# Experimental Test on the abc-Conjecture 

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#### Abstract

This thesis focuses on finding a counter-example to the $a b c$-Conjecture by comparing different databases.


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## Chapter 1

## Introduction

The following will be a quick overview of the development of mathematics and logic since its beginning. It will show the history of rigorousness and proofs in mathematics.

### 1.1 Mathematics and Logic in Ancient Times

Several thousand years ago, during the beginning of modern civilisation, mathematics began as the mere concept of counting, its sole purpose being its usage for trading as well as keeping track of time. One of the earliest artifacts used for mathematical purposes is the Ishango bone which was found in modernday Congo. It dates back at least 20,000 years and was probably used either as a list of prime numbers or as a six-month lunar calendar.[1] It is believed that the earliest traces of geometry can be found in old Egyptian artifacts from around $5,000 \mathrm{BCE}$ or in monuments in Great Britain from around $3,000 \mathrm{BCE}$. Nevertheless, these ideas are disputed. It is however undisputed that the oldest known mathematical documents emerged from Babylonian and Egyptian sources.
The Babylonians used a numeral system with base 60 , complete with multiplication and division, and developed a measure system from as early as $3,000 \mathrm{BCE}$. It is most astounding that the Babylonians were even able to solve quadratic and cubic equations. The Egyptians probably used a base 10 number system and had mathematical knowledge in both geometry and algebra. For instance, they were able to determine the surface area of three-dimensional objects. However, both the Egyptians and the Babylonians only used mathematics as a tool and thus did not know the concept of proving their results. The idea


Figure 1.1: Egyptian numeral system of mathematical concepts having to be proven before being considered to be true only emerged with the independent development of logic in Europe and Asia.

In Europe, the study of logic began with Greek philosophers distinguishing between different kinds of sentences, as well as developing techniques for discus-
sions based on arguments at around 500 BCE.[2] Even though Pythagoras, as well as Thales, had already established proofs to known statements, the most famous certainly being Pythagoras' Theorem: In a right triangle, the square of the length of the hypotenuse is the same as the sum of the squares of the two other sides., Greek logic did not yet include the need for proofs. Pythagoras' proof consists of a simple proof by rearrangement. Around 400 BCE, Plato introduced the notion of definitions and asked about the connection between assumptions and conclusions, which one can view as an early idea of the further notion of a theorem.[3][4]

His disciple, Aristotle, was the first to systematically study logic. In his Organon, he developed logic as a tool for the theoretical sciences, revolving around the notion of deduction. He gives detailed ideas about affirmations, denials and contradictions and divides proofs into two different categories; those based on "complete" deductions that don't need proofs, which can be viewed today as axioms, and those based on "incomplete" deductions which need to be proven and follow from the "complete" ones.[5] This is the first description of the concept of a proof. Aristotle also proves that counterexamples disprove a statement. However his school of logic was rivaled by the school of logic, which was developed by the


Figure 1.2: Aristotle Stoic philosophers. Both schools differentiate in that Aristotle's logic is based on "terms"; the subjects and predicates of assertions, while Stoic logic is based on "assertibles"; the equivalent of a proposition.[6]

However, not many texts on Stoic logic have sur-


Figure 1.3: Euclid vived, as Aristotle's logic became the dominating form of logic by the end of antiquity. Around 300 BCE, Euclid wrote one of the most important, if not the most important book in the history of mathematics: his Elements. It originally was a collection of 13 books containing definitions, postulates and propositions including their respective proofs in the areas of Euclidean geometry, number theory and algebra. It is universally accepted that Euclid was not the author of most of the theorems and their proofs, but the content of the Elements as well as the mathematical methods used have influenced all of mathematics. [7] His geometric proofs are based on 5 axioms (as cited by wikiversity.org) [8]

- A line can be drawn from a point to any other point.
- A finite line can be extended indefinitely.
- A circle can be drawn, given a center and a radius.
- All right angles are ninety degrees.
- If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles,
then the lines meet on that side and not on the other side.
These are the five postulates on which Euclidean geometry is based.
In Asia, logic developed independently in two different regions: in India and in China. Indian philosophy was divided into several major schools, of which some developed concepts of logic, the most influential being the Nyaya school's system of logic which later shaped Indian logic.[9] It is based on epistemology, the philosophy treating the theory of knowledge, and is thus not based on a strict mathematical study of propositions like Western logic. Instead, Indian logic tries to provide known scientific arguments with rigor. It does not admit statements which are known to be false, which leads to proofs by contradiction not being accepted in Indian logic.[10] The most notable Ancient Indian mathematician was Pingala [11], whose work contains the first known description of a binary numeral system as well as a version of the Binomial Theorem which describes the expansion of powers of a binomial.[12] However, since Indian logic was not as well-developed as Greek logic at that moment, Indian mathematicians did not yet put much rigor into proving their arguments.

In China, the development of logic began with the founding of the Mohist school by Mozi


Figure 1.4: An excerpt from The Nine Chapters on the Mathematical Art around 400 BCE.[4] It was however the introduction of Buddhism and the following introduction of Indian logic that the development of Buddhist logic appeared in China. As it originated from Indian logic, it is also based on epistemology.[13] Ancient Chinese mathematicians treated the areas of geometry as well as algebra, with magic squares dating back as far as 650 BCE and Gaussian elimination being used in The Nine Chapters on the Mathematical Art, an important book in Chinese mathematics finished around 200 BCE. A detailed commentary by Chinese mathematician Liu Hui from the 3rd century gave proofs to the problems in the book.[14]

### 1.2 Mathematics and Logic in the Middle Ages

Around 600 CE, the rise of Islam brought with it Islamic Philosophy and its own take on logic which placed significance on standards of arguments, though this method was quickly overthrown by the Mu'tazili philosophers who favored old Hellenistic philosophy. Several Persian logicians were however dissatisfied with Hellenistic philosophy and corrected it or developed their own forms of logic. Beginning in the 10th century, Islamic scholars began to base their works on clerical authority, which resulted in a decline of Aristotelian logic in Islamic science.[15] The most important aspect of Islamic logic is certainly the development of Avicennian logic which introduced the need for citation as well as inductive logic, i.e. deducting that a conclusion may be true based
on some evidence. The period from the 9 th to the 15 th century is considered to be the Golden Age of Islamic mathematics and saw several Islamic mathematicians proving theorems with the use of arithmetic and algebra.[16] Al-Khwarizmi introduced the Hindu numerical system, consisting of the numbers $0-10$, which proved to be a great tool for improving the efficiency of Islamic mathematics. Around 1000 CE, Al-Karaji developed one of the first known proofs by induction to prove the previously mentioned Binomial Theorem. Islamic scholars proved several other theorems during the following centuries.[17] They furthermore occupied themselves amongst other things with trigonometry as well as solving cubic equations and geometry.

In Europe, following the Dark Ages, Boethius, a Christian philosopher, was the most important contributor to logic of his time. As he was only familiar with some of Aristotle's works and none of the Stoics', Aristotelian logic became the predominant form of logic in medieval Europe. Around 1300, the theory of consequences emerged, which is an im-


Figure 1.5: AlKhwarizmi, the "father of algebra" portant aspect of later mathematical proofs. As Europe started trade relationships with the East, Eastern mathematics started to spread. In the 12th century, Al-Khwarizmi's Compendious Book on Calculation by Completion and Balancing was translated into Latin by Robert of Chester. With Johannes Gutenberg's invention of the printing press in 1439, knowledge became more easily transmittable, which enabled mathematicians to further extend their areas of expertise.[18] In the 13th century, with the help of Italian mathematician Fibonacci, the Hindu numerical system was introduced in Europe and quickly became the prevalent numerical system. Mathematicians began to become interested in infinite series, with French mathematician Nicole Oresme proving that the series $1+1 / 2+1 / 3+\ldots$ diverges.

In China, mathematics experienced their Golden Age during the 13th century, with over 30 reputable


Figure 1.6: Pascal's triangle in the Jade Mirror of the Four Unknowns mathematics schools existing around all of China.[19] Qin Jiushao introduced a method of solving high order equations, going as far as to solving a 10th order equation. Li Zhi collected hundreds of formulas and problems on inscribing circles in triangles in his SeaMirror of the Circle Measurments. Zhu Shijie's Jade Mirror of the Four Unknowns contains the formula $1^{2}+2^{2}+\ldots+n^{2}=\frac{n(n+1)(2 n+1)}{3!}$, although it is missing a proof. In the 13th century, trigonometry had finally become an important area in Chinese mathematics.

It was also in India that mathematics experienced their Golden Age during the Middle Ages. Around 500 CE , Arybhata gave the formula 1.2 in his Aryabhatiya, as well as a table of sines, a section covering several types of numerical equations and an ap-
proximation of pi which is accurate to four decimal places.[20] Brahmagupta gave a solution to the quadratic equation $a x^{2}+b x=c$ around 650 CE. His Brahmasphutasiddhanta contains one of the first appearance of the four fundamental numerical operations, as well as rules for calculations involving zero. He also gave a formula for finding Pythagorean triples and a formula for the length of the diagonals of any isosceles trapezoid, named Brahmagupta's Theorem.[21] In the 12th century, Bhaskara II discovered differential and infinitesimal calculus by approximating, more precisely he noticed that $\sin y^{\prime}-\sin y \approx\left(y-y^{\prime}\right) \cos y$ for $y^{\prime}$ close to $y$, which translates in today's notation to $\frac{d}{d x} \sin y=\cos y$.[22] He found a proof to Pythagoras' Theorem as well as solutions of equations of the type $a x^{2}+b=y^{2}$ and he stated an early version of Rolle's Theorem which is a special case of the Mean Value Theorem.

### 1.3 Mathematics and logic in modern times

Due to the Scientific Revolution, huge advancements were made in nearly every scientific field. Astronomers began discovering new celestial bodies and made some serious improvements in the study of planetary motion. In the 17th century, Gottfried Leibniz and Isaac Newton independently developed differential and integral calculus, the basis of so many later fields of mathematics as well as of other sciences, and the beginning of modern physics. John Napier was the first to introduce logarithms in 1614. Blaise Pascal and Pierre de Fermat founded probability theory, whose foundations had been set by Gerolamo Cardano in the 16th century.[23] Fermat wrote his Last Theorem without giving a proof, which might be the most well-known case of a theorem that was not proven by its author. Its content, as well as a proof of a special case will be discussed later. In the 18th century, calculus was in full development with Leonhard Euler introducing the exponential function and Euler's formula from which the famous $e^{i \pi}+1=0$ is derived.

He made significant contributions to graph theory by solving the "Seven Bridges of Königsberg" problem,


Figure 1.7: Leonhard Euler (1707-1783) to number theory by proving a number of theorems, as well as to several other areas of mathematics and physics.[24] During the 19th century, mathematics started getting more abstract with elliptic and hyperbolic geometry being developped, Niels Abel and Evariste Galois laying the fundaments of group theory, and Carl Friedrich Gauss developing, among others, complex analysis. At the end of the century, Giuseppe Peano and David Hilbert were among the mathematicians who founded mathematical logic.[25] At the beginning of the 20th century, Ernst Zermelo proposed his set theory which, with the help of Abraham Fraenkel, became the Zermelo-Fraenkel Set Theory, which is the most accepted foundation of today's mathematics. Hilbert founded Hilbert's Program whose goal it was to construct a finite and complete set of axioms on which mathematics were to be based.[26] However, in 1931 Kurt Gödel published his Incompleteness Theorem showing that this was in fact not possible; in such a set of axioms, statements can be
constructed which are true but unprovable.[27] Einstein's General Relativity Theory caused the rise of differential geometry, and the beginning of computer science led to advancements in algebra and mathematical logic. In the second half of the century, several long-standing mathematical problems were finally proven, among others Fermat's Last Theorem in 1994 by Andrew Wiles as well as the Four Color Theorem in 1976 by Wolfgang Haken and Kenneth Appel, and the Classification Of Finite Simple Groups, one of the largest mathematical proofs ever written filling several tens of thousands of pages.

In Asia, mathematical progress started to decline. Even though there were still remarkable Indian and


Figure 1.8: Ramanujan (1887-1920) Chinese mathematicians after the 16 th century, mathematics did not flourish as much as they did during the Middle Ages. With colonization, western mathematics were introduced in Asia even though some theorems had already been well-known to the local scholars for a long time. Around the beginning of the 20th century Indian mathematician Ramanujan, without previous mathematical training, made several important discoveries in many mathematical fields, and established theorems and conjectures in those fields. As of today, the age of Information has connected the whole world, and several Asian mathematics departments rank among the best in the world.

One of the most important events in the 20th and 21st centuries were however the publications of two lists of unsolved mathematical problems. In 1900, Hilbert published a list of 23 unsolved mathematical problems and in 2000, the Clay Mathematics Institute published the Millenium Prize Problems, a list of seven problems endowed by a million dollars each.[28][29] Some of the problems on Hilbert's list are:


Figure 1.9: David Hilbert (1862-1943)

- The Riemann Hypothesis. Still open.
- The Goldbach Conjecture: Every even integer bigger than 2 is the sum of two primes. Still open.
- What is the densest sphere packing? Probably resolved by Thomas Callister Hales in 1998.
- Is $a^{b}$ transcendental if $a \neq(0,1)$ is algebraic and $b$ is algebraic and irrational? In 1934, Aleksandr Gelfond and Theodor Schneider independently proved the answer to be "yes".

The Riemann Hypothesis is also one of the Millenium Prize Problems. Of the seven problems, only the Poincaré conjecture has been proven, in 2003 by Grigori Perelman. Other problems on the list are:

- P vs NP, a theoretical computer science problem on algorithm execution times.
- The Navier-Stokes Existence And Smoothness Problem which asks if certain solutions to the Navier-Stokes equations exist under asserted conditions.
- The Birch And Swinnerton-Dyer Conjecture which asks for a way to tell if elliptic curves over the rational numbers have finite or infinite number of rational solutions.

Beside these problems, there are still several big mathematical problems that have been open for a long time, such as the Hadamard Matrix Conjecture, whether the Euler constant is irrational, the Collartz Conjecture, and the ABCConjecture, around which this thesis revolves.[30]

### 1.4 Motivation

The previous section showed the most important aspect of today's mathematical philosophy: A statement is only true if it has been proven. It might have been acceptable in ancient mathematics to assume the truthfulness of statements, but today's mathematics would collapse without this rigor, since anyone could claim anything to be true. That's why the task of mathematical researchers is not only to develop new concepts and formulas, but also to prove them as well. Without a proof, a formula might still hold to a certain degree in applied mathematics, but mathematically it would be considered invalid. With the help of computers, one can often find a huge number of examples validating a conjecture, but without proof the conjecture is mathematically not valid. Finally proving the statement is really hard, as the case of Fermat's Last Theorem exemplifies. Fermat never gave a proof for his theorem even though he claimed to have found it. Mathematicians tried to find a counterexample for centuries with various methods, studying bigger and bigger numbers. In the end, Andrew Wiles published its proof after over 350 years, going down into the annals of mathematical history.[31] The $a b c$-Conjecture does seem to hold for a huge amount of numbers as computations show. However, this doesn't mean much in the mathematical point of view, since there could always exist a counterexample bigger than all the numbers tested so far. The goal of this thesis is to find a counterexample to the $a b c$-Conjecture. This would have a big impact on mathematics, since the Conjecture is not only assumed by many to be true, but there are many conjectures which can be proven with the help of the $a b c$-Conjecture. Mathematicians would have to find new ways and develop new ideas to prove those conjectures. For me this subject is particularly interesting as it not only includes the thrill of an unsolved mathematical problem, but also a programming aspect which is something I wanted to be a part of my Bachelor thesis. The subject helps me develop my programming skills, while it also gives me the opportunity to thoroughly explore the $a b c$-Conjecture.

## Chapter 2

## The $a b c$-Conjecture

### 2.1 The Conjecture

[32]
Definition 1. Let $n \in \mathbb{N}$. The radical $\operatorname{rad}(n)$ is defined as the product of the distinct prime factors of $n$.

Lemma 1. Let $a, b, c \in \mathbb{N}$. If $a, b$ and $c$ are coprime, then $\operatorname{rad}(a b c)=\operatorname{rad}(a) \operatorname{rad}(b) \operatorname{rad}(c)$.
Proof. The proof follows directly from the assumption of coprimality.
Lemma 2. There are infinitely many triples $(a, b, c)$ of coprime positive integers such that $a+b=c$.

Proof. For $n \in \mathbb{N}_{\geq 2}$, we have that $1, n-1$ and $n$ are coprime.
Definition 2. Let $a, b, c \in \mathbb{N}$. The quality of $a, b, c$ is defined as $q(a b c)=$ $\log (c) / \log (\operatorname{rad}(a b c))$.

The ABC-Conjecture. [33] For every $\epsilon \in \mathbb{R}_{>0}$, there exist only finitely many triples (a,b,c) satisfying $a+b=c$ with $\operatorname{gcd}(a, b)=1$, such that

$$
\begin{equation*}
c>\operatorname{rad}(a b c)^{1+\epsilon} . \tag{2.1}
\end{equation*}
$$

or, equivalently:

$$
\begin{equation*}
c<K_{\epsilon} \operatorname{rad}(a b c)^{1+\epsilon} \text { forsome } K_{\epsilon} . \tag{2.2}
\end{equation*}
$$

A triplet of this form will be called an abc-triplet.
This implies :
For all $\epsilon \in \mathbb{R}$, there exist only finitely many triples (a,b,c) satisfying $a+b=c$ with $\operatorname{gcd}(a, b)=1$, such that

$$
\begin{equation*}
q(a, b, c)>1+\epsilon \tag{2.3}
\end{equation*}
$$

This follows directly from the definition of the quality in Definition 2.
Remark 1. The statement $a+b=c$ with $\operatorname{gcd}(a, b)=0$ implies that $a, b$ and $c$ are coprime. This can easily be checked by assuming that $\operatorname{gcd}(a, c) \neq 0$ which leads to the contradiction $\operatorname{gcd}(a, b) \neq 0$.

### 2.2 History of the ABC Conjecture

During his research in 1983, Richard Mason realised that a special case of Fermat's Last Theorem had never been explored; First powers with restrictions on the variables, that is, the equation $a+b=c$ with certain restrictions on $a, b$ and $c$. Without restrictions on $a, b$ and $c$, the equation can easily be solved in terms of two of the variables. However, Mason started by analysing the case of $a, b$ and $c$ being polynomials over a field.[30] This led him to the rediscovery of the Mason-Stothers theorem, published by W.Wilson Stothers in 1981:[34]

Theorem 1. Let $\mathbb{F}$ be a field and $a(X), b(X), c(X) \in \mathbb{F}[X]$ be relatively prime polynomials with $a+b=c$ and $a^{\prime}, b^{\prime}$ and $c^{\prime}$ being not all zero. Then,

$$
\max (\operatorname{deg}(a), \operatorname{deg}(b), \operatorname{deg}(c)) \leq \operatorname{deg}(\operatorname{rad}(a b c))-1
$$

where rad(abc) is the product of the distinct prime factors of abc in $\mathbb{F}[X]$. This is equivalent to the statement:

$$
\operatorname{deg}(c) \leq n_{0}(a b c)-1
$$

where $n_{0}$ is the number of distinct zeroes of a polynomial.
The following proof was proposed in "N. Snyder, An Alternate Proof of Mason's Theorem, Elemente der Mathematik, Birkhäuser Verlag, Basel, 2000, Ausgabe 55" .[35] The proof is based off the subsequent lemma which can also be found in Snyder's paper:

Lemma 3. Let $f(X) \neq 0, f \in \mathbb{F}[X]$. Then

$$
\operatorname{deg}(f) \leq \operatorname{deg}\left(g c d\left(f, f^{\prime}\right)\right)+n_{0}(f)
$$

where $n_{0}(f)$ is the number of zeroes of $f$ in $\mathbb{F}$.
Proof. Let $f(X)=c \prod_{i=1}^{n_{0}(f)}\left(X-\alpha_{i}\right)^{m_{i}}$. By derivation, we get:
$f^{\prime}(X)=c m_{1}\left(X-\alpha_{1}\right)^{m_{1}-1} \prod_{i=2}^{n_{0}(f)}\left(X-\alpha_{i}\right)^{m_{i}}+c\left(X-\alpha_{1}\right)^{a_{1}} \frac{d}{d X} \prod_{i=2}^{n_{0}(f)}\left(X-\alpha_{i}\right)^{m_{i}}$
This gives $\left(X-\alpha_{1}\right)^{m_{1}-1} \mid g c d\left(f, f^{\prime}\right)$. Or similarly, $\left(X-\alpha_{i}\right)^{m_{i}-1} \mid g c d\left(f, f^{\prime}\right)$, which leads to $\left(\prod_{i=1}^{n_{0}(f)}\left(X-\alpha_{i}\right)^{m_{i}-1}\right) \mid g c d\left(f, f^{\prime}\right)$. Or $\operatorname{deg}(f)=\operatorname{deg}\left(\prod_{i=1}^{n_{0}(f)}\left(X-\alpha_{i}\right)^{m_{i}-1}\right)+$ $n_{0}(f)$. Thus $\operatorname{deg}(f)-n_{0}(f) \leq \operatorname{deg}\left(g c d\left(f, f^{\prime}\right)\right)$, which proves the lemma.

From this follows the proof of the Mason-Stothers theorem:
Proof. We have: $a+b=c \Longrightarrow a^{\prime}+b^{\prime}=c^{\prime}$, and by multiplying the left hand side by $a^{\prime}$ and the right hand side by $a$, we get: $a a^{\prime}+b a^{\prime}=c a^{\prime}$ and $a^{\prime} a+b^{\prime} a=c^{\prime} a$. This leads to $a^{\prime} b-a b^{\prime}=a^{\prime} c-a c^{\prime}$, which shows that $\operatorname{gcd}\left(a, a^{\prime}\right), \operatorname{gcd}\left(b, b^{\prime}\right)$ and $\operatorname{gcd}\left(c, c^{\prime}\right)$ all divide $a^{\prime} b-a b^{\prime}$. As $a, b$ and $c$, and thus $a^{\prime}, b^{\prime}$ and $c^{\prime}$ are relatively prime, we get that

$$
\begin{equation*}
\operatorname{gcd}\left(a, a^{\prime}\right) \operatorname{gcd}\left(b, b^{\prime}\right) \operatorname{gcd}\left(c, c^{\prime}\right) \mid a^{\prime} b-a b^{\prime} \tag{2.4}
\end{equation*}
$$

Or $a^{\prime} b-a b^{\prime} \neq 0$, as this would lead to $a \mid a^{\prime} b$, which gives $a \mid a^{\prime} \Longrightarrow a^{\prime}=0$ as $a$ and $b$ are relatively prime. The same arguments lead to $b^{\prime}=0$ and $c^{\prime}=0$, which contradicts the assumptions. The equation 2.4 now gives us

$$
\begin{equation*}
\operatorname{deg}\left(g c d\left(a, a^{\prime}\right)\right)+\operatorname{deg}\left(\operatorname{gcd}\left(b, b^{\prime}\right)\right)+\operatorname{deg}\left(g c d\left(c, c^{\prime}\right)\right) \leq \operatorname{deg}(a)+\operatorname{deg}(b)-1 \tag{2.5}
\end{equation*}
$$

This comes from the fact that $\operatorname{deg}\left(a^{\prime} b-a b^{\prime}\right)=\max \left(\operatorname{deg}\left(a^{\prime} b\right), \operatorname{deg}\left(a b^{\prime}\right)\right)=$ $\operatorname{deg}(a)+\operatorname{deg}(b)-1$. Adding $\operatorname{deg}(c)$ to both sides in equation 2.5 leads to
$\operatorname{deg}(c) \leq \operatorname{deg}(a)-\operatorname{deg}\left(g c d\left(a, a^{\prime}\right)\right)+\operatorname{deg}(b)-\operatorname{deg}\left(g c d\left(b, b^{\prime}\right)\right)+\operatorname{deg}(c)-\operatorname{deg}\left(g c d\left(c, c^{\prime}\right)\right)-1$.
By applying the lemma, we get that

$$
\operatorname{deg}(c) \leq n_{0}(a)+n_{0}(b)+n_{0}(c)-1=n_{o}(a b c)-1
$$

as $a, b$ and $c$ are relatively prime.
Mason realized that the integer analogue of this theorem is the following: Let $a+b=c$, where $a, b$ and $c$ are relatively prime, then the number of prime factors of each of $a, b$ and $c$ is strictly less than that of distinct prime factors of $a b c$. Unfortunately, this statement is false as for example $a=1, b=7, c=8$ is a counterexample. In 1985 however, Masser and Osterlé corrected the statement and came up with a conjecture that they found no numerical contradiction for. This conjecture is known as the $a b c$-conjecture and the main subject of this Bachelor thesis.

### 2.3 Attempted proofs and Consequences

Throughout the years, there have been several attempted proofs of the $a b c$ Conjecture. In 2007 for example, during a conference at Columbia University in New York, French mathematician Lucien Szipro presented a proof attempt which was shortly thereafter found to be false.[36][37]
The most important proof attempt yet was published in 2012 by Japanese mathematician Shinichi Mochizuki in a paper introducing a new theory of mathematics called Inter-Universal Teichmüller Theory(IUT).[38] Mochizuki claims that with IUT, he found proofs for various number theoretical conjectures including Szipro's Conjecture and Vojta's Conjecture, as well as the abc-Conjecture. However, Mochizuki's proof has not yet been accepted, as several mathematicians claim that a proof in one of the paper's corollaries (Cor. 3.12) is fundamentally flawed with mathematicians Peter Scholze and Jakob Stix describing it as a "serious, unfixable gap". Since the release of the paper, mathematicians from all over the world are trying to get a grasp on Mochizuki's proof, but its size and difficulty are a major obstacle for most of them. Only a handful of mathematicians, which are all in "Mochizuki's orbit", as Brian Conrad of Stanford University stated, have accepted the proof so far. Despite several workshops on Inter-Universal Teichmüller Theory and a personal meeting between Mochizuki, Scholze and Stix, there has been no outcome as to the correctness of the paper, and many mathematicians consider the $a b c$-Conjecture to still be an open problem.[39]
Proving the $a b c$-Conjecture would not only prove other conjectures, but it would also lead to much simpler proofs of several important theorems.

### 2.3.1 Simpler Proofs of existing Theorems

Some of the theorems which can be proven with the $a b c$-Conjecture include:

- The Mordell conjecture, proven in 1983 by German mathematician Gerd Faltings.
- Fermat's Last Theorem for $n \geq 6$, proven in 1994 by English mathematician Andrew Wiles.
- Roth's Theorem, proven in 1955 by Harold Davenport and Klaus Roth.

The $a b c$-Conjecture leads to a proof of Fermat's Last Theorem[40] with $n \geq$ 6: [41]

Theorem 2 (Fermat's Last Theorem with $n \geq 6$ ). Let $n \in \mathbb{N}, n \geq 6$. Then there exist no strictly positive integer solutions for the equation $a^{n}+b^{n}=c^{n}$.

Proof. Assume the $a b c$-Conjecture, and let $a^{n}+b^{n}=c^{n}$ and, without loss of generality, $a, b$ (and thus $c$ ) be coprime. By the $a b c$-Conjecture, we have that:

$$
c^{n}<\operatorname{rad}\left(a^{n} b^{n} c^{n}\right)^{2}=\operatorname{rad}(a b c)^{2} \leq(a b c)^{2}<\left(c^{3}\right)^{2}=c^{6}
$$

Thus, $c^{n}<c^{6}$, which implies that for all integer solutions of the equation, $n<6$. This proves Fermat's Last Theorem for $n \geq 6$.

### 2.3.2 Conjectures based on the $a b c$-Conjecture

The proof of several other conjectures would follow from the $a b c$-Conjecture:

- The weak form of Hall's Conjecture, proposed by Stark and Trotter around 1980.[42]
- Marius Overhold proved in 1993 that the abc-Conjecture implies that Brocard's Equation has only finitely many solutions.
- The Fermat-Catalan Conjecture, which asks for integer solutions of the equation $a^{m}+b^{n}=c^{k}$.[43]
- The Erdös-Woods Conjecture.


## Chapter 3

## The search for a counterexample

### 3.1 Our method

The intent of this paper is to find a counterexample to the $a b c$-Conjecture, i.e. to construct an infinite sequence of triples with quality higher than $1+\epsilon$ for $\epsilon>0$. To reach this goal, we use the databank of the ABC@Home project which contains all $a b c$-triples where $c$ has less than 19 digits. We then compare the entries of this databank with the databank of the On-Line Encyclopedia of Integer Sequences (OEIS), a site containing hundreds of thousands of known integer sequences. Our goal is to find a sequence which contains many $a b c$ triplets, and to check if other elements in the sequence yield more $a b c$-triplets. If there were such a sequence which would give us an infinity of triplets with quality more than $1+\epsilon$ for a fixed $\epsilon>0$, the abc-Conjecture would be proven false. In the following sections, we will take a more in-depth look at the databanks, as well as the algorithm calculating the $a b c$-triplets for the ABC@Home project.

### 3.1.1 The ABC@Home project and its Databank

[44] In 2007, following a 2005 number theory seminar, Bart de Smith of the University of Leiden started a project called "Reken mee met ABC" ("Take ABC into account") with the intent of promoting sciences.[45] It contained the ABC@Home network, which was a network computing project whose goal it was to find all $a b c$-triplets with $c$ having less than 19 digits. This goal was achieved in 2011, with the project discovering 14482065 valid triplets. In the following years, the project continued, aiming to find all triplets with $c$ between $10^{1} 8$ and $2^{6} 3$. After reaching this goal in 2015, the search for greater triplets has been dismissed by the ABC@Home project. The project produced a total of 23827716 triplets, which are available for download on the project's homepage.

### 3.1.2 The OEIS-Databank

[46] The idea for a databank of all known integer sequences started in 1964 when Neil Sloane, the creator of the OEIS-project, encountered a certain sequence
in works which he couldn't find in any book. While looking through books in the hopes of finding his sequence, he started writing all of the sequences he found on cards and arranged them in lexicographic order. In 1973, the sequences he collected were put in a book. The book was in high demand, as many mathematicians found it very useful and in 1995, a revised edition was published with the help of Canadian mathematician Simon Plouffe, containing more that double the sequences of the initial book. In 1996, Sloane, who worked for AT\&T at the time, launched the On-Line Encyclopedia of Integer Sequences. During the following years, the database grew heavily, which lead to Sloane not being able to process all the new submissions he got every day. Thus, in 2009, he decided to set up the non-profit OEIS Foundation Inc., which allowed him to maintain the database with the help of a group of editors. As of today, the database contains over 300.000 sequences, each sequence containing also information on its apparition, author and code to calculate the sequence in Maple.[47]

### 3.1.3 The ABC@Home Algorithm

The following is a short summary of the algorithm used in the ABC@Home network, as described in the 7 th chapter of the PhD thesis of Willem Jan Palenstijn of Leiden University.[48]
He starts by giving an upper bound on the number of $a b c$-triples where $c$ is less than a certain integer $N$. He concludes that for every $\epsilon>0$, that number of triples is at most $O\left(N^{2 / 3+\epsilon}\right)$, i.e. the number of triples grows at most with exponent $\frac{2}{3}$. He proves this using a theorem from one of the earlier chapters which gives a relation between the number $N$ and the cardinal of $\left\{x<N \mid \operatorname{rad}(x)<N^{\alpha}\right\}$ for some $0<\alpha<1$. Furthermore, he cites a theorem by Sander Dahmen of the Vrije Universiteit Amsterdam setting $\exp \left((\log (N))^{1 / 2-\epsilon}\right)$ as the lower bound of the aforementioned $a b c$-triples.
He then gives an algorithm to enumerate all square-free integers $x$ with $a \leq x \leq b$ with their corresponding prime factors:

- Create two lists, one for the integers $x$, and one for their corresponding prime factorization list.
- Run over the primes $p^{2}<b$ and check if they divide $x$. If $p$ divides $x$, and $p^{2}$ doesn't, add $p$ to the list of factors of $x$.
This is only the basic version of this algorithm which does not contain all the necessary steps. For a more detailed version, one can consult section 7.7 of the PhD thesis, which contains this algorithm under the name 'Algorithm 7.7'.[48]

Subsequently, Palenstijn describes the ABC@Home algorithm which, given integers $N$, as well as some bounds $m_{x}, M_{x}, m_{y}, M_{y}$ on $a$ and $b$, and square-free integers $t$ and $g$ with $t \mid g$, computes all $a b c$-triples with $c<N, \operatorname{rad}(a)$ and $\operatorname{rad}(b)$ bounded by $m_{x}, M_{x}$ and $m_{y}, M_{y}$ respectively, and $\operatorname{gcd}(x, t)=g$ :

- Compute a list $P$ of all prime numbers $p<N\left(\frac{1}{3}\right)$.
- Compute lists $L_{x}$ and $L_{y}$ of square-free integers between the bounds of $a$ and $b$.
- Generate sorted lists $X$ and $Y$ of integers less than $N$ with radical in $L_{x}$ and $L_{y}$ respectively.
- Partition $X$ and $Y$ into subsets.
- Loop over pairs $\left(X_{i}, Y_{j}\right)$ of these subsets:
- Generate a table $T\left(X_{i}, Y_{j}\right)$ which for all $(x, y) \in\left(X_{i}, Y_{j}\right)$ sets $r=1$, $u=x+y$.
- Loop over $P$ :
- Loop over $y \in Y_{j}$ and find all $x \in X_{i}$ such that $p \mid x+y$.
- Divide all factors $p$ from $u$.
- Multiply $r$ with $p$.
- Now, loop over all $x+y$ in $T$.
- Skip the triple if $x$ and $y$ are not coprime or $r>c / r a d(x y)$.
- By dividing by primes in $P$, filter out all the triples which are no $a b c$-triples.
This algorithm returns a list of all $a b c$-triples with $c<N$. The integer $t$ is used to speed up the process by getting rid of all $x$ and $y$ which are coprime with $t$. Setting it to $t=2 * 3 * 5 * 7=210$ seems, by experimentation, to be a good value.


Figure 3.1: $\log _{10}$ of the number of triples below $10^{N}$

### 3.2 Our computations

The goal of this thesis is to find a sequence of $a b c$-triples with constant or increasing quality. If we were to find such a sequence, we would have a valid counter-
example to the conjecture. Our search consists of comparing the ABC@HomeDatabank with the OEIS-Databank and finding sequences which contain a certain amount of $c$-entries of the databank. Now, one can match each element of the sequence to its corresponding triples and try to find a subsequence which has at least constant quality.
First, one has to get both databanks in an adequate text format. The code was written in Python 2 with the use of the Canopy-IDE.[49]

### 3.2.1 Getting the files ready

We first wrote the following program which extracts only the $c$-entries of the ABC@Home-Database. It gets rid of duplicates and writes a file with all the $c$ entries that are left. This new file, 'c_no_duplicates.txt', will be used to match entries with the OEIS-Database file.

```
def create_only_c():
    file1 = open("abc_databank.txt", "r+");
    file2 = open("all_c.txt", "w");
    file2 = open("c_no_duplicates.txt", "w");
    #extracts only the first entry (c)
    for line in file1:
        list1 = line.split(" ");
        file2.write(list1[0] + "\n");
    #gets rid of duplicate entries
    seen_c = set()
    for line in file2:
        if line not in seen_c:
            file2.write(line);
            seen_c.add(line);
```

| 1 | 9 |
| :--- | :--- |
| 2 | 32 |
| 3 | 49 |
| 4 | 64 |
| 5 | 81 |
| 6 | 81 |
| 7 | 125 |
| 8 | 128 |
| 9 | 225 |
| 10 | 243 |
| 11 | 245 |
| 12 | 250 |

Figure 3.2: First lines of c__no__duplicates.txt
We now have to do the same thing with the OEIS-Databank. We wrote the program create_OEIS(), which first converts the OEIS-Database into a more easily readable format and then extracts all the duplicates in each line, and sorts each line by integer values.
def create_OEIS():

```
file1 = open("OEIS-Databank.txt", "r+");
file2 = open("Converted_OEIS.txt", "w");
for line in file1:
    #replaces the comma from the original file with spaces
    sequence = line.replace(",", " ");
    list1 = sequence.split(" ");
        #gets rid of the beginning and the end of the line which are
            not numbers
    del list1[0];
    del list1[0];
    list1 = list1[:-1];
        #puts the line entries in a dictionary format and sorts them
                with duplicate removal
    list2 = dict.fromkeys(list1);
    list2 = map(int,list2);
    list2 = sorted(list2);
        #if there are less than 3 entries, we get rid of the line
    if len(list2) < 3:
        pass;
    else:
        file2.write(" ".join(map(str, list2)));
        file2.write(" "+"\n");
```



```
    1
    1
```



```
5
6
7 1 2 4 4
8
9
```



```
11
```



Figure 3.3: First lines of Converted_OEIS.txt

### 3.2.2 The search algorithm

Now, we will explain the search algorithm. For optimisation purposes, we wrote the search algorithm so that it can use all cores of the computing system, as Canopy usually only runs on one core. Therefore, as the computing system we used has 16 cores, we split the Converted_OEIS.txt file into 16 different files:

```
#divides the Converted_OEIS.txt file into files the number of cores
def dividefile():
    cores = mult.cpu_count()
    for i in range(cores):
        file1 = open("Converted_OEIS"+str(i)+".txt", "w");
        j = 0;
        with open("Converted_OEIS.txt", "r+") as file2:
            #as the Converted_OEIS.txt has 157448 lines, we split it in
                    chunks of length 157448/8 = 19681.
```

```
for line in islice(file2, i*19681, (i+1)*19681):
    file1.write(line);
```

Now that all files are in an adequate format to work with, we wrote the search algorithm which compares the two files with each other. For every ( $i<$ number of cores), the algorithm takes a sequence of the Converted_OEISi.txt file and, for each element in the sequence, checks if it is contained in the c__no_duplicates.txt file. If it is, it increases the corresponding count of the sequence by one. If the total count of the sequence is over 15 , the sequence is then written to the file Validsequencesi.txt with its corresponding place number in the Converted_OEIS.txt file.

```
#The search algorithm for the file Converted\_OEISi
def search(i):
    file1 = open("Converted_OEIS" + str(i) + ".txt", "r");
    sequencenumber = int(i)*19681;
    resultlist = [];
    for sequence in file1:
        sequencenumber += 1;
        count = 0;
        list1 = sequence.split(" ");
        validentries = [];
        for value in list1:
            if value == "\n":
                pass;
        elif (int(value) > 10**18): #as the ABC@Home Databank goes up
                    to 10^18, checking for bigger numbers is pointless
                pass;
            else:
                file2 = open("c\_no\_duplicates.txt", "r");
                for line in file2:
                    if int(line) == int(value):
                        validentries.append(value);
                        count+=1;
                if int(line) > int(value):
                                    break; #as the file is ordered by size, if we are
                                    already past the sequence entry, we will not
                                    find a corresponding entry in the
                                    c\_no\_duplicates.txt file.
        if count > 15:
        resultlist.append(validentries);
        with open("Validsequences"+ str(i) +".txt", "a") as f:
            f.write("sequence number: " + str(sequencenumber) + " /
                        ");
                for item in validentries:
                f.write(str(item)+ " ");
            f.write("\n");
```

```
#The following distributes the search algorithm on all cores.
if __name__ == '__main__':
    cores = mult.cpu_count();
    __spec__ = "ModuleSpec(name='builtins', loader=<class
            '_frozen_importlib.BuiltinImporter'>)"
    for k in range(cores):
        p = mult.Process(target = search, args = (str (k)))
        p.start()
```

```
sequence number: 72 / 32 64 128 256 512 2048 4096 8192 16384 32768 65536 13
sequence number: 237 / 9 81 243 729 2187 6561 19683 59049 177147 531441 159
sequence number: 283 / 9 49 64 81 225 256 289 625 676 729 961 1369 1681 230
sequence number: 295 / 64 256 4096 16384 65536 262144 1048576 4194304 16777
sequence number: 344 / 125 625 3125 15625 78125 390625 1953125 9765625 4882
sequence number: 413 / 49 343 2401 16807 117649 823543 5764801 40353607 282
sequence number: 526 / 9 32 128 512 2048 3584 6144 8192 24576 32768 131072
sequence number: 570 / 64 125 343 512 729 1331 2197 4096 4913 5832 6859 800
sequence number: 575 / 81 256 625 2401 4096 6561 10000 14641 28561 50625 65
sequence number: 576 / 32 243 3125 16807 32768 59049 100000 161051 371293 5
sequence number: 784 / 9 81 243 729 2187 6561 8748 19683 59049 78732 118098
sequence number: 1006 / 64 729 4096 15625 46656 117649 262144 531441 100000
```

Figure 3.4: First lines of Converted_OEIS0.txt
As we now have 16 different files, with each file containing a certain amount of sequences that have more than 15 entries corresponding with known abctriples, we only have to write all of these files into one file. Since we realised that the number of sequences is still relatively large, we only kept the sequences with at least 30 entries.

```
def Collect_sequences():
    file1 = open("Validsequences.txt", "ゅ");
    for i in range(10):
        file2 = open("Validsequences" + str(i) + ".txt", "r");
        for line in file2:
            list1 = line.split(" ");
            if len(list1) >= 34: #the first 4 entries are no members of
            the sequence
            for j in range(len(list1)):
                    file1.write(list1[j] + " ");
            file1.write("\n");
```

This finally leaves us with 37 different sequences that have at least 30 entries corresponding to the $c$-part of known $a b c$-triples. Of course, only having the $c$-entries does not yet provide a useful way for extracting a subsequence with certain characteristics on its $a b c$-quality. That's why we first need to assign each entry in the Validsequences.txt all of its known $a b c$-triples with their relative prime decomposition.
As we did not find a good method for calculating prime decomposition in Python, we used the PARI/GP program instead. This program allowed us to quickly compute the prime decomposition of all our $a b c$-triples.
PARI/GP returns the factorization of a number by matrix representation. There are three forms of returns from PARI/GP for the factorization of a number $N$ :

- If $N=1$, then PARI/GP returns "matrix $(0,2)$ ".
- If $N=p^{n}$ for $p$ prime, then PARI/GP returns "Mat([p, n])".
- If $N=\prod_{i=1}^{m} p_{i}{ }^{n_{i}}$ for $p_{i}$ prime, then PARI/GP returns " $\left[p_{1}, n_{1} ; \ldots ; p_{m}\right.$, $\left.n_{m}\right]^{\prime \prime}$.

With the goal of making a file that PARI/GP is able to read, the following algorithm first assigns the entries in Validsequences.txt to their corresponding $a b c$-triples and then writes the prime decomposition command for PARI/GP into the PARI-file.txt file. This file will then be read by PARI/GP, which will then write the prime decomposition to the file abcFactors.txt.

```
def assigntriple():
    file1 = open("Validsequences.txt", "r");
    for line in file1:
        list1 = line.split(" ");
        if len(list1) == 2: #gets rid of empty lines
            pass;
        else:
            del list1[-1];
            del list1[3];
            del list1[2];
            del list1[1];
            del list1[0];
                #gets rid of the sequence name and number
            for value in list1:
                file2 = open("c_no_duplicates.txt", "r");
                for double in file2:
                list2 = double.split(" ");
                if int(value) == int(list2[0]):
                    a = int(list2[1][:-1]);
                    c = int(list2[0]);
                    b = c-a;
                    file3 = open("PARI-file.txt", "a");
                    file3.write("factor("+str(c)+")"+"\n"+r"\w
                                    C:\Users\username\abcFactors.txt"+"\n"+
                    "factor("+str(a)+")"+"\n"+r"\w
                            C:\Users\username\abcFactors.txt"+"\n"+
                    "factor("+str(b)+")"+"\n"+r"\w
                            C:\Users\username\abcFactors.txt"+"\n")
                elif int(value) < int(list2[0]):
                    break;
```

This now gives us the file abcFactors.txt in which the factorisations of all triples corresponding to the entries in Validsequences.txt are stored. This will now be modified by algorithm 4.1 to give us a final file abcFinal.txt which will include the sequences of triples with their corresponding factorization as well as their quality and their merit.

```
factor(9)
\w C:\Users\username\abcFactors.txt
factor(1)
\w C:\Users\username\abcFactors.txt
factor(8)
\w C:\Users\username\abcFactors.txt
factor(32)
\w C:\Users\username\abcFactors.txt
factor(5)
\w C:\Users\username\abcFactors.txt
factor(27)
\w C:\Users\username\abcFactors.txt
```

Figure 3.5: First lines of PARI-file.txt


Figure 3.6: How to input PARI-file.txt into PARI/GP

### 3.3 Our results

Our goal was to extract a subsequence with increasing quality and checking if the quality would increase further. To do so, we wrote the following algorithm qualitysequences(). It builds a sequence with increasing quality out of the abcFinal.txt-sequences. Out of all the triples with highest $c$, it takes the one with the biggest quality and then inductively repeats this step for all the triples with less high $c$-entry. This gives a subsequence of the original sequence which has increasing quality. To make the algorithm easier, it operates on the file abcFinalreversed.txt, which is the abcFinal.txt file reversed.

```
def qualitysequences():
    file1 = open("abcFactorsreversed.txt", "r");
    file2 = open("Qualityincrease.txt", "a");
    C = 10**19;
    listquality = [100]; #initial quality, should be higher than all the
        other qualities in the sequence
    listC = [0];
    i = 0;
    linelist = [("sequence " + str(28-i) + ": \n")]; #it goes from
        sequence 28 to sequence 1
    for line in file1:
        list1 = line.split("=");
        c = int(list1[1][1:][:-1]);
        quality = float(list1[8][1:][:-7]);
        if c > C: #a new sequence starts
```

```
Mat([3, 2])
matrix(0,2)
Mat([2, 3])
Mat ([2, 5])
Mat([5, 1])
Mat ([3, 3])
Mat([2, 6])
matrix (0,2)
[3, 2; 7, 1]
Mat([3, 4])
matrix (0,2)
[2, 4; 5, 1]
```

Figure 3.7: An extract of the abcFactors.txt file

```
c=9 = Mat ([3, 2]) b = 1 = matrix (0,2) a = 8 = Mat ([2, 3]) radical = 6 quality = 1.226294385530917 merit = 0.05351103517746218
= 32 = Mat ([2, 5]) b = 5 = Mat ([5, 1]) a = 27 = Mat ([3, 3]) radical = 30 quality = 1.018975235452531 merit = 0.0014991077590693783
= 64=Mat ([2,6]) b = = = matrix (0,2) a = 63 = [3, 2; 7, 1] radical = 42 quality = 1.1126941404922133 merit = 0.06258515100661385
* ( )
c=128= Mat([2,7]) b = = = Mat([3, 1]) a = 125= Mat([5, 3]) radical = 30 quality = 1.4265653296335432 merit = 0.7575819399084418
= 243 = Mat([3, 5]) b = 1 = matrix(0,2) a = 242 = [2, 1; 11, 2] radical = 66 quality = 1.311101219926227 merit = 0.580914433583302
= 256 = Mat([2, 8]) b = 13 = Mat([13, 1]) a = 243 = Mat([3, 5]) radical = 78 quality = 1.2727904629543532 merit = 0.4771348987168572
c=256=\operatorname{Mat}([2,8]) b = 81 = Mat ([3, 4]) a = 175 = [5, 2; 7, 1] radical = 210 quality = 1.0370424407259895 merit = 0.012300877988305719
c=512= Mat([2, 9]) b = 5 = Mat([5, 1]) a = 507 = [3, 1; 13, 2] radical = 390 quality = 1.0456203807611666 merit = 0.02217776739979556
= 512 = Mat([2, 9]) b = 169= Mat([13, 2]) a = 343= Mat([7, 3]) radical = 182 quality = 1.198754152359422 merit = 0.33908131024981064
```

Figure 3.8: An extract of the abcFinal.txt file

```
    i += 1;
    C = c;
    Quality = quality;
    plt.figure(i);
    plt.plot(listC[1:], listquality[1:], "r--");
    plt.title(listC, fontsize = 6);
    plt.figure(i).show();
    plt.savefig("qualityincrease"+str(i)+".png");
    listquality = [100, quality]; #reset
    listC = [0, c]; #reset
    for value in linelist:
        file2.write(str(value));
    file2.write("\n");
    linelist = [("sequence " + str(28-i) + ": \n"), 0];
if c < C:
    if quality < listquality[len(listquality)-1]: #if the quality
        is less than that of the previous c
        listquality.append(quality);
        listC.append(c);
        Quality = quality;
        C = c;
        linelist.append(line);
    else:
        C = c;
        Quality = quality;
if c == C :
    if (quality > Quality) and (quality <=
                listquality[len(listquality)-2]):
```

```
    Quality = quality;
    listquality[len(listquality)-1] = quality;
    linelist[len(linelist)-1] = line;
else:
    pass;
```

This gives us the file Qualityincrease.txt which contains the subsequences of abcFinal.txt with maximal increasing quality. Disappointingly, there is no apparent pattern in the $a$ and $b$ entries of the subsequences, and none of those are sequences contained in the OEIS-Databank. This essentially concludes the project, since we were not able to extract a formula for a sequence of increasing quality.


Figure 3.9: Increasing quality in sequence nr. 26
We were nevertheless not fully satisfied with having no result, which is why we tried to at least construct a subsequence of increasing merit. These subsequences can easily be computed by changing "float(list1[8][1:][:-7])" to "float(list1[9][1:][:1])" in the previous algorithm, which instead of the quality now extracts the merit. This gave us a whole new file of subsequences with increasing merit, of which disappointingly none could be matched to the OEIS-Databank, just as in the previous case with the quality. This finally concludes the project.

We believe that our method has fully been investigated with no result. Therefore, further exploration of the method seems futile. This leaves the $a b c$ Conjecture still standing and future proofs will most likely have to include newly developed areas of mathematics to get results.

## Chapter 4

## Appendix

### 4.1 The conversion algorithm

The following algorithm converts the abcFactors.txt file, created by PARI/GP, into the nicely structured abcFinal.txt file.

```
def convert():
    file1 = open("abcFactors.txt", "r+");
    file3 = open("abcFinal.txt", "ए");
    i = 0; #counts to three to always get the corresponding triple
        values together
    primes = []; #saves all the primes of the current triple
    C = 0;
    for word in file1:
        if word[0] == "M": #is of the kind Mat([a, b])
            control = False;
            list1 = word.split(",");
            list1[0] = list1[0][5:];
            list1[1] = list1[1][:-3];
            print(list1);
        elif word[0] == "m": #is of the kind matrix(a, b)
            control = False;
            list1 = word.split(",");
            list1[0] = list1[0][7:];
            list1[1] = list1[1][:-2];
            print(list1);
        else: #is of the kind [a, b; ... ;c, d]
            control = True;
            list1 = []
            list2 = word.split(";");
            l = len(list2);
            list2[0] = list2[0][1:];
            list2[l-1] = list2[l-1][:-2];
            for value in list2:
                    list3 = value.split(",");
                    list1.append(list3);
```

```
    print(list1);
length = len(list1);
a = 1;
if control == True: #if the line is of the kind [a, b; ... ;c, d]
    if i == 0: # c of the abc-triple
        for j in range(0, length, +1):
                primes.append(int(list1[j][0]));
                a = a*(int(list1[j][0])**int(list1[j][1]));
            C = a; #saves the value of c for further computation of
                the quality
            file3.write("c = " + str(a) + " = ")
            file3.write(word[:-1] + " ");
            i += 1;
    elif i == 1: # b of the abc-triple
            for j in range(0, length, +1):
                primes.append(int(list1[j][0]));
                a = a*(int(list1[j][0])**int(list1[j][1]));
            file3.write("b = " + str(a) + " = ")
            file3.write(word[:-1] + " ");
            i += 1;
    elif i == 2: # a of the abc-triple
        for j in range(0, length, +1):
            primes.append(int(list1[j][0]));
            a = a*(int(list1[j][0])**int(list1[j][1]));
        radical = 1;
        for k in range(len(primes)):
            radical = radical*primes[k];
        quality = (math.log(C))/(math.log(radical));
        merit =
            ((quality-1)**2)*(math.log(radical))*(math.log(math.log(radical)));
        file3.write("a = " + str(a) + " = ")
        file3.write(word[:-1] + " ");
        file3.write(" radical = " +str(radical) + " quality = "
            +str(quality) + " merit = " +str(merit));
        file3.write("\n");
        i = 0;
        primes = [];
else:
    if i == 0: # c of the abc-triple
        if int(list1[0]) == 0:
            a = 1;
        else:
            a = (int(list1[0])**int(list1[1]));
            primes.append(int(list1[0]));
        C = a;
        file3.write("c = " + str(a) + " = ")
        file3.write(word[:-1] + " ");
        i += 1;
    elif i == 1: # b of the abc-triple
```

```
    if int(list1[0]) == 0:
        a = 1;
    else:
        a = (int(list1[0])**int(list1[1]));
        primes.append(int(list1[0]));
    file3.write("b = " + str(a) + " = ")
    file3.write(word[:-1] + " ");
    i += 1;
elif i == 2: # a of the abc-triple
    if int(list1[0]) == 0:
        a = 1;
    else:
        a = (int(list1[0])**int(list1[1]));
        primes.append(int(list1[0]));
    radical = 1;
    for k in range(len(primes)):
        radical = radical*primes[k];
    quality = (math.log(C))/(math.log(radical));
    merit =
            ((quality-1)**2)*(math.log(radical))*(math.log(math.log(radical)));
    file3.write("a = " + str(a) + " = ")
    file3.write(word[:-1] + " ");
    file3.write(" radical = " +str(radical) + " quality = "
        +str(quality) + " merit = " +str(merit));
    file3.write("\n");
    i = 0;
    primes = [];
```


### 4.2 Computed results

The following are the OEIS-Number and -Description of 12 extracted sequences with significant enough size, their subsequence of increasing quality, as well as the figures of their increasing quality and merit. The PARI/GP factorization has been explained in section 3.2.

### 4.2.1 Sequence 1

A065771 Prime powers n such that both $\tau\left(n^{2}\right)$ and $\sigma\left(n^{2}\right)$ are composite numbers.

[^0]

Figure 4.1: Increasing quality in sequence nr. 3


Figure 4.2: Increasing merit in sequence nr. 3

### 4.2.2 Sequence 2

A076469 Perfect powers $m^{k}$ where $m$ is an integer and $k>4$. or A246550 Prime powers $p^{e}$ where $p$ is a prime and $e>=4$.
$\mathrm{c}=531441=\operatorname{Mat}([3,12]) \mathrm{b}=49=\operatorname{Mat}([7,2]) \mathrm{a}=531392=[2,6 ; 19,2 ; 23,1]$ radical $=18354$ quality $=1.342827542997735$ merit $=2.635643220813526$
$=524288=\operatorname{Mat}([2,19]) \quad \mathrm{b}=4913=\operatorname{Mat}([17,3]) \mathrm{a}=519375=[3,1 ; 5,4 ; 277,1]$ radical $=141270$ quality $=1.1105853299516497$ merit $=0.35863535917689343$
$\mathrm{c}=390625=\operatorname{Mat}([5,8]) \mathrm{b}=117649=\operatorname{Mat}([7,6]) \mathrm{a}=272976=[2,4 ; 3,1 ; 11,2 ; 47,1]$ radical $=108570$ quality $=1.1104214132202936$ merit $=0.346460002140961$
$c=371293=\operatorname{Mat}([13,5]) \quad \mathrm{b}=1539=[3,4 ; 19,1] \mathrm{a}=369754=[2,1 ; 7,5 ; 11,1]$ radical $=114114$ quality $=1.1013137224794165$ merit $=0.2934291819756031$

$\mathrm{c}=117649=\operatorname{Mat}([7,6]) \mathrm{b}=12500=[2,5,5] \mathrm{a}=105149=[11,3 ; 79,1]$ radical $=60830$ quality $=1.059879466729859$ merit $=0.0947685263276362$
$\mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=6487=[13,1 ; 499,1] \mathrm{a}=59049=\operatorname{Mat}([3,10])$ radical $=38922$ quality $=1.049297420915307$ merit $=0.060566258521041194$
$\mathrm{c}=59049=\operatorname{Mat}([3,10]) \mathrm{b}=15625=\operatorname{Mat}([5,6]) \mathrm{a}=43424=[2,5 ; 23,1 ; 59,1]$ radical $=40710$ quality $=1.0350372922734312$ merit $=0.030779769126492332$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$=2187=\operatorname{Mat}([3,7]) \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193$ merit $=0.0019000255575361094$
$=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.3: Increasing quality in sequence nr. 9


Figure 4.4: Increasing merit in sequence nr. 9

### 4.2.3 Sequence 3

A076470 Perfect powers $m^{k}$ where $m$ is an integer and $k>5$.
$\mathrm{c}=1771561=\operatorname{Mat}([11,6]) \mathrm{b}=91=[7,1 ; 13,1] \mathrm{a}=1771470=[2,1 ; 3,11 ; 5,1]$ radical $=30030$ quality $=1.3954838404823704$ merit $=3.7622641764419975$
$\mathrm{c}=1594323=\operatorname{Mat}([3,13]) \mathrm{b}=7424=[2,8 ; 29,1] \mathrm{a}=1586899=[17,4 ; 19,1]$ radical $=56202$ quality $=1.305873782653318$ merit $=2.4476817174888095$
$\mathrm{c}=1048576=\operatorname{Mat}([2,20]) \mathrm{b}=50575=[5,2 ; 7,1 ; 17,2] \mathrm{a}=998001=[3,6 ; 37,2]$ radical $=132090$ quality $=1.1756986573787018$ merit $=0.8981073965362553$
$c=823543=\operatorname{Mat}([7,7]) \quad \mathrm{b}=3072=[2,10 ; 3,1] \mathrm{a}=820471=[17,3 ; 167,1]$ radical $=119238$ quality $=1.1653276277840765$ merit $=0.7855216932161503$
$\mathrm{c}=531441=\operatorname{Mat}([3,12]) \mathrm{b}=191=\operatorname{Mat}([191,1]) \mathrm{a}=531250=[2,1 ; 5,6 ; 17,1]$ radical $=97410$ quality $=1.1477069698739997$ merit $=0.6117837913212401$

$\mathrm{c}=177147=\operatorname{Mat}([3,11]) \mathrm{b}=46075=[5,2,191 ; 971] \mathrm{a}=131072=\operatorname{Mat}([2,17])$ radical $=55290$ quality $=1.1066255312066344 \mathrm{merit}=0.2968047377434422$
$\mathrm{c}=131072=\operatorname{Mat}([2,17]) \mathrm{b}=95=[5,1 ; 19,1] \mathrm{a}=130977=[3,5 ; 7,2 ; 11,1]$ radical $=43890$ quality $=1.102349618350441$ merit $=0.2653014130816672$
$c=131072=\operatorname{Mat}([2,17]) \quad b=95=[5,1 ; 19,1] a=130977=[3,5 ; 7,2 ; 11,1] \quad$ radical $=43890$ quality $=1.102349618350441$ merit $=0.2653014130816672$
$c=117649=\operatorname{Mat}([7,6]) \quad b=324=[2,2 ; 3,4] a=117325=[5,2 ; 13,1 ; 19,2]$ radical $=51870$ quality $=1.07543548461615$ merit $=0.1473283668442908$
$\mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=7=\operatorname{Mat}([7,1]) \mathrm{a}=65529=[3,4 ; 809,1]$ radical $=33978$ quality $=1.0629595375106713$ merit $=0.0969835532642618$
$\mathrm{c}=59049=\operatorname{Mat}([3,10]) \mathrm{b}=485=[5,1 ; 97,1] \mathrm{a}=58564=[2,2 ; 11,4]$ radical $=32010$ quality $=1.0590255296303734$ merit $=0.08454748856131326$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$\mathrm{c}=2187=\operatorname{Mat}([3,7]) \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193 \operatorname{merit}=0.001900025557536109$
$c=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.5: Increasing quality in sequence nr. 10


Figure 4.6: Increasing merit in sequence nr. 10

### 4.2.4 Sequence 4

A095887 Triangle read by rows: $T(n, k)=(n-k+1)^{n}, n>=1,1<=k<=n$.
$\mathrm{c}=387420489=\operatorname{Mat}([3,18]) \mathrm{b}=1515625=[5,6 ; 97,1] \mathrm{a}=385904864=[2,5 ; 31,1 ; 73,3]$ radical $=6585330$ quality $=1.259527009976646$ merit $=2.9119813790027185$ $=134217728=\operatorname{Mat}([2,27]) b=926757=[3,2 ; 13,1 ; 89,2] \mathrm{a}=133290971=[11,1 ; 59,4]$ radical $=4505358$ quality 1.2215420118507678 merit $=2.05224852489132$ $c=40353607=\operatorname{Mat}([7,9]) \mathrm{b}=42823=[11,1 ; 17,1 ; 229,1] \mathrm{a}=40310784=[2,11 ; 3,9]$ radical $=1798566$ quality $=1.215982714893403 \mathrm{merit}=1.7921086084880353$
$\mathrm{c}=16777216=\operatorname{Mat}([2,24]) \mathrm{b}=2197=\operatorname{Mat}([13,3]) \mathrm{a}=16775019=[3,6 ; 23011,1]$ radical $=1794858$ quality $=1.155210286177332$ merit $=0.9252983022149146$
$c=10077696=[2,9 ; 3,9] b=14125=[5,3 ; 113,1] \mathrm{a}=10063571=[7,2 ; 59,3]$ radical $=1400070$ quality $=1.1394713012993065$ merit $=0.7294762868997512$
$\mathrm{c}=5764801=\operatorname{Mat}([7,8]) \mathrm{b}=648=[2,3 ; 3,4] \mathrm{a}=5764153=[73,1 ; 281,2]$ radical $=861546$ quality $=1.1390846029894408$ merit $=0.6913177200951904$
$=823543=\operatorname{Mat}([7,7) \mathrm{b}=371293=\operatorname{Mat}([1,5]) \mathrm{a}=452250=[2,1 ; 3,3 ; 5,3 ; 67,1]$ radical $=182910$ quality $=1.1241$.
$\mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=709=\operatorname{Mat}([709,1]) \mathrm{a}=64827=[3,3 ; 7,4]$ radical $=29778$ quality $=1.0765740735526552$ merit $=0.140879546089314$
$c=46656=[2,6 ; 3,6] \mathrm{b}=4913=\operatorname{Mat}([17,3]) \mathrm{a}=41743=[13,3 ; 19,1]$ radical $=25194$ quality $=1.0608026156104984$ merit $=0.08676941245716602$
$\mathrm{c}=2187=\operatorname{Mat}([3,7]) \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193$ merit $=0.0019000255575361094$
$c=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762 \operatorname{merit}=0.0008682588486994485$


Figure 4.7: Increasing quality in sequence nr. 13


Figure 4.8: Increasing merit in sequence nr. 13

### 4.2.5 Sequence 5

## A098293 Powers of 2 alternating with powers of 3 .

$\mathrm{c}=387420489=\operatorname{Mat}([3,18]) \mathrm{b}=1515625=[5,6 ; 97,1] \mathrm{a}=385904864=[2,5 ; 31,1 ; 73,3]$ radical $=6585330$ quality $=1.259527009976646$ merit $=2.9119813790027185$ $\mathrm{c}=129140163=\operatorname{Mat}([3,17]) \mathrm{b}=8000000=[2,9 ; 5,6] \mathrm{a}=121140163=[29,3 ; 4967,1]$ radical $=4321290$ quality $=1.2223529048295212$ merit $=2.0596112874736856$
$\mathrm{c}=43046721=\operatorname{Mat}([3,16]) \mathrm{b}=1=\operatorname{matrix}(0,2) \mathrm{a}=43046720=[2,6 ; 5,1 ; 17,1 ; 41,1 ; 193,1]$ radical $=4035630$ quality $=1.1556225447925967$ merit $=1.0027234075306528$ $\mathrm{c}=14348907=\operatorname{Mat}([3,15]) \mathrm{b}=619115=[5,1 ; 7,3 ; 19,2] \mathrm{a}=13729792=[2,15 ; 419,1]$ radical $=1671810$ quality $=1.150024724364855 \mathrm{merit}=0.8586448729415037$
$\mathrm{c}=4782969=\operatorname{Mat}([3,14]) \mathrm{b}=2725=[5,2 ; 109,1] \mathrm{a}=4780244=[2,2 ; 7,2 ; 29,3]$ radical $=663810$ quality $=1.1473114607778128$ merit $=0.7551200767869274$
$\mathrm{c}=1594323=\operatorname{Mat}([3,13]) \mathrm{b}=90508=[2,2 ; 11,3 ; 17,1] \mathrm{a}=1503815=[5,1 ; 67,3]$ radical $=375870$ quality $=1.112562225625076$ merit $=0.4151317221515415$
$c=531441=\operatorname{Mat}([3,12]) \quad b=240625=[5,5 ; 7,1 ; 11,1] a=290816=[2,12 ; 71,1]$ radical $=164010$ quality $=1.0979093814132679$ merit $=0.28610782629987797$
$\mathrm{c}=524288=\operatorname{Mat}(2,1]) \mathrm{b}=131072=\operatorname{Mat}([2,17]) \mathrm{b}=32657=[17,2 ; 113,1] \mathrm{a}=98415=[3,9 ; 5,1]$ radical $=57630$ quality $=1.074960648269333$ merit $=0.1474851032648427$
$c=65536=\operatorname{Mat}([2,16]) b=7=\operatorname{Mat}([7,1]) \mathrm{a}=65529=[3,4 ; 809,1] \quad$ radical $=33978$ quality $=1.0629595375106713 \operatorname{merit}=0.0969835532642618$
$\mathrm{c}=65049=\operatorname{Mat}([3,10]) \mathrm{b}=485=[5,1 ; 97,1] \mathrm{a}=58564=[2,2 ; 11,4]$ radical $=32010$ quality $=1.0590255296303734$ merit $=0.08454748856131326$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$c=2187=\operatorname{Mat}([3,7]) \quad \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193$ merit $=0.001900025557536109$
$\mathrm{c}=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762 \operatorname{merit}=0.0008682588486994485$


Figure 4.9: Increasing quality in sequence nr. 14


Figure 4.10: Increasing merit in sequence nr. 14

### 4.2.6 Sequence 6

A075363 Triangle read by rows, in which the $n$-th row gives $n$ smallest powers of $n$. or
A099615 Triangle read by rows, $2<=k<=n$ : $T(n, k)=$ denominator of $(1+1 / n)^{k}-(1+k / n)$ and of $(1-1 / n)^{k}-(1-k / n)$. or
A095896 Triangle read by rows: $\left.T(n, k)=n^{( } n-k+1\right), n>=1,1<=k<=n$.
$\mathrm{c}=387420489=\operatorname{Mat}([3,18]) \mathrm{b}=1515625=[5,6 ; 97,1] \mathrm{a}=385904864=[2,5 ; 31,1 ; 73,3]$ radical $=6585330$ quality $=1.259527009976646$ merit $=2.9119813790027185$ $c=43046721=\operatorname{Mat}([3,16]) \quad \mathrm{b}=1=\operatorname{matrix}(0,2) \mathrm{a}=43046720=[2,6 ; 5,1 ; 17,1 ; 41,1 ; 193,1]$ radical $=4035630$ quality $=1.1556225447925967 \mathrm{merit}=1.0027234075306528$ $\mathrm{c}=16777216=\operatorname{Mat}([2,24]) \mathrm{b}=2197=\operatorname{Mat}([13,3]) \mathrm{a}=16775019=[3,6 ; 23011,1]$ radical $=1794858$ quality $=1.155210286177332$ merit $=0.9252983022149146$
$\mathrm{c}=4782969=\operatorname{Mat}([3,14]) \mathrm{b}=2725=[5,2 ; 109,1] \mathrm{a}=4780244=[2,2 ; 7,2 ; 29,3]$ radical $=663810$ quality $=1.1473114607778128 \mathrm{merit}=0.7551200767869274$
$=531441=\operatorname{Mat}([3,12]) \mathrm{b}=240625=[5,5 ; 7,1 ; 11,1] \mathrm{a}=290816=[2,12 ; 71,1]$ radial 164010 quity $=1.0979093814132679 \mathrm{merit}=0.28610782629987797$
$\mathrm{c}=117649=\operatorname{Mat}([7,6]) \mathrm{b}=324=[2,2 ; 3,4] \mathrm{a}=117325=[5,2 ; 13,1 ; 19,2] \quad$ radical $=51870$ quality $=1.07543548461615$ merit $=0.1473283668442908$
$\mathrm{c}=59049=\operatorname{Mat}([3,10]) \mathrm{b}=10000=[2,4 ; 5,4] \mathrm{a}=49049=[7,3 ; 11,1 ; 13,1]$ radical $=30030$ quality $=1.065584273830386$ merit $=0.10346446825949088$
$\mathrm{c}=46656=[2,6 ; 3,6] \mathrm{b}=4913=\operatorname{Mat}([17,3]) \mathrm{a}=41743=[13,3 ; 19,1]$ radical $=25194$ quality $=1.0608026156104984$ merit $=0.08676941245716602$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$ $\mathrm{c}=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.11: Increasing quality in sequence nr. 15


Figure 4.12: Increasing merit in sequence nr. 15

### 4.2.7 $\quad$ Sequence 7

A246549 Prime powers $p^{e}$ where $p$ is a prime and $e>=3$ (prime powers without 1 , the primes, or the squares of primes). or
A100391 Prime powers which are local minima of A006530.
$\mathrm{c}=130321=\operatorname{Mat}([19,4]) \mathrm{b}=12672=[2,7 ; 3,2 ; 11,1] \mathrm{a}=117649=\operatorname{Mat}([7,6])$ radical $=8778$ quality $=1.2971091295739554$ merit $=1.7682278466791408$ $\mathrm{c}=117649=\operatorname{Mat}([7,6]) \mathrm{b}=28561=\operatorname{Mat}([13,4]) \mathrm{a}=89088=[2,10 ; 3,1 ; 29,1]$ radical $=15834$ quality $=1.2074005954439029$ merit $=0.9438019575269713$
$\mathrm{c}=83521=\operatorname{Mat}([17,4]) \mathrm{b}=1152=[2,7 ; 3,2] \mathrm{a}=82369=[7,2 ; 41,2]$ radical $=29274$ quality $=1.1019400973819453$ merit $=0.2490837312668208$
$\mathrm{c}=79507=\operatorname{Mat}([43,3]) \mathrm{b}=4096=\operatorname{Mat}([2,12]) \mathrm{a}=75411=[3,4 ; 7,2 ; 19,1]$ radical $=34314$ quality $=1.0804622037917706$ merit $=0.1586144815276661$
$\mathrm{c}=68921=\operatorname{Mat}([41,3]) \mathrm{b}=800=[2,5 ; 5,2] \mathrm{a}=68121=[3,4 ; 29,2]$ radical $=35670$ quality $=1.0628359874003266$ merit $=0.09724557089118555$
$\mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=6487=[13,1 ; 499,1] \mathrm{a}=59049=\operatorname{Mat}([3,10])$ radical $=38922$ quality $=1.049297420915307$ merit $=0.060566258521041194$
$\mathrm{c}=59049=\operatorname{Mat}([3,10]) \mathrm{b}=15625=\operatorname{Mat}([5,6]) \mathrm{a}=43424=[2,5 ; 23,1 ; 59,1]$ radical $=40710$ quality $=1.0350372922734312$ merit $=0.030779769126492332$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$c=2187=\operatorname{Mat}([3,7]) \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193$ merit $=0.001900025557536109$
$c=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.13: Increasing quality in sequence nr. 16


Figure 4.14: Increasing merit in sequence nr. 16

### 4.2.8 Sequence 8

A111261 Numbers of the form $\left.(m / n)^{( } m-n\right)$ with $m>=n$.
$\mathrm{c}=43046721=\operatorname{Mat}([3,16]) \mathrm{b}=83521=\operatorname{Mat}([17,4]) \mathrm{a}=42963200=[2,8 ; 5,2 ; 7,2 ; 137,1]$ radical $=489090$ quality $=1.3417856234336722$ merit $=3.9370152947185657$ $\mathrm{c}=33554432=\operatorname{Mat}([2,25]) \mathrm{b}=9156027=[3,1 ; 1747,2] \mathrm{a}=24398405=[5,1 ; 47,4]$ radical $=2463270$ quality $=1.1774600236737605$ merit $=1.2462687499946097$
$c=16777216=\operatorname{Mat}([2,24]) \quad \mathrm{b}=2197=\operatorname{Mat}([13,3]) \mathrm{a}=16775019=[3,6 ; 23011,1]$ radical $=1794858$ quality $=1.155210286177332$ merit $=0.9252983022149146$
$c=8388608=\operatorname{Mat}([2,23]) \quad b=30375=[3,5 ; 5,3] \mathrm{a}=8358233=[13,2 ; 19,2 ; 137,1]$ radical $=1015170$ quality $=1.152692117618954$ merit $=0.8470589366203329$
$\mathrm{c}=4782969=\operatorname{Mat}([3,14]) \mathrm{b}=2725=[5,2 ; 109,1] \mathrm{a}=4780244=[2,2 ; 7,2 ; 29,3]$ radical $=663810$ quality $=1.1473114607778128 \mathrm{merit}=0.755120076786927$
$\mathrm{c}=4194304=\operatorname{Mat}([2,22]) \mathrm{b}=34929=[3,2 ; 3881,1] \mathrm{a}=4159375=[5 ; 5 ; 11,3]$ radical $=1280730$ quality $=1.0843562669114515$ merit $=0.2645435856557991$
$c=4194304=\operatorname{Mat}([2,22]) \mathrm{b}=34929=[3,2 ; 3881,1] \mathrm{a}=4159375=[5,5 ; 11,3]$ radical $=1280730$ quality $=1.0843562669114515$ merit $=0.2645435856557991$
$c=2097152=\operatorname{Mat}([2,21]) \mathrm{b}=83349=[3,5 ; 7,3] \mathrm{a}=2013803=[11,3 ; 17,1 ; 89,1]$ radical $=699006$ quality $=1.0816409554990716$ merit $=0.23316999425645868$
$\mathrm{c}=531441=\operatorname{Mat}([3,12]) \quad \mathrm{b}=1=\operatorname{matrix}(0,2) \mathrm{a}=531440=[2,4 ; 5,1 ; 7,1 ; 13,1 ; 73,1]$ radical $=199290$ quality $=1.080379415869545$ merit $=0.19722605195982415$
$\mathrm{c}=390625=\operatorname{Mat}([5,8]) \mathrm{b}=2209=\operatorname{Mat}([47,2]) \mathrm{a}=388416=[2,6 ; 3,1 ; 7,1 ; 17,2]$ radical $=167790$ quality $=1.0702412233273266 \mathrm{merit}=0.14764533113282224$
$\mathrm{c}=131072=\operatorname{Mat}([2,17]) \mathrm{b}=6413=[11,2 ; 53,1] \mathrm{a}=124659=[3,8 ; 19,1]$ radical $=66462$ quality $=1.0611574973269269$ merit $=0.09998420588198229$
$c=117649=\operatorname{Mat}([7,6]) \mathrm{b}=12500=[2,2 ; 5,5] \mathrm{a}=105149=[11,3 ; 79,1]$ radical $=60830$ quality $=1.059879466729859$ merit $=0.0947685263276362$
$c=65536=\operatorname{Mat}([2,16]) \mathrm{b}=6487=[1,1 ; 499,1] \mathrm{a}=59049=\operatorname{Mat}([3,10])$ radical $=38922$ quality $=1.04929742091507$ merit $=0.060566255521041194$
$\begin{aligned} & \mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=6487=[13,1 ; 499,1] \mathrm{a}=59049=\operatorname{Mat}([3,10]) \text { radical }=38922 \text { quality }=1.049297420915307 \text { merit }=0.060566258521041194 \\ &=59049=\operatorname{Mat}([3,10]) \mathrm{b}=15625=\operatorname{Mat}([5,6]) \mathrm{a}=43424=[2,5 ; 23,1 ; 59,1] \quad \text { radical }=40710 \text { quality }=1.0350372922734312 \text { merit }=0.030779769126492332\end{aligned}$
$=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3]) \quad$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$c=729=\operatorname{Mat}([3,6]) b=200=[2,3 ; 5,2] a=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.15: Increasing quality in sequence nr. 18


Figure 4.16: Increasing merit in sequence nr. 18

### 4.2.9 Sequence 9

A120458 Triangle read by rows: row 0 is 1 ; for $n>0$, row n gives $1^{n}$, $\operatorname{prime}(1)^{n}$, prime $(2)^{n}, \ldots, \operatorname{prime}(n)^{n}$. or
A319075 Square array $T(n, k)$ read by antidiagonal upwards in which row $n$ lists the $n$-th powers of primes, hence column $k$ lists the powers of the $k$-th prime, $n>=0, k>=1$.
$\mathrm{c}=815730721=\operatorname{Mat}([13,8]) \mathrm{b}=25=\operatorname{Mat}([5,2]) \mathrm{a}=815730696=[2,3 ; 3,3 ; 11,2 ; 23,2 ; 59,1]$ radical $=5821530$ quality $=1.3172945889635699$ merit $=4.306060599870464$ $\mathrm{c}=410338673=\operatorname{Mat}([17,7]) \mathrm{b}=566048=[2,5 ; 7,2 ; 19,2] \mathrm{a}=409772625=[3,1 ; 5,3 ; 103,3]$ radical $=6986490$ quality $=1.258447758491067$ merit $=2.902646548075581$ $\mathrm{c}=214358881=\operatorname{Mat}([11,8]) \mathrm{b}=2307361=[7,4 ; 31,2] \mathrm{a}=212051520=[2,6 ; 3,8 ; 5,1 ; 101,1]$ radical $=7232610$ quality $=1.2145769242411055 \mathrm{merit}=2.00684200337074$ $\mathrm{c}=19487171=\operatorname{Mat}([11,7]) \mathrm{b}=5815296=[2,10 ; 3,2 ; 631,1] \mathrm{a}=13671875=[5,9 ; 7,1]$ radical $=1457610$ quality $=1.1827016523205258$ merit $=1.256684226720755$ $\mathrm{c}=5764801=\operatorname{Mat}([7,8]) \mathrm{b}=648=[2,3 ; 3,4] \mathrm{a}=5764153=[73,1 ; 281,2]$ radical $=861546$ quality $=1.1390846029894408$ merit $=0.6913177200951904$
$\mathrm{c}=4826809=\operatorname{Mat}([13,6]) \mathrm{b}=75449=[11,1 ; 19,3] \mathrm{a}=4751360=[2,15 ; 5,1 ; 29,1]$ radical $=787930$ quality $=1.133498537786449$ merit $=0.6311527941829186$
$\mathrm{c}=1771561=\operatorname{Mat}([11,6]) \mathrm{b}=65536=\operatorname{Mat}([2,16]) \mathrm{a}=1706025=[3,1 ; 5,2 ; 23,2 ; 43,1]$ radical $=326370$ quality $=1.133239841890956$ merit $=0.5727679244037188$
$\mathrm{c}=823543=\operatorname{Mat}([7,7]) \mathrm{b}=371293=\operatorname{Mat}([13,5]) \mathrm{a}=452250=[2,1 ; 3,3 ; 5,3 ; 67,1]$ radical $=182910$ quality $=1.124176994405736$ merit $=0.4660874159105764 \mathrm{~s}$
$\mathrm{c}=390625=\operatorname{Mat}([5,8]) \mathrm{b}=117649=\operatorname{Mat}([7,6]) \mathrm{a}=272976=[2,4 ; 3,1 ; 11,2 ; 47,1]$ radical $=108570$ quality $=1.1104214132202936$ merit $=0.346460002140961$
$\mathrm{c}=161051=\operatorname{Mat}([11,5]) \mathrm{b}=12096=[2,6 ; 3,3 ; 7,1] \mathrm{a}=148955=[5,1 ; 31,3]$ radical $=71610$ quality $=1.0725008570014634$ merit $=0.14185106903747513$
$c=117649=\operatorname{Mat}([7,6]) \quad \mathrm{b}=34128=[2,4 ; 3,3 ; 79,1] \mathrm{a}=83521=\operatorname{Mat}([17,4])$ radical $=56406$ quality $=1.0671945018820044$ merit $=0.1181794318841646$
$c=2187=\operatorname{Mat}([3,7]) \mathrm{b}=512=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1] \quad$ radical $=2010$ quality $=1.011096139914193 \mathrm{merit}=0.0019000255575361094$
$\mathrm{c}=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.17: Increasing quality in sequence nr. 22


Figure 4.18: Increasing merit in sequence nr. 22

### 4.2.10 Sequence 10

A076470 Perfect powers $m^{k}$ where m is an integer and $\mathrm{k}>5$. or A122494 Set of numbers of the form $a^{b}$ with $2<=a<=b$, sorted.
$=1594323=\operatorname{Mat}([3,13]) \mathrm{b}=7424=[2,8 ; 29,1] \mathrm{a}=1586899=[17,4 ; 19,1]$ radical $=56202$ quality $=1.305873782653318$ merit $=2.4476817174888095$ $c=1048576=\operatorname{Mat}([2,20]) \mathrm{b}=50575=[5,2 ; 7,1 ; 17,2] \mathrm{a}=998001=[3,6 ; 37,2]$ radical $=132090$ quality $=1.1756986573787018$ merit $=0.8981073965362553$ $\mathrm{c}=823543=\operatorname{Mat}([7,7]) \mathrm{b}=3072=[2,10 ; 3,1] \mathrm{a}=820471=[17,3 ; 167,1] \quad$ radical $=119238$ quality $=1.1653276277840765$ merit $=0.7855216932161503$

$\mathrm{c}=390625=\operatorname{Mat}([5,8]) \mathrm{b}=117649=\operatorname{Mat}([7,6]) \mathrm{a}=272976=[2,4 ; 3,1 ; 11,2 ; 47,1]$ radical $=108570$ quality $=1.1104214132202936$ merit $=0.346460002140961$
$\mathrm{c}=177147=\operatorname{Mat}([3,11]) \mathrm{b}=46075=[5,2 ; 19,1 ; 97,1] \mathrm{a}=131072=\operatorname{Mat}([2,17])$ radical $=55290$ quality $=1.1066255312066344$ merit $=0.2968047377434422$
$c=131072=\operatorname{Mat}([2,17]) \quad \mathrm{b}=95=[5,1 ; 19,1] \mathrm{a}=130977=[3,5 ; 7,2 ; 11,1] \quad$ radical $=43890$ quality $=1.102349618350441$ merit $=0.2653014130816672$
$\mathrm{c}=65536=\operatorname{Mat}([2,16]) \mathrm{b}=709=\operatorname{Mat}([709,1]) \mathrm{a}=64827=[3,3 ; 7,4]$ radical $=29778$ quality $=1.0765740735526552$ merit $=0.140879546089314$
$c=59049=\operatorname{Mat}([3,10]) \mathrm{b}=10000=[2,4 ; 5,4] \mathrm{a}=49049=[7,3 ; 11,1 ; 13,1]$ radical $=30030$ quality $=1.065584273830386$ merit $=0.10346446825949088$
$c=46656=[2,6 ; 3,6] \mathrm{b}=4913=\operatorname{Mat}([17,3]) \mathrm{a}=41743=[13,3 ; 19,1]$ radical $=25194$ quality $=1.0608026156104984$ merit $=0.08676941245716602$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.03446093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=8192=\operatorname{Mat}([2,13]) \mathrm{b}=11=\operatorname{Mat}([11,1]) \mathrm{a}=8181=[3,4 ; 101,1]$ radical $=6666$ quality $=1.0234120783723573$ merit $=0.010498233280691932$
$=729=\operatorname{Mat}([3,6]) \mathrm{b}=500=\operatorname{Mat}([2,9]) \mathrm{a}=1675=[5,2 ; 67,1]$ radical $=2010$ quality $=1.011096139914193$ merit $=0.001900025557536109$


Figure 4.19: Increasing quality in sequence nr. 24


Figure 4.20: Increasing merit in sequence nr. 24

### 4.2.11 Sequence 11

A128686 Least number of the form semiprime - 1 which is the product of exactly n primes.
$\mathrm{c}=24189255811072=[2,41 ; 11,1] \mathrm{b}=196367346709=[19,3 ; 31,5] \mathrm{a}=23992888464363=[3,1 ; 7,8 ; 13,2 ; 8209,1]$ radical $=29039616606$ quality $=1.2791392693573205$ merit $=5.973058523906151$ $\mathrm{c}=20890720927744=[2,40 ; 19,1] \mathrm{b}=291355693369=[13,1 ; 59,2 ; 6438373,1] \mathrm{a}=20599365234375=[3,3 ; 5,17]$ radical $=2814792291870$ quality $=1.0699233564092074$ merit $=0.4703211412429409$ $\mathrm{c}=1099511627776=\operatorname{Mat}([2,40]) \mathrm{b}=239256535903=[7,11 ; 11,2] \mathrm{a}=860255091873=[3,7 ; 251,1 ; 311,1 ; 5039,1]$ radical $=181727413098$ quality $=1.0694333590257687$ merit $=0.4068654104394979$
$\mathrm{c}=927712935936=[2,35 ; 3,3] \mathrm{b}=268091351=[11,4 ; 18311,1] \mathrm{a}=927444844585=[5,1 ; 7,1 ; 23,6 ; 179,1]$ radical $=174142553970$ quality $=1.0646308022996973$ merit $=0.35177026101038533$
$\mathrm{c}=343597383680=[2,36 ; 5,1] \mathrm{b}=1744821=[3,4 ; 13,1 ; 1657,1] \mathrm{a}=343595638859=[11,3 ; 16067,2]$ radical $=114212751510$ quality $=1.0432580575191963$ merit $=0.15423373046642333$
$\mathrm{c}=171798691840=[2,35 ; 5,1] \mathrm{b}=3388476321=[3,5 ; 19,4 ; 107,1] \mathrm{a}=168410215519=[7,1 ; 13,3 ; 29,3 ; 449,1]$ radical $=72267721890$ quality $=1.0346327852291926$ merit $=0.09653882761119742$ $\mathrm{c}=115964116992=[2,32 ; 3,3] \mathrm{b}=36879991031=[13,4 ; 1291271,1] \mathrm{a}=79084125961=[19,6 ; 41,2]$ radical $=78460208502$ quality $=1.015574082320194$ merit $=0.019606515824163756$
$=8388698=[2,27 ; 3,2] b=47176427=[23,1 ; 29,5] a=736203125=[5,6 ; 7,1 ; 53,1 ; 127,1]$ radical $=942811170$ quality $=1.0119927100095067$ merit $=0.0090006075822568$
$=3145728=[2,20 ; 3,1] \mathrm{b}=55451=[11,1 ; 71,2] \mathrm{a}=3090277=[17,4 ; 37,1]$ radical $=2947494$ quality $=1.0043694943916004 \mathrm{merit}=0.0007682280132241918$
$c=131072=\operatorname{Mat}([2,17]) \mathrm{b}=65219=[7,2 ; 11,3] \mathrm{a}=65853=[3,5 ; 271,1]$ radical $=125202$ quality $=1.0039035263414824$ merit $=0.00044048038050787064$


Figure 4.21: Increasing quality in sequence nr. 26


Figure 4.22: Increasing merit in sequence nr. 26

### 4.2.12 Sequence 12

A132722 Numbers with $k$ decimal digits that are also $k$-th powers.
$c=2541865828329=\operatorname{Mat}([3,26]) \quad b=1012703329=[11,4 ; 263,2] a=2540853125000=[2,3 ; 5,8 ; 23,2 ; 29,1 ; 53,1]$ radical $=3068113290$ quality $=1.307612605913739$ merit $=6.374601882382574$
$=282429536481=\operatorname{Mat}([3,24]) \quad \mathrm{b}=47136567731=[11,5 ; 541,2] \mathrm{a}=235292968750=[2,1 ; 5,10 ; 7,1 ; 1721,1]$ radical $=2150750910$ quality $=1.2269809315279852$ merit $=3.3961555696523513$
$=31381059609=\operatorname{Mat}([3,22]) \mathrm{b}=25=\operatorname{Mat}([5,2]) \mathrm{a}=31381059584=[2,11 ; 7,1 ; 173,1 ; 12653,1]$ radical $=459683490$ quality $=1.2117422683160224$ merit $=2.6765984556511415$
$=3486784401=\operatorname{Mat}([3,20]) \mathrm{b}=4332433=[7,3 ; 17,1 ; 743,1] \mathrm{a}=3482451968=[2,17 ; 163,2]$ radical $=86471826$ quality $=1$. $202290012040235 \mathrm{merit}=2.172915033924577$
chat
$c=1073741824=\operatorname{Mat}([2,30]) \mathrm{b}=1369=\operatorname{Mat}([37,2]) \mathrm{a}=1073740455=[3,8 ; 5,1 ; 71,1 ; 461,1]$ radical $=36331410$ quality $=1.194518877448107$ merit $=1.8818217103144523$
$=387420489=\operatorname{Mat}([3,18]) \mathrm{b}=16540489=[7,4,83,2] \mathrm{a}=370880000=[2,9 ; 5,4,19,1 ; 61,1] \mathrm{radical}=20201370$ quality $=1.1755968364891651 \mathrm{merit}=1.4640232465616403$
$=134217728=\operatorname{Mat}([2,27]) \mathrm{b}=173889=[3,2 ; 139,2] \mathrm{a}=134043839=[23,4 ; 479,1]$ radical $=9188178$ quality $=1.1672471802876274$ merit $=1.244389242499289$
$=43046721=\operatorname{Mat}([3,16]) \mathrm{b}=1=\operatorname{matrix}(0,2) \mathrm{a}=43046720=[2,6 ; 5,1 ; 17,1 ; 41,1 ; 193,1]$ radical $=4035630$ quality $=1.1556225447925967$ merit $=1.0027234075306528$
$=16777216=\operatorname{Mat}([2,24]) \mathrm{b}=2197=\operatorname{Mat}([13,3]) \mathrm{a}=16775019=[3,6 ; 23011,1]$ radical $=1794858$ quality $=1.155210286177332$ merit $=0.9252983022149146$
$\begin{aligned} &=4782969=\operatorname{Mat}([3,14]) \quad \mathrm{b}=2725=[5,2 ; 109,1] \mathrm{a}=4780244=[2,2 ; 7,2 ; 29,3] \text { radical }=663810 \text { quality }=1.1473114607778128 \text { merit }=0.7551200767869274 \\ &=2097152=\operatorname{Mat}([2,21]) \quad \mathrm{b}=144027=[3,2 ; 13,1 ; 1231,1] \mathrm{a}=1953125=\operatorname{Mat}([5,9]) \text { radical }=480090 \text { quality }=1.1127039049372605 \text { merit }=0.42724965386318153\end{aligned}$
$=531441=\operatorname{Mat}([3,12]) \mathrm{b}=240625=[5,5 ; 7,1 ; 11,1] \mathrm{a}=290816=[2,12 ; 71,1]$ radical $=164010$ quality $=1.0979093814132679$ merit $=0.28610782629987797$
$=117649=\operatorname{Mat}([7,6]) \mathrm{b}=324=[2,2 ; 3,4] \mathrm{a}=117325=[5,2 ; 13,1 ; 19,2]$ radical $=51870$ quality $=1.07543548461615$ merit $=0.1473283668442908$
$=59049=\operatorname{Mat}([3,10]) \mathrm{b}=10000=[2,4 ; 5,4] \mathrm{a}=49049=[7,3 ; 11,1 ; 13,1]$ radical $=30030$ quality $=1.065584273830386$ merit $=0.1034644682594908$
$\mathrm{c}=32768=\operatorname{Mat}([2,15]) \mathrm{b}=8379=[3,2 ; 7,2 ; 19,1] \mathrm{a}=24389=\operatorname{Mat}([29,3])$ radical $=23142$ quality $=1.0346093426206775$ merit $=0.027776098986953608$
$\mathrm{c}=729=\operatorname{Mat}([3,6]) \mathrm{b}=200=[2,3 ; 5,2] \mathrm{a}=529=\operatorname{Mat}([23,2])$ radical $=690$ quality $=1.0084113092374762$ merit $=0.0008682588486994485$


Figure 4.23: Increasing quality in sequence nr. 28


Figure 4.24: Increasing merit in sequence nr. 28

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[^0]:    $c=16777216=\operatorname{Mat}([2,24]) b=2197=\operatorname{Mat}([13,3]) a=16775019=[3,6 ; 23011,1] \quad$ radical $=1794858$ quality $=1.155210286177332$ merit $=0.9252983022149146$
    $\mathrm{c}=13845841=\operatorname{Mat}([61,4]) \mathrm{b}=133632=[2,9 ; 3,2 ; 29,1] \mathrm{a}=13712209=[7,2 ; 23,4]$ radical $=1708854$ quality $=1.1457817090087676$ merit $=0.8124692544875347$
    $\mathrm{c}=12117361=\operatorname{Mat}([59,4]) \mathrm{b}=4092529=[7,2 ; 17,4] \mathrm{a}=8024832=[2,8 ; 3,6 ; 43,1]$ radical $=1811418$ quality $=1.1318930814861106$ merit $=0.6687525976855672$
    $c=7890481=\operatorname{Mat}([53,4]) \quad \mathrm{b}=9882=[2,1 ; 3,4 ; 61,1] \mathrm{a}=7880599=\operatorname{Mat}([199,3])$ radical $=3860202$ quality $=1.0471400987029473$ merit $=0.09163875979161587$
    $\mathrm{c}=4879681=\operatorname{Mat}([47,4]) \mathrm{b}=1024000=[2,13 ; 5,3] \mathrm{a}=3855681=[3,7 ; 41,1 ; 43,1]$ radical $=2485830$ quality $=1.0458011594962668 \mathrm{merit}=0.08308689730242558$
    $\mathrm{c}=1874161=\operatorname{Mat}([37,4]) \mathrm{b}=9=\operatorname{Mat}([3,2]) \mathrm{a}=1874152=[2,3 ; 7,3 ; 683,1] \operatorname{radical}=1061382$ quality $=1.0409791627953058$ merit $=0.06128202432704795$
    $\mathrm{c}=1594323=\operatorname{Mat}([3,13]) \mathrm{b}=34322=[2,1 ; 131,2] \mathrm{a}=1560001=\operatorname{Mat}([1249,2])$ radical $=981714$ quality $=1.035145503116223$ merit $=0.04472651867249758$
    $c=707281=\operatorname{Mat}([29,4]) b=2048=\operatorname{Mat}([2,11]) \mathrm{a}=705233=[79,2 ; 113,1]$ radical $=517766$ quality $=1.0237058624176838$ merit $=0.019054072691593378$
    $\mathrm{c}=\mathrm{c}^{2}$
    $\mathrm{c}=279841=\operatorname{Mat}([23,4]) \mathrm{b}=289=\operatorname{Mat}([17,2]) \mathrm{a}=279552=[2,10 ; 3,1 ; 7,1 ; 13,1]$ radical $=213486$ quality $=1.0220555069737798$ merit $=0.01496669485358114$
    $\mathrm{c}=131072=\operatorname{Mat}([2,17]) \mathrm{b}=41405=[5,1 ; 7,2 ; 13,2] \mathrm{a}=89667=[3,7 ; 41,1]$ radical $=111930$ quality $=1.0135797482655287$ merit $=0.005259385277340935$
    $\mathrm{c}=130321=\operatorname{Mat}([19,4]) \mathrm{b}=46800=[2,4 ; 3,2 ; 5,2 ; 13,1] \mathrm{a}=83521=\operatorname{Mat}([17,4])$ radical $=125970$ quality $=1.0028914709915537$ merit $=0.0002418622392246741$
    $\mathrm{c}=83521=\operatorname{Mat}([17,4]) \mathrm{b}=25921=[7,2 ; 23,2] \mathrm{a}=57600=[2,8 ; 3,2 ; 5,2]$ radical $=82110$ quality $=1.001505705544449$ merit $=6.224332362096628 \mathrm{e}-05$

