

and

$$\begin{aligned} d\Psi \quad = \quad & d(\omega_1 \wedge \eta) - (\partial_t(\omega_1 \wedge \eta) + d\omega_2) \wedge dt + \\ & i\, d(\omega_2 \wedge \eta) - i\,(\partial_t(\omega_2 \wedge \eta) - d\omega_1) \wedge dt. \end{aligned}$$

The forms F^2 and Ψ are both closed if and only if $(\eta(t), \omega_1(t), \omega_2(t), \omega_3(t))$ is a balanced $\mathrm{SU}(2)$-structure for any $t \in I$, and satisfies equations (8).

$\square$

Explicit solutions

Lie algebra (0,0,0,12,14):
The family of balanced $\mathrm{SU}(2)$-structures

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satisfies the evolution equations for $t \in \mathbb{R} - \{2/3\}$.

Explicit solutions

Lie algebra (0,0,0,12,14):

The family of balanced $\mathrm{SU}(2)$-structures

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Explicit solutions

Let $I = (-\infty, 2/3)$ and G be the simply-connected nilpotent Lie group with Lie algebra $(0, 0, 0, 12, 14)$.

The basis of 1-forms on the product manifold $G \times I$ given by

$$\alpha^1 = e^2, \alpha^2 = e^3\,, \alpha^3 = \sqrt[3]{\tfrac{2-3t}{2}}\, e^4,$$

$$\alpha^4 = \tfrac{1}{2}\sqrt[3]{\tfrac{2}{2-3t}}(e^2 + 2e^5) - \tfrac{2-3t}{4}e^2\,, \alpha^5 = \sqrt[3]{\tfrac{2-3t}{2}}\, e^1\,, \alpha^6 = dt$$

is orthonormal with respect to the Riemannian metric associated with the balanced SU(3)-structure on $G \times I$.

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Explicit solutions

The Hermitian balanced structure on $G \times I$ is given by

$$\begin{aligned}
F &= e^{23} - \tfrac{1}{2}e^{24} + e^{45} + \tfrac{2-3t}{4}\sqrt[3]{\tfrac{2-3t}{2}}e^{24} + \sqrt[3]{\tfrac{2-3t}{2}}e^{1} \wedge dt, \\
\Psi_+ &= \tfrac{1}{2}e^{123} - e^{135} - \tfrac{2-3t}{4}\sqrt[3]{\tfrac{2-3t}{2}}e^{123} + \sqrt[3]{\tfrac{(2-3t)^2}{4}}e^{124} + \\
&\quad \left(\sqrt[3]{\tfrac{2}{2-3t}}e^{25} + \sqrt[3]{\tfrac{2-3t}{2}}e^{34}\right) \wedge dt, \\
\Psi_- &= e^{125} + \sqrt[3]{\tfrac{(2-3t)^2}{4}}e^{134} + \left(\tfrac{1}{2}\sqrt[3]{\tfrac{2}{2-3t}}e^{23} - \tfrac{2-3t}{4}e^{23} + \right. \\
&\quad \left. \sqrt[3]{\tfrac{2-3t}{2}}e^{24} - \sqrt[3]{\tfrac{2}{2-3t}}e^{35}\right) \wedge dt.
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Examples of manifolds with full holonomy

Theorem M. Fernández, —, L. Ugarte, R. Villacampa, *J. Math. Phys.* '09)
Any 3-dimensional complex-parallelizable (non-abelian) solvable Lie group has a Hermitian metric such that the holonomy of its Bismut connection is equal to $\mathrm{SU}(3)$.

Examples of manifolds with full holonomy

Example Consider the Lie algebra defined by the complex structure equations

$$d\varphi^1 = d\varphi^2 = 0, \quad d\varphi^3 = -\varphi^{12}, \quad d\varphi^4 = -2\varphi^{13}.$$

Let $\varphi^j = e^{2j-1} + i\, e^{2j}$, $j = 1, \ldots, 4$.

Then

$$de^1 = de^2 = de^3 = de^4 = 0,\ de^5 = -e^{13} + e^{24},$$

$$de^6 = -e^{14} - e^{23},\ de^7 = -2(e^{15} - e^{26}),$$

$$de^8 = -2(e^{16} + e^{25}).$$

Examples of manifolds with full holonomy

J complex structure given by

$$Je^1 = -e^2, \quad Je^2 = e^1, \quad Je^3 = -e^4, \quad Je^4 = e^3,$$
$$Je^5 = -e^6, \quad Je^6 = e^5, \quad Je^7 = -e^8, \quad Je^8 = e^7.$$

The fundamental form F associated with the J-Hermitian metric $g = \sum_{i=1}^{8} e^i \otimes e^i$ is given by

$$F = \sum_{j=1}^{4} e^{2j-1} \wedge e^{2j},$$

Examples of manifolds with full holonomy

- g is balanced and the torsion T is given by

$$T = JdF = e^{135} + e^{146} + 2e^{157} + 2e^{168} + e^{236} - e^{245} + 2e^{258} - 2e^{267}.$$

The following curvature forms of the Bismut connection are linearly independent:

Examples of manifolds with full holonomy

$$\Omega^1_3 = -e^{13} - e^{24},\ \Omega^1_4 = -e^{14} + e^{23},\ \Omega^1_5 = -4(e^{15} + e^{26}),$$
$$\Omega^1_6 = -4(e^{16} - e^{25}),\ \Omega^3_4 = 2e^{12},\ \Omega^5_6 = 2(3e^{12} - e^{34}),$$
$$\Omega^5_7 = -2(e^{35} + e^{46}),\ \Omega^5_8 = -2(e^{36} - e^{45}),$$
$$\Omega^7_8 = -8(e^{12} + e^{56}).$$

This gives a 9-dimensional space.
By computing the covariant the derivative of the curvature it follows that $\mathrm{Hol}(\nabla^B)=\mathrm{SU}(4)$.