Contributions in Differential Geometry
a Round Table

Abstracts

Dimitri Alekseevsky (Hamburg), Homogeneous Lorentzian manifolds

We discuss the structure of homogeneous Lorentzian manifolds $M = G/H$.

In the case of a proper homogeneous manifold $M = G/H$, where the action of G on M is proper, or, equivalenty, the stabilizer H is compact, we give a necessary and sufficient condition for the manifold $M = G/H$ to admit an invariant Lorentzian metric. In the case of a non-compact simple group G, we indicate a way for a description of all proper Lorentzian homogeneous manifolds for the group G.

We recall the results by N. Kowalsky and others about a description of Lorentzian manifolds M with isometric nonproper action of a semisimple Lie group G , in particular, nonproper homogeneous manifolds $M = G/H$ for a semisimple group.

We give some constructions of nonproper homogeneous Lorentzian manifolds for a non semisimple Lie group G. In particular, we classify nonproper homogeneous Lorentzian manifolds with completely reducible isotropy group or with irreducible screen representation. The last classification is based on a description of noncompact connected subgroups of the Lorentz group, which is a generalization of the description by L. Berard Bergery and A. Ikemakhen of weakly irreducible subgroups of the Lorentz group.

Bernd Ammann (Regensburg), A surgery formula for the (smooth) Yamabe number

The smooth Yamabe number of a compact manifold M is defined as

$$\sigma(M) := \sup \inf \int_M \mathrm{scal}^g \, dv^g$$

where the supremum runs over all conformal classes $[g_0]$ on M and the infimum runs over all metrics g of volume 1 in $[g_0]$.

We prove: If N is obtained by surgery of codimension ≥ 3 from M, then

$$\sigma(N) \geq \min\{\sigma(M), \Lambda_n\},$$

where $\Lambda_n > 0$ only depends on $n = \dim M$. This formula unifies and generalizes previous formulas by Gromov-Lawson, Schoen-Yau, Kobayashi, Petean-Yun and allows many conclusions by using bordism theory.

Cyriaque Atindogbe (Abomey-Calavi), Normalization and prescribed extrinsic scalar curvature on lightlike hypersurfaces

One of the main challenges today in the study of the null geometry of lightlike submanifolds is the lack of canonical normalization in general. In this talk, we briefly present how various quantities calculated noncanonically for a lightlike hypersurface vary with a change in normalization, and then turn back to the concept of extrinsic scalar curvature on lightlike hypersurfaces we considered in a recent work. This scalar quantity has been studied on lightlike hypersurfaces equipped with a given normalization. But a very important problem was left open: How can one characterize the set of all normalizations admitting a prescribed extrinsic scalar curvature? We present here various reponses to this question, supported by examples.

Helga Baum (HU Berlin), Holonomy groups of Lorentzian manifolds — a status report

The classification of the holonomy groups of (simply connected) Riemannian manifolds is know for a long time. The classification of the holonomy groups of (simply connected) Lorentzian manifolds was a long time open problem. It was achieved only recently. Main steps of this classification due to Lionel Bérard-Bergery and his coworkers. In the first part of the talk I will review this algebraic classification. In the second art I will explain the construction of Lorentzian metrics with special holonomy - local ones as well as metrics with special global geometric properties. In particular, I will describe the holonomy groups of Lorentzian manifolds with parallel spinors, and will present geodesically complete as well as globally hyperbolic Lorentzian manifolds with complete Cauchy surfaces and parallel spinors.

Lionel Bérard Bergery, Geometries on total spaces of fiber bundles

We will study some examples of geometric structures on the total spaces of some fiber bundles and especially bundles associated with the tangent bundle on a manifold. We will construct various linear connexions on the total spaces which are compatible with those geometries. A general setting will be given , then we will study some class of particular examples.

First examples come from the theory of holonomy of torsion-free connexions, in the non-reducible case. As explicit examples, we will consider the total space of the tangent bundle or the cotangent bundle of a manifold
(Joint work with T. Krantz)

Other interesting examples come from the study of pseudo-riemannian manifolds. here we consider the total space of the bundle of "Sylvester-decompositions", which admits canonically a one-parameter family of Riemannian metrics
(Joint work with M.A. Lawn)

Gérard Besson (Grenoble), Ricci flow on open 3-manifolds

The study of the Riemannian Geometry of non compact 3-manifolds is completely open. We shall present a structure theorem for the manifolds with bounded geometry and scalar curvature bounded below. It uses the full strength of Perelman's work on the Ricci flow with surgery, suitably modified.

Charles Boubel (Strasbourg), The algebra of the parallel endomorphisms of a pseudo-Riemannian metric

A Kähler metric is a Riemannian metric admitting an almost complex structure J which is parallel. Among metrics that are not locally a product, Kähler and hyperkähler metrics are the only Riemannian metrics admitting parallel endomorphism fields, ie sections of $\mathrm{End}(TM)$, non proportional to the identity. This is not true for pseudo-Riemannian metrics: those may admit an algebra of parallel endomorphism fields of any dimension. I explore this situation. This algebra is (classically) the sum of a semi-simple algebra and of its radical. The former may be of eight different types, while the latter may be non trivial. I give some classification results and parametrise the space of germs of metrics corresponding to some significant cases.

Jost Eschenburg (Augsburg), Indefinite extrinsic symmetric spaces

Let V be a real vector space with non-degenerate inner product. A full non-degenerate submanifold $M \subset V$ is called extrinsic symmetric if the reflection at each of its affine normal spaces keeps M invariant. Clearly, M is an orbit under the group K generated by these reflections. For definite inner products Dirk Ferus proved that (V, K) is the isotropy representation of another symmetric space G/K, and further $M = K.v$ for some $v \in V \subset L(G)$ such that the corresponding inner derivation $D = ad(v)$ satisfies $D^3 = -D$.

In the general case, each extrinsic isometry is affine, i.e. linear plus translation, and we could show that the linear parts again form the isotropy representation of a symmetric space G/K, but we could not see what the orbit is. However, unless the shape operator S_H with respect to the mean curvature normal squares to 0, we could show that the extrinsic isometry group has indeed a fixed point, that is, it is linear, and we recover precisely Ferus' result.

Ines Kath has recently solved the remaining case; in particular, she understood the affine part of the group by constructing again a derivation D (not necessarily inner) with $D^3 = -D$. I will briefly report about the new ideas.
(joint work with Jong Ryul Kim)

Anton Galaev (Brno) On the holonomy groups of Riemannian supermanifolds

I introduce the notion of the holonomy group for a linear connection on a supermanifold. I show that it satisfies the main properties of the usual holonomy group of a smooth manifold. Then I classify possible irreducible connected holonomy groups (satisfying an additional condition) of Riemannian supermanifolds and I discuss the corresponding geometries.

Marc Herzlich (Montpellier) A differential geometric description of the Cartan connection for CR structures

We give a simple differntial geometric construction of the canonical Cartan (or tractor) bundle and connection in (strictly pseudo-convex) Cauchy-Riemann geometry. This offers an alternative route to the usual, more abstract, definition through a Lie algebraic approach.

Aziz Ikemakhen (Marrakech), On a class of indecomposable reducible pseudo-Riemannian manifolds

We provide the tangent bundle TM of pseudo-Riemannian manifold (M, g) with the Sasaki metric g^s and with the complete metric g^c. First we show that the holonomy groups H^s and H^c respectively of (TM, g^s) and (TM, g^c) contain the holonomy group of (M, g). This allows us to show that if the basis manifold (M, g) is indecomposable reducible, then (TM, g^s) is also indecomposable reducible. We find conditions on the base manifold under which (TM, g^s) and (TM, g^c) are Kählerian, locally symmetric or Einstein manifolds.

(TM, g^c) is always reducible. We show that it is indecomposable if (M, g) is irreducible.

Ines Kath (Greifswald) Indefinite symmetric spaces and quadratic extensions

Symmetric spaces constitute an important class of (pseudo-)Riemannian manifolds. On the one hand they are sufficiently complicated to serve as examples for many geometric phenomena and on the other hand they are simple enough to calculate various geometric quantities explicitly. The study of symmetric spaces began with the work of É. Cartan who classified Riemannian symmetric spaces completely.

Here we will be interested mainly in pseudo-Riemannian symmetric spaces. In this case it is impossible to give a full classification in the sense of a list. We present a structure theory for symmetric spaces that allows a systematic construction and that gives a "recipe" how to get an explicit classification under suitable addiditional conditions, e.g., for small index of the metric.

The theory is based on an idea of L. Bérard-Bergery.

Claude LeBrun (Stony Brook) On Four-dimensional Einstein manifolds

Andrei Moroianu (É. Polytechnique) Index-theoretical obstructions to the existence of weakly complex structures

We prove that compact quaternionic-Kähler manifolds of positive scalar curvature admit no almost complex structure, even in the weak sense, except for the Grassmannians of complex planes $\mathrm{Gr}_2(\mathbb{C}^{n+2})$. We also prove that an inner symmetric space of compact type is weakly complex if and only if it is a product of spheres and Hermitian symmetric spaces.

Philippe Nabonnand (Nancy), Autour de la notion de courbure

Andrea Sambusetti (Roma Sapienza), On the geometry of rays and the Gromov compactification of negatively curved manifolds

The notion of Busemann function was originally introduced by Herbert Busemann in the fifties as a tool to develop a theory of parallels on geodesic spaces (e.g. complete Riemannian manifolds).

The Busemann functions captures the idea of "angle at infinity" between infinite geodesic rays, and this idea played an important role in the study of the topology complete noncompact Riemannian manifolds. In particular this notion has a special place in the geometry of Hadamard spaces (simply connected manifolds with nonpositive curvature) and in the dynamics of Kleinian groups.

For a Hadamard manifold X, the Busemann functions yield a useful compactification of the space, which was originally introduced by M. Gromov; this compactification has the topology of a sphere and is easily understood in terms of rays.

This nice picture breaks down for non-simply connected manifolds. The aim of the talk is to explain the main differences between the "visual" description of the Gromov compactification for Hadamard spaces and the non-simply connected case.

I will give some examples of the main interesting pathologies in the non-simply connected case, such as:

1. divergent rays having the same Busemann functions;
2. points on the Gromov boundary which are not Busemann functions of any ray;
3. discontinuity of the Busemann functions with respect to the initial conditions.

I will also explain that, restricting to geometrically finite manifolds (the simplest class of non-simply connected, negatively curved manifolds) all the pathologies disappear and we recover a simple description of the Gromov boundary.
(joint work with F. Dal'bo and M. Peigne')

Lorenz Schwachhöfer (Dortmund), Extrinsic symmetric spaces

We investigate extrinsic symplectic symmetric spaces (e.s.s.s.). Similar as in the (pseudo-)Riemannian case, this is an symmetric space which is immersed into a symplectic vector space with parallel second fundamental form. However, there are fundamental differences of the algebraic and geometric behaviour to the (pseudo-)Riemannian case which we shall discuss in detail. Also, we shall briefly describe the significance of e.s.s.s. for deformation quantization.