Invariance groups of finite functions

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The invariance group of an $n$-variable function is the group of permutations of its variables that leave the function invariant. It is easy to see that every subgroup of the symmetric group $S_n$ is the invariance group of some $n$-ary function with a sufficiently large domain and codomain. However, representability of permutation groups as invariance groups of functions $f : A^n \to B$ with given sets $A$ and $B$ is a nontrivial problem. This problem has been studied previously mainly in the Boolean case $A = \{0, 1\}$; here we propose a natural generalization. Let us say that a group $G \leq S_n$ is $(k, m)$-representable if there exists a function $f : A^n \to B$ with $|A| = k$ and $|B| = m$ such that the invariance group of $f$ is $G$. Furthermore, we call a group $(k, \infty)$-representable if it is $(k, m)$-representable for some natural number $m$.

Clote and Kranakis investigated $(2, m)$-representability of groups, and they applied it in the study of circuit complexity of Boolean functions and languages. It has been claimed that the intersection of $(2,2)$-representable groups is again $(2,2)$-representable. This claim has been disproved by Kisielewicz, by showing that the Klein four-group is the intersection of two $(2,2)$-representable groups, but it is not $(2,2)$-representable.

On the other hand, the class of $(k, \infty)$-representable subgroups of $S_n$ for any given $k$ is easily seen to be closed under intersection; in fact, a group $G \leq S_n$ is $(k, \infty)$-representable if and only if it is the intersection of invariance groups of operations $f : A^n \to A$ on a $k$-element set $A$. We introduce a Galois connection between permutations of $\{1, \ldots, n\}$ and $n$-ary operations on $\{1, \ldots, k\}$ such that a subgroup $G$ of $S_n$ is Galois closed if and only if it is $(k, \infty)$-representable. The study of the Galois closures yields the explicit description of $(k, \infty)$-representable groups for $k = n - 1$ and $k = n - 2$, and we also obtain a characterization in the general case $k = n - d$ under the additional assumption that $n$ is much larger than $d$. (Note that the case $k \geq n$ is trivial, as in this case all subgroups of $S_n$ are $(k, \infty)$-representable.)

Our Galois connection is closely related to orbit equivalence of permutation groups. From the description of orbit equivalent pairs of primitive groups obtained by Seress we deduce that all primitive groups except the alternating groups are $(3, \infty)$-representable.

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