

Experimental Test on the abc-Conjecture

Arno GEIMER under the supervision of Alexander D. RAHM, PhD

Bachelor thesis at the University of Luxembourg

June 2019

Abstract

This thesis focuses on finding a counter-example to the abc-Conjecture by comparing different databases.

Contents

1	Intr	oducti	on	5
	1.1	Mathe	matics and Logic in Ancient Times	5
	1.2	Mathe	matics and Logic in the Middle Ages	7
	1.3	Mathe	matics and logic in modern times	9
	1.4	Motiva	tion	1
2	The	abc-Co	onjecture 1	2
	2.1	The C	onjecture	2
	2.2	History	y of the ABC Conjecture	3
	2.3	Attem	pted proofs and Consequences	4
		2.3.1	Simpler Proofs of existing Theorems	5
		2.3.2	Conjectures based on the <i>abc</i> -Conjecture	5
3	The	search	n for a counterexample 1	6
	3.1	Our m	ethod	6
		3.1.1	The ABC@Home project and its Databank 1	6
		3.1.2	The OEIS-Databank	6
		3.1.3	The ABC@Home Algorithm	7
	3.2	Our co	$\tilde{1}$ mputations \ldots $\tilde{1}$	8
		3.2.1	Getting the files ready	9
		3.2.2	The search algorithm	20
	3.3	Our re	sults	4
4	Apr	oendix	2	7
	4.1	The co	ponversion algorithm	7
	4.2	Compi	ited results	0
		4.2.1	Sequence 1	0
		4.2.2	Sequence 2	1
		4.2.3	Sequence 3	2
		4.2.4	Sequence 4	3
		4.2.5	Sequence 5	4
		4.2.6	Sequence 6	5
		4.2.7	Sequence 7	6
		4.2.8	Sequence 8	$\overline{7}$
		4.2.9	Sequence 9	8
		4.2.10	Sequence 10	9
		4.2.11	Sequence 11	0
		4.2.12	Sequence 12	1

List of Figures

Chapter 1

Introduction

The following will be a quick overview of the development of mathematics and logic since its beginning. It will show the history of rigorousness and proofs in mathematics.

1.1 Mathematics and Logic in Ancient Times

Several thousand years ago, during the beginning of modern civilisation, mathematics began as the mere concept of counting, its sole purpose being its usage for trading as well as keeping track of time. One of the earliest artifacts used for mathematical purposes is the Ishango bone which was found in modernday Congo. It dates back at least 20,000 years and was probably used either as a list of prime numbers or as a six-month lunar calendar.[1] It is believed that the earliest traces of geometry can be found in old Egyptian artifacts from around 5,000 BCE or in monuments in Great Britain from around 3,000 BCE. Nevertheless, these ideas are disputed. It is however undisputed that the oldest known mathematical documents emerged from Babylonian and Egyptian sources.

The Babylonians used a numeral system with base 60, complete with multiplication and division, and developed a measure system from as early as 3,000 BCE. It is most astounding that the Babylonians were even able to solve quadratic and cubic equations. The Egyptians probably used a base 10 number system and had mathematical knowledge in both geometry and algebra. For instance, they were able to determine the surface area of three-dimensional objects. However, both the Egyptians and the Babylonians only used mathematics as a tool and thus did not know the concept of proving their results. The idea of mathematical concepts having to be proven before



Figure 1.1: Egyptian numeral system

being considered to be true only emerged with the independent development of logic in Europe and Asia.

In Europe, the study of logic began with Greek philosophers distinguishing between different kinds of sentences, as well as developing techniques for discussions based on arguments at around 500 BCE.[2] Even though Pythagoras, as well as Thales, had already established proofs to known statements, the most famous certainly being **Pythagoras' Theorem:** In a right triangle, the square of the length of the hypotenuse is the same as the sum of the squares of the two other sides., Greek logic did not yet include the need for proofs. Pythagoras' proof consists of a simple proof by rearrangement. Around 400 BCE, Plato introduced the notion of definitions and asked about the connection between assumptions and conclusions, which one can view as an early idea of the further notion of a theorem.[3][4]

His disciple, Aristotle, was the first to systematically study logic. In his *Organon*, he developed logic as a tool for the theoretical sciences, revolving around the notion of deduction. He gives detailed ideas about affirmations, denials and contradictions and divides proofs into two different categories; those based on "complete" deductions that don't need proofs, which can be viewed today as axioms, and those based on "incomplete" deductions which need to be proven and follow from the "complete" ones.[5] This is the first description of the concept of a proof. Aristotle also proves that counterexamples disprove a statement. However his school of logic was rivaled by the school of logic, which was developed by the



Figure 1.2: Aristotle

Stoic philosophers. Both schools differentiate in that Aristotle's logic is based on "terms"; the subjects and predicates of assertions, while Stoic logic is based on "assertibles"; the equivalent of a proposition.[6]



However, not many texts on Stoic logic have survived, as Aristotle's logic became the dominating form of logic by the end of antiquity. Around 300 BCE, Euclid wrote one of the most important, if not the most important book in the history of mathematics: his *Elements*. It originally was a collection of 13 books containing definitions, postulates and propositions including their respective proofs in the areas of Euclidean geometry, number theory and algebra. It is universally accepted that Euclid was not the author of most of the theorems and their proofs, but the content of the *Elements* as well as the mathematical methods used have influenced all of mathematics.[7] His geometric proofs are based on 5 axioms (as cited by wikiversity.org)[8]

Figure 1.3: Euclid

- A line can be drawn from a point to any other point.
- A finite line can be extended indefinitely.
- A circle can be drawn, given a center and a radius.
- All right angles are ninety degrees.
- If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles,

then the lines meet on that side and not on the other side.

These are the five postulates on which Euclidean geometry is based.

In Asia, logic developed independently in two different regions: in India and in China. Indian philosophy was divided into several major schools, of which some developed concepts of logic, the most influential being the Nyaya school's system of logic which later shaped Indian logic.[9] It is based on epistemology, the philosophy treating the theory of knowledge, and is thus not based on a strict mathematical study of propositions like Western logic. Instead, Indian logic tries to provide known scientific arguments with rigor. It does not admit statements which are known to be false, which leads to proofs by contradiction not being accepted in Indian logic.[10] The most notable Ancient Indian mathematician was Pingala [11], whose work contains the first known description of a binary numeral system as well as a version of the Binomial Theorem which describes the expansion of powers of a binomial.[12] However, since Indian logic was not as well-developed as Greek logic at that moment, Indian mathematicians did not yet put much rigor into proving their arguments.



Figure 1.4: An excerpt from *The Nine Chapters on the Mathematical Art*

In China, the development of logic began with the founding of the Mohist school by Mozi around 400 BCE.[4] It was however the introduction of Buddhism and the following introduction of Indian logic that the development of Buddhist logic appeared in China. As it originated from Indian logic, it is also based on epistemology.[13] Ancient Chinese mathematicians treated the areas of geometry as well as algebra, with magic squares dating back as far as 650 BCE and Gaussian elimination being used in The Nine Chapters on the Mathematical Art, an important book in Chinese mathematics finished around 200 BCE. A detailed commentary by Chinese mathematician Liu Hui from the 3rd century gave proofs to the problems in the book.[14]

1.2 Mathematics and Logic in the Middle Ages

Around 600 CE, the rise of Islam brought with it Islamic Philosophy and its own take on logic which placed significance on standards of arguments, though this method was quickly overthrown by the Mu'tazili philosophers who favored old Hellenistic philosophy. Several Persian logicians were however dissatisfied with Hellenistic philosophy and corrected it or developed their own forms of logic. Beginning in the 10th century, Islamic scholars began to base their works on clerical authority, which resulted in a decline of Aristotelian logic in Islamic science.[15] The most important aspect of Islamic logic is certainly the development of Avicennian logic which introduced the need for citation as well as inductive logic, i.e. deducting that a conclusion may be true based on some evidence. The period from the 9th to the 15th century is considered to be the Golden Age of Islamic mathematics and saw several Islamic mathematicians proving theorems with the use of arithmetic and algebra.[16] Al-Khwarizmi introduced the Hindu numerical sys-

Ai-Khwarizini introduced the Hindu numerical system, consisting of the numbers 0-10, which proved to be a great tool for improving the efficiency of Islamic mathematics. Around 1000 CE, Al-Karaji developed one of the first known proofs by induction to prove the previously mentioned Binomial Theorem. Islamic scholars proved several other theorems during the following centuries.[17] They furthermore occupied themselves amongst other things with trigonometry as well as solving cubic equations and geometry.

In Europe, following the Dark Ages, Boethius, a Christian philosopher, was the most important contributor to logic of his time. As he was only familiar with some of Aristotle's works and none of the Stoics', Aristotelian logic became the predominant form of logic in medieval Europe. Around 1300, the theory of consequences emerged, which is an important aspect of later mathematical proofs. As Eu-



Figure 1.5: Al-Khwarizmi, the "father of algebra"

rope started trade relationships with the East, Eastern mathematics started to spread. In the 12th century, Al-Khwarizmi's *Compendious Book on Calculation by Completion and Balancing* was translated into Latin by Robert of Chester. With Johannes Gutenberg's invention of the printing press in 1439, knowledge became more easily transmittable, which enabled mathematicians to further extend their areas of expertise.[18] In the 13th century, with the help of Italian mathematician Fibonacci, the Hindu numerical system was introduced in Europe and quickly became the prevalent numerical system. Mathematicians began to become interested in infinite series, with French mathematician Nicole Oresme proving that the series 1 + 1/2 + 1/3 + ... diverges.



Figure 1.6: Pascal's triangle in the Jade Mirror of the Four Unknowns

In China, mathematics experienced their Golden Age during the 13th century, with over 30 reputable mathematics schools existing around all of China.[19] Qin Jiushao introduced a method of solving high order equations, going as far as to solving a 10th order equation. Li Zhi collected hundreds of formulas and problems on inscribing circles in triangles in his *Sea-Mirror of the Circle Measurments*. Zhu Shijie's *Jade Mirror of the Four Unknowns* contains the formula $1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{3!}$, although it is missing a proof. In the 13th century, trigonometry had finally become an important area in Chinese mathematics.

It was also in India that mathematics experienced their Golden Age during the Middle Ages. Around 500 CE, Arybhata gave the formula 1.2 in his *Aryabhatiya*, as well as a table of sines, a section covering several types of numerical equations and an approximation of pi which is accurate to four decimal places.[20] Brahmagupta gave a solution to the quadratic equation $ax^2 + bx = c$ around 650 CE. His Brahmasphutasiddhanta contains one of the first appearance of the four fundamental numerical operations, as well as rules for calculations involving zero. He also gave a formula for finding Pythagorean triples and a formula for the length of the diagonals of any isosceles trapezoid, named Brahmagupta's Theorem.[21] In the 12th century, Bhaskara II discovered differential and infinitesimal calculus by approximating, more precisely he noticed that $\sin y' - \sin y \approx (y - y') \cos y$ for y' close to y, which translates in today's notation to $\frac{d}{dx} \sin y = \cos y$.[22] He found a proof to Pythagoras' Theorem as well as solutions of equations of the type $ax^2 + b = y^2$ and he stated an early version of Rolle's Theorem which is a special case of the Mean Value Theorem.

1.3 Mathematics and logic in modern times

Due to the Scientific Revolution, huge advancements were made in nearly every scientific field. Astronomers began discovering new celestial bodies and made some serious improvements in the study of planetary motion. In the 17th century, Gottfried Leibniz and Isaac Newton independently developed differential and integral calculus, the basis of so many later fields of mathematics as well as of other sciences, and the beginning of modern physics. John Napier was the first to introduce logarithms in 1614. Blaise Pascal and Pierre de Fermat founded probability theory, whose foundations had been set by Gerolamo Cardano in the 16th century. [23] Fermat wrote his Last Theorem without giving a proof, which might be the most well-known case of a theorem that was not proven by its author. Its content, as well as a proof of a special case will be discussed later. In the 18th century, calculus was in full development with Leonhard Euler introducing the exponential function and Euler's formula from which the famous $e^{i\pi} + 1 = 0$ is derived.



Figure 1.7: Leonhard Euler (1707 - 1783)

He made significant contributions to graph theory by solving the "Seven Bridges of Königsberg" problem, to number theory by proving a number of theorems, as well as to several other areas of mathematics and physics. [24] During the 19th century, mathematics started getting more abstract with elliptic and hyperbolic geometry being developped, Niels Abel and Evariste Galois laying the fundaments of group theory, and Carl Friedrich Gauss developing, among others, complex analysis. At the end of the century, Giuseppe Peano and David Hilbert were among the mathematicians who founded mathematical logic. [25] At the beginning of the 20th century, Ernst Zermelo proposed his set theory which, with the help of Abraham Fraenkel, became the Zermelo-Fraenkel Set Theory, which is the most accepted foundation of today's

mathematics. Hilbert founded Hilbert's Program whose goal it was to construct a finite and complete set of axioms on which mathematics were to be based.[26] However, in 1931 Kurt Gödel published his Incompleteness Theorem showing that this was in fact not possible; in such a set of axioms, statements can be constructed which are true but unprovable.^[27] Einstein's General Relativity Theory caused the rise of differential geometry, and the beginning of computer science led to advancements in algebra and mathematical logic. In the second half of the century, several long-standing mathematical problems were finally proven, among others Fermat's Last Theorem in 1994 by Andrew Wiles as well as the Four Color Theorem in 1976 by Wolfgang Haken and Kenneth Appel, and the Classification Of Finite Simple Groups, one of the largest mathematical proofs ever written filling several tens of thousands of pages.



Figure 1.8: Ramanujan (1887 - 1920)

In Asia, mathematical progress started to decline. Even though there were still remarkable Indian and Chinese mathematicians after the 16th century, mathematics did not flourish as much as they did during the Middle Ages. With colonization, western mathematics were introduced in Asia even though some theorems had already been well-known to the local scholars for a long time. Around the beginning of the 20th century Indian mathematician Ramanujan, without previous mathematical training, made several important discoveries in many mathematical fields, and established theorems and conjectures in those fields. As of today, the age of Information has connected the whole world, and several Asian mathematics depart-

ments rank among the best in the world.

One of the most important events in the 20th and 21st centuries were however the publications of two lists of unsolved mathematical problems. In 1900, Hilbert published a list of 23 unsolved mathematical problems and in 2000, the Clay Mathematics Institute published the Millenium Prize Problems, a list of seven problems endowed by a million dollars each.[28][29] Some of the problems on Hilbert's list are:

- The Continuum Hypothesis: There is no set whose cardinality is between that of N and R. Proven to be unprovable.
- The Riemann Hypothesis. Still open.
- The Goldbach Conjecture: Every even integer bigger than 2 is the sum of two primes. Still open.
- What is the densest sphere packing? Probably resolved by Thomas Callister Hales in 1998.
- Is a^b transcendental if $a \neq (0, 1)$ is algebraic and b is algebraic and irrational? In 1934, Aleksandr Gelfond and Theodor Schneider independently proved the answer to be "yes".

The Riemann Hypothesis is also one of the Millenium Prize Problems. Of the seven problems, only the Poincaré conjecture has been proven, in 2003 by Grigori Perelman. Other problems on the list are:



Figure 1.9: David Hilbert (1862 - 1943)

- P vs NP, a theoretical computer science problem on algorithm execution times.
- The Navier-Stokes Existence And Smoothness Problem which asks if certain solutions to the Navier-Stokes equations exist under asserted conditions.
- The Birch And Swinnerton-Dyer Conjecture which asks for a way to tell if elliptic curves over the rational numbers have finite or infinite number of rational solutions.

Beside these problems, there are still several big mathematical problems that have been open for a long time, such as the Hadamard Matrix Conjecture, whether the Euler constant is irrational, the Collartz Conjecture, and the ABC-Conjecture, around which this thesis revolves.[30]

1.4 Motivation

The previous section showed the most important aspect of today's mathematical philosophy: A statement is only true if it has been proven. It might have been acceptable in ancient mathematics to assume the truthfulness of statements, but today's mathematics would collapse without this rigor, since anyone could claim anything to be true. That's why the task of mathematical researchers is not only to develop new concepts and formulas, but also to prove them as well. Without a proof, a formula might still hold to a certain degree in applied mathematics, but mathematically it would be considered invalid. With the help of computers, one can often find a huge number of examples validating a conjecture, but without proof the conjecture is mathematically not valid. Finally proving the statement is really hard, as the case of Fermat's Last Theorem exemplifies. Fermat never gave a proof for his theorem even though he claimed to have found it. Mathematicians tried to find a counterexample for centuries with various methods, studying bigger and bigger numbers. In the end, Andrew Wiles published its proof after over 350 years, going down into the annals of mathematical history.[31] The abc-Conjecture does seem to hold for a huge amount of numbers as computations show. However, this doesn't mean much in the mathematical point of view, since there could always exist a counterexample bigger than all the numbers tested so far. The goal of this thesis is to find a counterexample to the *abc*-Conjecture. This would have a big impact on mathematics, since the Conjecture is not only assumed by many to be true, but there are many conjectures which can be proven with the help of the abc-Conjecture. Mathematicians would have to find new ways and develop new ideas to prove those conjectures. For me this subject is particularly interesting as it not only includes the thrill of an unsolved mathematical problem, but also a programming aspect which is something I wanted to be a part of my Bachelor thesis. The subject helps me develop my programming skills, while it also gives me the opportunity to thoroughly explore the *abc*-Conjecture.

Chapter 2

The *abc*-Conjecture

2.1 The Conjecture

[32]

Definition 1. Let $n \in \mathbb{N}$. The *radical* rad(n) is defined as the product of the distinct prime factors of n.

Lemma 1. Let $a, b, c \in \mathbb{N}$. If a, b and c are coprime, then rad(abc) = rad(a)rad(b)rad(c).

Proof. The proof follows directly from the assumption of coprimality. \Box

Lemma 2. There are infinitely many triples (a, b, c) of coprime positive integers such that a + b = c.

Proof. For $n \in \mathbb{N}_{\geq 2}$, we have that 1, n-1 and n are coprime.

Definition 2. Let $a, b, c \in \mathbb{N}$. The quality of a, b, c is defined as q(abc) = log(c)/log(rad(abc)).

The ABC-Conjecture. [33] For every $\epsilon \in \mathbb{R}_{>0}$, there exist only finitely many triples (a,b,c) satisfying a + b = c with gcd(a, b) = 1, such that

$$c > rad(abc)^{1+\epsilon}.$$
(2.1)

or, equivalently:

$$c < K_{\epsilon} rad(abc)^{1+\epsilon} for some K_{\epsilon}.$$
(2.2)

A triplet of this form will be called an *abc-triplet*.

This implies :

For all $\epsilon \in \mathbb{R}$, there exist only finitely many triples (a,b,c) satisfying a + b = c with gcd(a, b) = 1, such that

$$q(a,b,c) > 1 + \epsilon. \tag{2.3}$$

This follows directly from the definition of the quality in Definition 2.

Remark 1. The statement a + b = c with gcd(a, b) = 0 implies that a, b and c are coprime. This can easily be checked by assuming that $gcd(a, c) \neq 0$ which leads to the contradiction $gcd(a, b) \neq 0$.

2.2 History of the ABC Conjecture

During his research in 1983, Richard Mason realised that a special case of Fermat's Last Theorem had never been explored; First powers with restrictions on the variables, that is, the equation a + b = c with certain restrictions on a, band c. Without restrictions on a, b and c, the equation can easily be solved in terms of two of the variables. However, Mason started by analysing the case of a, b and c being polynomials over a field.[30] This led him to the rediscovery of the **Mason–Stothers theorem**, published by W.Wilson Stothers in 1981:[34]

Theorem 1. Let \mathbb{F} be a field and a(X), b(X), $c(X) \in \mathbb{F}[X]$ be relatively prime polynomials with a + b = c and a', b' and c' being not all zero. Then,

$$max(deg(a), deg(b), deg(c)) \le deg(rad(abc)) - 1$$

where rad(abc) is the product of the distinct prime factors of abc in $\mathbb{F}[X]$. This is equivalent to the statement:

$$deg(c) \le n_0(abc) - 1,$$

where n_0 is the number of distinct zeroes of a polynomial.

The following proof was proposed in "N. SNYDER, An Alternate Proof of Mason's Theorem, *Elemente der Mathematik*, Birkhäuser Verlag, Basel, 2000, Ausgabe 55".[35] The proof is based off the subsequent lemma which can also be found in Snyder's paper:

Lemma 3. Let $f(X) \neq 0, f \in \mathbb{F}[X]$. Then

$$deg(f) \le deg(gcd(f, f')) + n_0(f)$$

where $n_0(f)$ is the number of zeroes of f in \mathbb{F} .

Proof. Let $f(X) = c \prod_{i=1}^{n_0(f)} (X - \alpha_i)^{m_i}$. By derivation, we get:

$$f'(X) = cm_1(X - \alpha_1)^{m_1 - 1} \prod_{i=2}^{n_0(f)} (X - \alpha_i)^{m_i} + c(X - \alpha_1)^{a_1} \frac{d}{dX} \prod_{i=2}^{n_0(f)} (X - \alpha_i)^{m_i}$$

This gives $(X - \alpha_1)^{m_1 - 1} | gcd(f, f')$. Or similarly, $(X - \alpha_i)^{m_i - 1} | gcd(f, f')$, which leads to $\left(\prod_{i=1}^{n_0(f)} (X - \alpha_i)^{m_i - 1}\right) | gcd(f, f')$. Or $deg(f) = deg(\prod_{i=1}^{n_0(f)} (X - \alpha_i)^{m_i - 1}) + n_0(f)$. Thus $deg(f) - n_0(f) \leq deg(gcd(f, f'))$, which proves the lemma. \Box

From this follows the proof of the Mason–Stothers theorem:

Proof. We have: $a+b=c \implies a'+b'=c'$, and by multiplying the left hand side by a' and the right hand side by a, we get: aa'+ba'=ca' and a'a+b'a=c'a. This leads to a'b-ab'=a'c-ac', which shows that gcd(a,a'), gcd(b,b') and gcd(c,c') all divide a'b-ab'. As a, b and c, and thus a', b' and c' are relatively prime, we get that

$$gcd(a, a')gcd(b, b')gcd(c, c')|a'b - ab'.$$
(2.4)

Or $a'b - ab' \neq 0$, as this would lead to a|a'b, which gives $a|a' \implies a' = 0$ as a and b are relatively prime. The same arguments lead to b' = 0 and c' = 0, which contradicts the assumptions. The equation 2.4 now gives us

$$deg(gcd(a, a')) + deg(gcd(b, b')) + deg(gcd(c, c')) \le deg(a) + deg(b) - 1 \quad (2.5)$$

This comes from the fact that deg(a'b - ab') = max(deg(a'b), deg(ab')) = deg(a) + deg(b) - 1. Adding deg(c) to both sides in equation 2.5 leads to

$$deg(c) \leq deg(a) - deg(gcd(a, a')) + deg(b) - deg(gcd(b, b')) + deg(c) - deg(gcd(c, c')) - 1 - deg(gcd(c, c')) - 1 - deg(gcd(c, c')) - de$$

By applying the lemma, we get that

$$deg(c) \le n_0(a) + n_0(b) + n_0(c) - 1 = n_o(abc) - 1$$

as a, b and c are relatively prime.

Mason realized that the integer analogue of this theorem is the following: Let a + b = c, where a, b and c are relatively prime, then the number of prime factors of each of a, b and c is strictly less than that of distinct prime factors of *abc*. Unfortunately, this statement is false as for example a = 1, b = 7, c = 8 is a counterexample. In 1985 however, Masser and Osterlé corrected the statement and came up with a conjecture that they found no numerical contradiction for. This conjecture is known as the *abc*-conjecture and the main subject of this Bachelor thesis.

2.3 Attempted proofs and Consequences

Throughout the years, there have been several attempted proofs of the *abc*-Conjecture. In 2007 for example, during a conference at Columbia University in New York, French mathematician Lucien Szipro presented a proof attempt which was shortly thereafter found to be false.[36][37]

The most important proof attempt yet was published in 2012 by Japanese mathematician Shinichi Mochizuki in a paper introducing a new theory of mathematics called Inter-Universal Teichmüller Theory(IUT).[38] Mochizuki claims that with IUT, he found proofs for various number theoretical conjectures including Szipro's Conjecture and Vojta's Conjecture, as well as the abc-Conjecture. However, Mochizuki's proof has not yet been accepted, as several mathematicians claim that a proof in one of the paper's corollaries (Cor. 3.12) is fundamentally flawed with mathematicians Peter Scholze and Jakob Stix describing it as a "serious, unfixable gap". Since the release of the paper, mathematicians from all over the world are trying to get a grasp on Mochizuki's proof, but its size and difficulty are a major obstacle for most of them. Only a handful of mathematicians, which are all in "Mochizuki's orbit", as Brian Conrad of Stanford University stated, have accepted the proof so far. Despite several workshops on Inter-Universal Teichmüller Theory and a personal meeting between Mochizuki, Scholze and Stix, there has been no outcome as to the correctness of the paper, and many mathematicians consider the *abc*-Conjecture to still be an open problem.[39]

Proving the *abc*-Conjecture would not only prove other conjectures, but it would also lead to much simpler proofs of several important theorems.

2.3.1 Simpler Proofs of existing Theorems

Some of the theorems which can be proven with the *abc*-Conjecture include:

- The Mordell conjecture, proven in 1983 by German mathematician Gerd Faltings.
- Fermat's Last Theorem for $n \ge 6$, proven in 1994 by English mathematician Andrew Wiles.
- Roth's Theorem, proven in 1955 by Harold Davenport and Klaus Roth.

The *abc*-Conjecture leads to a proof of Fermat's Last Theorem[40] with $n \ge 6$:[41]

Theorem 2 (Fermat's Last Theorem with $n \ge 6$). Let $n \in \mathbb{N}$, $n \ge 6$. Then there exist no strictly positive integer solutions for the equation $a^n + b^n = c^n$.

Proof. Assume the *abc*-Conjecture, and let $a^n + b^n = c^n$ and, without loss of generality, a, b (and thus c) be coprime. By the *abc*-Conjecture, we have that:

$$c^{n} < rad(a^{n}b^{n}c^{n})^{2} = rad(abc)^{2} \le (abc)^{2} < (c^{3})^{2} = c^{6}$$

Thus, $c^n < c^6$, which implies that for all integer solutions of the equation, n < 6. This proves Fermat's Last Theorem for $n \ge 6$.

2.3.2 Conjectures based on the *abc*-Conjecture

The proof of several other conjectures would follow from the *abc*-Conjecture:

- The weak form of Hall's Conjecture, proposed by Stark and Trotter around 1980.[42]
- Marius Overhold proved in 1993 that the *abc*-Conjecture implies that Brocard's Equation has only finitely many solutions.
- The Fermat-Catalan Conjecture, which asks for integer solutions of the equation $a^m + b^n = c^k$.[43]
- The Erdös-Woods Conjecture.

Chapter 3

The search for a counterexample

3.1 Our method

The intent of this paper is to find a counterexample to the *abc*-Conjecture, i.e. to construct an infinite sequence of triples with quality higher than $1 + \epsilon$ for $\epsilon > 0$. To reach this goal, we use the databank of the ABC@Home project which contains all *abc*-triples where *c* has less than 19 digits. We then compare the entries of this databank with the databank of the On-Line Encyclopedia of Integer Sequences (OEIS), a site containing hundreds of thousands of known integer sequences. Our goal is to find a sequence which contains many *abc*-triplets, and to check if other elements in the sequence yield more *abc*-triplets. If there were such a sequence which would give us an infinity of triplets with quality more than $1 + \epsilon$ for a fixed $\epsilon > 0$, the abc-Conjecture would be proven false. In the following sections, we will take a more in-depth look at the databanks, as well as the algorithm calculating the *abc*-triplets for the ABC@Home project.

3.1.1 The ABC@Home project and its Databank

[44] In 2007, following a 2005 number theory seminar, Bart de Smith of the University of Leiden started a project called "Reken mee met ABC" ("Take ABC into account") with the intent of promoting sciences.[45] It contained the ABC@Home network, which was a network computing project whose goal it was to find all *abc*-triplets with *c* having less than 19 digits. This goal was achieved in 2011, with the project discovering 14482065 valid triplets. In the following years, the project continued, aiming to find all triplets with *c* between 10^{18} and 2^{63} . After reaching this goal in 2015, the search for greater triplets has been dismissed by the ABC@Home project. The project produced a total of 23827716 triplets, which are available for download on the project's homepage.

3.1.2 The OEIS-Databank

[46] The idea for a databank of all known integer sequences started in 1964 when Neil Sloane, the creator of the OEIS-project, encountered a certain sequence in works which he couldn't find in any book. While looking through books in the hopes of finding his sequence, he started writing all of the sequences he found on cards and arranged them in lexicographic order. In 1973, the sequences he collected were put in a book. The book was in high demand, as many mathematicians found it very useful and in 1995, a revised edition was published with the help of Canadian mathematician Simon Plouffe, containing more that double the sequences of the initial book. In 1996, Sloane, who worked for AT&T at the time, launched the On-Line Encyclopedia of Integer Sequences. During the following years, the database grew heavily, which lead to Sloane not being able to process all the new submissions he got every day. Thus, in 2009, he decided to set up the non-profit OEIS Foundation Inc., which allowed him to maintain the database with the help of a group of editors. As of today, the database contains over 300.000 sequences, each sequence containing also information on its apparition, author and code to calculate the sequence in Maple.[47]

3.1.3 The ABC@Home Algorithm

The following is a short summary of the algorithm used in the ABC@Home network, as described in the 7th chapter of the PhD thesis of Willem Jan Palenstijn of Leiden University.[48]

He starts by giving an upper bound on the number of *abc*-triples where *c* is less than a certain integer *N*. He concludes that for every $\epsilon > 0$, that number of triples is at most $O(N^{2/3+\epsilon})$, i.e. the number of triples grows at most with exponent $\frac{2}{3}$. He proves this using a theorem from one of the earlier chapters which gives a relation between the number *N* and the cardinal of $\{x < N | rad(x) < N^{\alpha}\}$ for some $0 < \alpha < 1$. Furthermore, he cites a theorem by Sander Dahmen of the Vrije Universiteit Amsterdam setting $exp((log(N))^{1/2-\epsilon})$ as the lower bound of the aforementioned *abc*-triples.

He then gives an algorithm to enumerate all square-free integers x with $a \le x \le b$ with their corresponding prime factors:

- Create two lists, one for the integers x, and one for their corresponding prime factorization list.
- Run over the primes $p^2 < b$ and check if they divide x. If p divides x, and p^2 doesn't, add p to the list of factors of x.

This is only the basic version of this algorithm which does not contain all the necessary steps. For a more detailed version, one can consult section 7.7 of the PhD thesis, which contains this algorithm under the name 'Algorithm 7.7'.[48]

Subsequently, Palenstijn describes the ABC@Home algorithm which, given integers N, as well as some bounds m_x, M_x, m_y, M_y on a and b, and square-free integers t and g with t|g, computes all *abc*-triples with c < N, rad(a) and rad(b) bounded by m_x, M_x and m_y, M_y respectively, and gcd(x, t) = g:

- Compute a list P of all prime numbers $p < N(\frac{1}{3})$.
- Compute lists L_x and L_y of square-free integers between the bounds of a and b.
- Generate sorted lists X and Y of integers less than N with radical in L_x and L_y respectively.

- Partition X and Y into subsets.
- Loop over pairs (X_i, Y_j) of these subsets:
 - Generate a table $T(X_i, Y_j)$ which for all $(x, y) \in (X_i, Y_j)$ sets r = 1, u = x + y.
 - Loop over P:
 - Loop over $y \in Y_j$ and find all $x \in X_i$ such that p|x+y.
 - Divide all factors p from u.
 - Multiply r with p.
 - Now, loop over all x + y in T.
 - Skip the triple if x and y are not coprime or r > c/rad(xy).
 - By dividing by primes in P, filter out all the triples which are no abc-triples.

This algorithm returns a list of all *abc*-triples with c < N. The integer t is used to speed up the process by getting rid of all x and y which are coprime with t. Setting it to t = 2 * 3 * 5 * 7 = 210 seems, by experimentation, to be a good value.



Figure 3.1: \log_{10} of the number of triples below 10^N

3.2 Our computations

The goal of this thesis is to find a sequence of *abc*-triples with constant or increasing quality. If we were to find such a sequence, we would have a valid counterexample to the conjecture. Our search consists of comparing the ABC@Home-Databank with the OEIS-Databank and finding sequences which contain a certain amount of c-entries of the databank. Now, one can match each element of the sequence to its corresponding triples and try to find a subsequence which has at least constant quality.

First, one has to get both databanks in an adequate text format. The code was written in Python 2 with the use of the Canopy-IDE.[49]

3.2.1 Getting the files ready

We first wrote the following program which extracts only the *c*-entries of the ABC@Home-Database. It gets rid of duplicates and writes a file with all the *c*-entries that are left. This new file, 'c_no_duplicates.txt', will be used to match entries with the OEIS-Database file.

```
def create_only_c():
```

```
file1 = open("abc_databank.txt", "r+");
file2 = open("all_c.txt", "w");
file2 = open("c_no_duplicates.txt", "w");
#extracts only the first entry (c)
for line in file1:
    list1 = line.split(" ");
    file2.write(list1[0] + "\n");
#gets rid of duplicate entries
seen_c = set()
for line in file2:
    if line not in seen_c:
        file2.write(line);
        seen_c.add(line);
```

	9
2	32
3	49
4	64
5	81
6	81
7	125
8	128
9	225
10	243
11	245
12	250

Figure 3.2: First lines of c_no_duplicates.txt

We now have to do the same thing with the OEIS-Databank. We wrote the program create_OEIS(), which first converts the OEIS-Database into a more easily readable format and then extracts all the duplicates in each line, and sorts each line by integer values.

def create_OEIS():

```
file1 = open("OEIS-Databank.txt", "r+");
file2 = open("Converted_OEIS.txt", "w");
for line in file1:
       #replaces the comma from the original file with spaces
   sequence = line.replace(",", " ");
   list1 = sequence.split(" ");
       #gets rid of the beginning and the end of the line which are
           not numbers
   del list1[0];
   del list1[0];
   list1 = list1[:-1];
       #puts the line entries in a dictionary format and sorts them
           with duplicate removal
   list2 = dict.fromkeys(list1);
   list2 = map(int,list2);
   list2 = sorted(list2);
       #if there are less than 3 entries, we get rid of the line
   if len(list2) < 3:</pre>
       pass;
   else:
       file2.write(" ".join(map(str, list2)));
       file2.write(" "+"\n");
```

```
0 1 2 3 4 5 6 10 12 13 14 15 50 51 52 267
    1 2 3 4 5 6 7 8 9 10 12
 3
    1 2 3 4 5 6 7 8 9 10 12
    1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
 4
    1 2 3 4 5 6 7 8 11 12 15 16 19 22 25 28 31 34 40 43 49 52 58 64 70 76 82 88
5
    1 2 3 4 5 6 8 10 12 15 18 22 27 32 38 46 54 64 76 89 104 122 142 165 192 22
 6
7
    1 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 16 \ 18 \ 20 \ 22 \ 24 \ 28 \ 30 \ 32 \ 36 \ 40 \ 42 \ 44 \ 46 \ 48 \ 52 \ 58 \ 60 \ 66
8
    1 2 4 8 9 18 23 44 63 122 190 362 612 1162 2056 3914 7155 13648 25482 48734
9
    1 2 4 8 10 20 30 56 94 180 316 596 1096 2068 3856 7316 13798 26272 49940 95
    0 1 2 4 5 10 14 26 42 78 132 249 445 842 1561 2988 5671 10981 21209 41472 8
10
    1 2 3 4 5 7 8 9 11 13 16 17 19 23 25 27 29 31 32 37 41 43 47 49 53 59 61 64
   1 2 4 6 10 16 30 52 94 172 316 586 1096 2048 3856 7286 13798 26216 49940 95
12
```

Figure 3.3: First lines of Converted_OEIS.txt

3.2.2 The search algorithm

Now, we will explain the search algorithm. For optimisation purposes, we wrote the search algorithm so that it can use all cores of the computing system, as Canopy usually only runs on one core. Therefore, as the computing system we used has 16 cores, we split the Converted_OEIS.txt file into 16 different files:

```
#divides the Converted_OEIS.txt file into files the number of cores
def dividefile():
    cores = mult.cpu_count()
    for i in range(cores):
        file1 = open("Converted_OEIS"+str(i)+".txt", "w");
        j = 0;
        with open("Converted_OEIS.txt", "r+") as file2:
            #as the Converted_OEIS.txt has 157448 lines, we split it in
            chunks of length 157448/8 = 19681.
```

```
for line in islice(file2, i*19681, (i+1)*19681):
    file1.write(line);
```

Now that all files are in an adequate format to work with, we wrote the search algorithm which compares the two files with each other. For every (i < number of cores), the algorithm takes a sequence of the Converted_OEIS*i*.txt file and, for each element in the sequence, checks if it is contained in the c_no_duplicates.txt file. If it is, it increases the corresponding count of the sequence by one. If the total count of the sequence is over 15, the sequence is then written to the file Validsequences*i*.txt with its corresponding place number in the Converted_OEIS.txt file.

```
#The search algorithm for the file Converted\_OEISi
def search(i):
   file1 = open("Converted_OEIS" + str(i) + ".txt", "r");
   sequencenumber = int(i)*19681;
   resultlist = [];
   for sequence in file1:
       sequencenumber += 1;
       count = 0;
       list1 = sequence.split(" ");
       validentries = [];
       for value in list1:
           if value == "\n":
              pass;
           elif (int(value) > 10**18): #as the ABC@Home Databank goes up
               to 10<sup>18</sup>, checking for bigger numbers is pointless
              pass;
           else:
              file2 = open("c\_no\_duplicates.txt", "r");
              for line in file2:
                  if int(line) == int(value):
                      validentries.append(value);
                      count+=1;
                  if int(line) > int(value):
                      break; #as the file is ordered by size, if we are
                          already past the sequence entry, we will not
                          find a corresponding entry in the
                          c\_no\_duplicates.txt file.
       if count > 15:
           resultlist.append(validentries);
           with open("Validsequences"+ str(i) +".txt", "a") as f:
              f.write("sequence number: " + str(sequencenumber) + " /
                   ");
              for item in validentries:
                  f.write(str(item)+ " ");
              f.write("\n");
```

```
#The following distributes the search algorithm on all cores.
if __name__ == '__main__':
    cores = mult.cpu_count();
    __spec__ = "ModuleSpec(name='builtins', loader=<class
        '_frozen_importlib.BuiltinImporter'>)"
    for k in range(cores):
        p = mult.Process(target = search, args = (str(k)))
        p.start()
```

```
sequence number: 72 / 32 64 128 256 512 2048 4096 8192 16384 32768 65536 13
    sequence number: 237 / 9 81 243 729 2187 6561 19683 59049 177147 531441 159
 3
    sequence number: 283 / 9 49 64 81 225 256 289 625 676 729 961 1369 1681 230
    sequence number: 295 / 64 256 4096 16384 65536 262144 1048576 4194304 16777
 4
    sequence number: 344 / 125 625 3125 15625 78125 390625 1953125 9765625 4882
    sequence number: 413 / 49 343 2401 16807 117649 823543 5764801 40353607 282
 6
    sequence number: 526 / 9 32 128 512 2048 3584 6144 8192 24576 32768 131072
8
    sequence number: 570 / 64 125 343 512 729 1331 2197 4096 4913 5832 6859 800
    sequence number: 575 / 81 256 625 2401 4096 6561 10000 14641 28561 50625 65
9
    sequence number: 576 / 32 243 3125 16807 32768 59049 100000 161051 371293 5
    sequence number: 784 / 9 81 243 729 2187 6561 8748 19683 59049 78732 118098
12 sequence number: 1006 / 64 729 4096 15625 46656 117649 262144 531441 100000
```

Figure 3.4: First lines of Converted_OEIS0.txt

As we now have 16 different files, with each file containing a certain amount of sequences that have more than 15 entries corresponding with known *abc*triples, we only have to write all of these files into one file. Since we realised that the number of sequences is still relatively large, we only kept the sequences with at least 30 entries.

```
def Collect_sequences():
    file1 = open("Validsequences.txt", "w");
    for i in range(10):
        file2 = open("Validsequences" + str(i) + ".txt", "r");
        for line in file2:
            list1 = line.split(" ");
            if len(list1) >= 34: #the first 4 entries are no members of
                the sequence
            for j in range(len(list1)):
                file1.write(list1[j] + " ");
            file1.write("\n");
            file1.writ
```

This finally leaves us with 37 different sequences that have at least 30 entries corresponding to the *c*-part of known *abc*-triples. Of course, only having the *c*-entries does not yet provide a useful way for extracting a subsequence with certain characteristics on its *abc*-quality. That's why we first need to assign each entry in the Validsequences.txt all of its known *abc*-triples with their relative prime decomposition.

PARI/GP returns the factorization of a number by matrix representation. There are three forms of returns from PARI/GP for the factorization of a number N:

As we did not find a good method for calculating prime decomposition in Python, we used the PARI/GP program instead. This program allowed us to quickly compute the prime decomposition of all our *abc*-triples.

- If N = 1, then PARI/GP returns "matrix(0,2)".
- If $N = p^n$ for p prime, then PARI/GP returns "Mat([p, n])".
- If $N = \prod_{i=1}^{m} p_i^{n_i}$ for p_i prime, then PARI/GP returns " $[p_1, n_1; ...; p_m, n_m]$ ".

With the goal of making a file that PARI/GP is able to read, the following algorithm first assigns the entries in Validsequences.txt to their corresponding *abc*-triples and then writes the prime decomposition command for PARI/GP into the PARI-file.txt file. This file will then be read by PARI/GP, which will then write the prime decomposition to the file abcFactors.txt.

```
def assigntriple():
   file1 = open("Validsequences.txt", "r");
   for line in file1:
       list1 = line.split(" ");
       if len(list1) == 2: #gets rid of empty lines
           pass;
       else:
           del list1[-1];
           del list1[3];
           del list1[2];
           del list1[1];
           del list1[0];
                #gets rid of the sequence name and number
           for value in list1:
              file2 = open("c_no_duplicates.txt", "r");
              for double in file2:
                  list2 = double.split(" ");
                  if int(value) == int(list2[0]):
                      a = int(list2[1][:-1]);
                      c = int(list2[0]);
                      b = c-a;
                      file3 = open("PARI-file.txt", "a");
                      file3.write("factor("+str(c)+")"+"\n"+r"\w
                          C:\Users\username\abcFactors.txt"+"\n"+
                      "factor("+str(a)+")"+"\n"+r"\w
                          C:\Users\username\abcFactors.txt"+"\n"+
                      "factor("+str(b)+")"+"\n"+r"\w
                          C:\Users\username\abcFactors.txt"+"\n")
                  elif int(value) < int(list2[0]):</pre>
                      break;
```

This now gives us the file abcFactors.txt in which the factorisations of all triples corresponding to the entries in Validsequences.txt are stored. This will now be modified by algorithm 4.1 to give us a final file abcFinal.txt which will include the sequences of triples with their corresponding factorization as well as their quality and their merit.

```
factor(9)
     \w C:\Users\username\abcFactors.txt
 3
    factor(1)
 4
    \w C:\Users\username\abcFactors.txt
 5
    factor(8)
 6
    \w C:\Users\username\abcFactors.txt
    factor(32)
 8
    \w C:\Users\username\abcFactors.txt
 9
    factor(5)
10
    \w C:\Users\username\abcFactors.txt
11
    factor(27)
12 \w C:\Users\username\abcFactors.txt
```

Figure 3.5: First lines of PARI-file.txt

```
Copyright (C) 2000-2018 The PARI Group
PARI/GP is free software, covered by the GNU General Public License, and comes
Type ? for help, \q to quit.
Type ?17 for how to get moral (and possibly technical) support.
parisize = 8000000, primelimit = 500000
(17:08) gp > \r C:\Users\username\PARI-file.txt
```

Figure 3.6: How to input PARI-file.txt into PARI/GP

3.3 Our results

Our goal was to extract a subsequence with increasing quality and checking if the quality would increase further. To do so, we wrote the following algorithm qualitysequences(). It builds a sequence with increasing quality out of the abcFinal.txt-sequences. Out of all the triples with highest c, it takes the one with the biggest quality and then inductively repeats this step for all the triples with less high c-entry. This gives a subsequence of the original sequence which has increasing quality. To make the algorithm easier, it operates on the file abcFinalreversed.txt, which is the abcFinal.txt file reversed.

```
def qualitysequences():
    file1 = open("abcFactorsreversed.txt", "r");
    file2 = open("Qualityincrease.txt", "a");
    C = 10**19;
    listquality = [100]; #initial quality, should be higher than all the
        other qualities in the sequence
    listC = [0];
    i = 0;
    linelist = [("sequence " + str(28-i) + ": \n")]; #it goes from
        sequence 28 to sequence 1
    for line in file1:
        list1 = line.split("=");
        c = int(list1[1][1:][:-1]);
        quality = float(list1[8][1:][:-7]);
        if c > C: #a new sequence starts
```

1	Mat([3, 2])
2	matrix(0,2)
3	Mat([2, 3])
4	Mat([2, 5])
5	Mat([5, 1])
6	Mat([3, 3])
7	Mat([2, 6])
8	<pre>matrix(0,2)</pre>
9	[3, 2; 7, 1]
10	Mat([3, 4])
11	<pre>matrix(0,2)</pre>
12	[2, 4; 5, 1]

Figure 3.7: An extract of the abcFactors.txt file

1	c =	9 = Mat([3, 2]) b = 1 = matrix(0,2) a = 8 = Mat([2, 3]) radical = 6 quality = 1.226294385530917 merit = 0.05351103517746218
2	с =	32 = Mat([2, 5]) b = 5 = Mat([5, 1]) a = 27 = Mat([3, 3]) radical = 30 quality = 1.018975235452531 merit = 0.0014991077590693783
3	с =	64 = Mat([2, 6]) b = 1 = matrix(0,2) a = 63 = [3, 2; 7, 1] radical = 42 quality = 1.1126941404922133 merit = 0.06258515100661385
4	с =	81 = Mat([3, 4]) b = 1 = matrix(0,2) a = 80 = [2, 4; 5, 1] radical = 30 quality = 1.292030029884618 merit = 0.3550696325974392
5	с =	81 = Mat([3, 4]) b = 32 = Mat([2, 5]) a = 49 = Mat([7, 2]) radical = 42 quality = 1.1757189916348774 merit = 0.15216194048019044
6	с =	128 = Mat([2, 7]) b = 3 = Mat([3, 1]) a = 125 = Mat([5, 3]) radical = 30 quality = 1.4265653296335432 merit = 0.7575819399084418
7	с =	243 = Mat([3, 5]) b = 1 = matrix(0,2) a = 242 = [2, 1; 11, 2] radical = 66 quality = 1.311101219926227 merit = 0.580914433583302
8	с =	256 = Mat([2, 8]) b = 13 = Mat([13, 1]) a = 243 = Mat([3, 5]) radical = 78 quality = 1.2727904629543532 merit = 0.4771348987168572
9	с =	256 = Mat([2, 8]) b = 81 = Mat([3, 4]) a = 175 = [5, 2; 7, 1] radical = 210 quality = 1.0370424407259895 merit = 0.01230087798830571
10	с =	512 = Mat([2, 9]) b = 5 = Mat([5, 1]) a = 507 = [3, 1; 13, 2] radical = 390 quality = 1.0456203807611666 merit = 0.02217776739979556
11	c =	512 = Mat([2, 9]) b = 169 = Mat([13, 2]) a = 343 = Mat([7, 3]) radical = 182 quality = 1.198754152359422 merit = 0.33908131024981064

Figure 3.8: An extract of the abcFinal.txt file

```
i += 1;
   C = c;
   Quality = quality;
   plt.figure(i);
   plt.plot(listC[1:], listquality[1:], "r--");
   plt.title(listC, fontsize = 6);
   plt.figure(i).show();
   plt.savefig("qualityincrease"+str(i)+".png");
   listquality = [100, quality]; #reset
   listC = [0, c]; #reset
   for value in linelist:
       file2.write(str(value));
   file2.write("\n");
   linelist = [("sequence " + str(28-i) + ": \n"), 0];
if c < C:
   if quality < listquality[len(listquality)-1]: #if the quality</pre>
       is less than that of the previous c
       listquality.append(quality);
       listC.append(c);
       Quality = quality;
       C = c;
       linelist.append(line);
   else:
       C = c;
       Quality = quality;
if c == C :
   if (quality > Quality) and (quality <=</pre>
        listquality[len(listquality)-2]):
```

```
Quality = quality;
listquality[len(listquality)-1] = quality;
linelist[len(linelist)-1] = line;
else:
   pass;
```

This gives us the file Quality increase txt which contains the subsequences of abcFinal.txt with maximal increasing quality. Disappointingly, there is no apparent pattern in the a and b entries of the subsequences, and none of those are sequences contained in the OEIS-Databank. This essentially concludes the project, since we were not able to extract a formula for a sequence of increasing quality.



Figure 3.9: Increasing quality in sequence nr. 26

We were nevertheless not fully satisfied with having no result, which is why we tried to at least construct a subsequence of increasing merit. These subsequences can easily be computed by changing "float(list1[8][1:][:-7])" to "float(list1[9][1:][:-1])" in the previous algorithm, which instead of the quality now extracts the merit. This gave us a whole new file of subsequences with increasing merit, of which disappointingly none could be matched to the OEIS-Databank, just as in the previous case with the quality. This finally concludes the project.

We believe that our method has fully been investigated with no result. Therefore, further exploration of the method seems futile. This leaves the *abc*-Conjecture still standing and future proofs will most likely have to include newly developed areas of mathematics to get results.

Chapter 4

Appendix

4.1 The conversion algorithm

The following algorithm converts the abcFactors.txt file, created by PARI/GP, into the nicely structured abcFinal.txt file.

```
def convert():
   file1 = open("abcFactors.txt", "r+");
   file3 = open("abcFinal.txt", "w");
   i = 0; #counts to three to always get the corresponding triple
       values together
   primes = []; #saves all the primes of the current triple
   C = 0;
   for word in file1:
       if word[0] == "M": #is of the kind Mat([a, b])
           control = False;
          list1 = word.split(",");
          list1[0] = list1[0][5:];
          list1[1] = list1[1][:-3];
          print(list1);
       elif word[0] == "m": #is of the kind matrix(a, b)
           control = False;
          list1 = word.split(",");
          list1[0] = list1[0][7:];
          list1[1] = list1[1][:-2];
          print(list1);
       else: #is of the kind [a, b; ... ;c, d]
          control = True;
          list1 = []
          list2 = word.split(";");
          l = len(list2);
          list2[0] = list2[0][1:];
          list2[1-1] = list2[1-1][:-2];
          for value in list2:
              list3 = value.split(",");
              list1.append(list3);
```

```
print(list1);
length = len(list1);
a = 1;
if control == True: #if the line is of the kind [a, b; ... ;c, d]
   if i == 0: # c of the abc-triple
       for j in range(0, length, +1):
          primes.append(int(list1[j][0]));
          a = a*(int(list1[j][0])**int(list1[j][1]));
       C = a; #saves the value of c for further computation of
           the quality
       file3.write("c = " + str(a) + " = ")
       file3.write(word[:-1] + " ");
       i += 1;
   elif i == 1: # b of the abc-triple
       for j in range(0, length, +1):
           primes.append(int(list1[j][0]));
           a = a*(int(list1[j][0])**int(list1[j][1]));
       file3.write("b = " + str(a) + " = ")
       file3.write(word[:-1] + " ");
       i += 1;
   elif i == 2: # a of the abc-triple
       for j in range(0, length, +1):
          primes.append(int(list1[j][0]));
           a = a*(int(list1[j][0])**int(list1[j][1]));
       radical = 1;
       for k in range(len(primes)):
          radical = radical*primes[k];
       quality = (math.log(C))/(math.log(radical));
       merit =
           ((quality-1)**2)*(math.log(radical))*(math.log(math.log(radical)));
       file3.write("a = " + str(a) + " = ")
       file3.write(word[:-1] + " ");
       file3.write(" radical = " +str(radical) + " quality = "
           +str(quality) + " merit = " +str(merit));
       file3.write("\n");
       i = 0;
       primes = [];
else:
   if i == 0: # c of the abc-triple
       if int(list1[0]) == 0:
          a = 1;
       else:
          a = (int(list1[0])**int(list1[1]));
          primes.append(int(list1[0]));
       C = a;
       file3.write("c = " + str(a) + " = ")
       file3.write(word[:-1] + " ");
       i += 1;
   elif i == 1: # b of the abc-triple
```

```
if int(list1[0]) == 0:
      a = 1;
   else:
      a = (int(list1[0])**int(list1[1]));
      primes.append(int(list1[0]));
   file3.write("b = " + str(a) + " = ")
   file3.write(word[:-1] + " ");
   i += 1;
elif i == 2: # a of the abc-triple
   if int(list1[0]) == 0:
      a = 1;
   else:
      a = (int(list1[0])**int(list1[1]));
      primes.append(int(list1[0]));
   radical = 1;
   for k in range(len(primes)):
       radical = radical*primes[k];
   quality = (math.log(C))/(math.log(radical));
   merit =
        ((quality-1)**2)*(math.log(radical))*(math.log(math.log(radical)));
   file3.write("a = " + str(a) + " = ")
   file3.write(word[:-1] + " ");
   file3.write(" radical = " +str(radical) + " quality = "
       +str(quality) + " merit = " +str(merit));
   file3.write("\n");
   i = 0;
   primes = [];
```

4.2 Computed results

The following are the OEIS-Number and -Description of 12 extracted sequences with significant enough size, their subsequence of increasing quality, as well as the figures of their increasing quality and merit. The PARI/GP factorization has been explained in section 3.2.

4.2.1 Sequence 1

A065771 Prime powers n such that both $\tau(n^2)$ and $\sigma(n^2)$ are composite numbers.

```
c = 16777216 = Mat([2, 24]) b = 2197 = Mat([13, 3]) a = 16775019 = [3, 6; 23011, 1] radical = 1794858 quality = 1.155210286177332 merit = 0.9252983022149146
c = 13845841 = Mat([61, 4]) b = 133632 = [2, 9; 3, 2; 29, 1] a = 13712209 = [7, 2; 23, 4] radical = 1708854 quality = 1.1457817990087676 merit = 0.8124692544875347
c = 12117361 = Mat([53, 4]) b = 9482 = [2, 1; 3, 4; 61, 1] a = 8024832 = [2, 8; 3, 6; 43, 1] radical = 1811418 quality = 1.1318930814861106 merit = 0.6687525976855677
c = 7890481 = Mat([54, 4]) b = 9482 = [2, 1; 3, 4; 61, 1] a = 7880599 = Mat([199, 3]) radical = 3860202 quality = 1.04710098702477 merit = 0.06183755971651587
c = 4879681 = Mat([47, 4]) b = 1024000 = [2, 13; 5, 3] a = 3855681 = [3, 7; 41, 1; 43, 1] radical = 2485830 quality = 1.0443078666480792 merit = 0.0618375979161587
c = 4194304 = Mat([2, 22]) b = 936279 = [3, 5; 3853, 1] a = 33258025 = [5, 2; 19, 4] radical = 2196210 quality = 1.0443078666480792 merit = 0.0618375297161587
c = 1874161 = Mat([37, 4]) b = 9 = Mat([3, 2]) a = 1874152 = [2, 3; 7, 3; 683, 1] radical = 1061382 quality = 1.044307866480792 merit = 0.0618302302432704755
c = 1594323 = Mat([3, 13]) b = 34322 = [2, 1; 131, 1] radical = 1061382 quality = 1.03270585217635058 merit = 0.04472661867249758
c = 707281 = Mat([23, 4]) b = 248 = Mat([17, 2]) a = 279552 = [2, 10; 3, 1; 7, 1; 13, 1] radical = 213468 quality = 1.020550569737788 merit = 0.01428651857814
c = 13072 = Mat([21, 4]) b = 248 = Mat([17, 2]) a = 279552 = [3, 7; 41, 1] radical = 1213486 quality = 1.020550569737788 merit = 0.0014286548585114
c = 130321 = Mat([19, 4]) b = 46800 = [2, 4; 3, 2; 5, 2; 13, 1] a = 63521 = Mat([17, 4]) b = 46800 = [2, 4; 3, 2; 5, 2; 13, 1] a = 63521 = Mat([17, 4]) b = 46800 = [2, 4; 3, 2; 5, 2; 13, 1] a = 83521 = Mat([17, 4]) b = 46800 = [2, 4; 3, 2; 5, 2; 13, 1] a = 63521 = Mat([17, 4]) b = 245921 = [7, 2; 23, 2] a = 57600 = [2, 8; 3, 2; 5, 2] radical = 82110 quality = 1.001505705544449 merit = 0.02243236296628e-05
```



Figure 4.1: Increasing quality in sequence nr. 3



Figure 4.2: Increasing merit in sequence nr. 3

4.2.2 Sequence 2

A076469 Perfect powers m^k where m is an integer and k > 4. or A246550 Prime powers p^e where p is a prime and e >= 4.

c = 531441 = Mat([3, 12]) b = 49 = Mat([7, 2]) a = 531392 = [2, 6; 19, 2; 23, 1] radical = 18354 quality = 1.342827542997735 merit = 2.635643220813526 c = 524288 = Mat([2, 19]) b = 4913 = Mat([17, 3]) a = 519375 = [3, 1; 5, 4; 277, 1] radical = 141270 quality = 1.1105853299516497 merit = 0.35863535917689343 c = 330625 = Mat([5, 8]) b = 117649 = Mat([7, 6]) a = 272976 = [2, 4; 3, 1; 11, 2; 47, 1] radical = 108570 quality = 1.110421432202936 merit = 0.3486460002140961 c = 371293 = Mat([13, 5]) b = 1539 = [3, 4; 19, 1] a = 369754 = [2, 1; 7, 5; 11, 1] radical = 108570 quality = 1.01137224794165 merit = 0.2942941819566031 c = 161051 = Mat([11, 5]) b = 12096 = [2, 6; 3, 3; 7, 1] a = 148955 = [5, 1; 31, 3] radical = 71610 quality = 1.0725008570014634 merit = 0.03942941819566031 c = 1161051 = Mat([11, 5]) b = 15200 = [2, 2; 5, 5] a = 105149 = [11, 3; 79, 1] radical = 66462 quality = 1.0611574973269209 merit = 0.0994820588198229 c = 117649 = Mat([7, 6]) b = 12500 = [2, 2; 5, 5] a = 105149 = [11, 3; 79, 1] radical = 60430 quality = 1.049297420915307 merit = 0.0994820588198229 c = 117649 = Mat([7, 6]) b = 12500 = [2, 2; 5, 5] a = 105149 = [11, 3; 79, 1] radical = 39322 quality = 1.049297420915307 merit = 0.0994820588198229 c = 117649 = Mat([2, 16]) b = 6487 = [13, 1; 499, 1] a = 59049 = Mat([3, 10]) radical = 39322 quality = 1.049297420915307 merit = 0.0307976912642932 c = 55049 = Mat([2, 16]) b = 63879 = [3, 2; 7, 2; 19, 1] a = 24389 = Mat([2, 3]) radical = 23142 quality = 1.030537292734312 merit = 0.037776912842332 c = 21768 = Mat([2, 15]) b = 8379 = [3, 2; 7, 2; 19, 1] a = 24389 = Mat([2, 3]) radical = 6064 quality = 1.01293742051537 merit = 0.0019408233280691530 c = 8192 = Mat([2, 13]) b = 512 = Mat([11, 1]) a = 8181 = [3, 4; 101, 1] radical = 6066 quality = 1.012461273373 merit = 0.0019408233280691530 c = 2187 = Mat([3, 6]) b = 512 = Mat([2, 9]) a = 1675 = [5, 2; 67, 1] radical = 6060 quality = 1.01036139914193 merit = 0.0019400255575561044 c = 729 = Mat([3, 6]) b = 200 = [2, 3; 5, 2] a = 529 = Mat([23, 2]) rad



Figure 4.3: Increasing quality in sequence nr. 9



Figure 4.4: Increasing merit in sequence nr. 9

4.2.3 Sequence 3

A076470 Perfect powers m^k where m is an integer and k > 5.

c = 1771661 = Mat([11, 6]) b = 91 = [7, 1; 13, 1] a = 1771470 = [2, 1; 3, 11; 5, 1] radical = 30030 quality = 1.3954838404823704 merit = 3.7622641764419975
c = 1594323 = Mat([3, 13]) b = 744 = [2, 8; 29, 1] a = 1586899 = [17, 4; 19, 1] radical = 5620 quality = 1.305873782653318 merit = 2.4476817174888095
c = 1048576 = Mat([2, 20]) b = 50575 = [5, 2; 7, 1; 17, 2] a = 998001 = [3, 6; 37, 2] radical = 132090 quality = 1.17659865737870158 merit = 0.4891073965362553
c = 823543 = Mat([7, 7]) b = 3072 = [2, 10; 3, 1] a = 820471 = [17, 3; 167, 1] radical = 97410 quality = 1.16750959737970158 merit = 0.4891073965362553
c = 531441 = Mat([3, 12]) b = 191 = Mat([19, 11]) a = 531250 = [2, 1; 5, 6; 17, 1] radical = 97410 quality = 1.1105853299516497 merit = 0.51865358917693343
c = 3390625 Mat([2, 19]) b = 4913 = Mat([17, 3]) a = 519375 = [3, 1; 5, 4; 277, 1] radical = 108570 quality = 1.1102451322936 merit = 0.35865358917693934
c = 3390625 Mat([5, 13]) b = 11749 = Mat([7, 6]) a = 272976 = [2, 4; 3, 1; 11, 2; 47, 1] radical = 108570 quality = 1.10421413220336 merit = 0.3546002140961
c = 177147 = Mat([3, 11]) b = 46075 = [5, 1; 19, 1; 97, 1] a = 131072 = Mat([2, 17]) radical = 55290 quality = 1.106265512066344 merit = 0.258060414991
c = 131072 Mat([2, 17]) b = 56 = [5, 1; 19, 1] a = 131072 = Mat([2, 17]) radical = 55290 quality = 1.102346183041 merit = 0.268047377434422
c = 117649 = Mat([7, 6]) b = 324 = [2, 2; 3, 4] a = 117325 = [5, 2; 13, 1; 19, 2] radical = 51870 quality = 1.102346183041 merit = 0.1473283668442908
c = 65536 = Mat([2, 16]) b = 7 = Mat([7, 1]) a = 65529 = [3, 4; 809, 1] radical = 33978 quality = 1.062595375106314 merit = 0.0864574895613126
c = 32768 = Mat([2, 16]) b = 7 = Mat([7, 1]) a = 56529 = [3, 4; 809, 1] radical = 33978 quality = 1.005259537516713 merit = 0.00863553242618
c = 55049 = Mat([2, 16]) b = 7 = Mat([7, 1]) a = 8638 = [2, 2; 11, 4] radical = 32104 quality = 1.03240893725073 merit = 0.00863553242618
c = 23768 = Mat([2, 15]) b = 4387 = [5, 2; 7, 2; 19, 1] a = 24389 = Mat([23, 3]) radical = 3



Figure 4.5: Increasing quality in sequence nr. 10



Figure 4.6: Increasing merit in sequence nr. 10

4.2.4 Sequence 4

A095887 Triangle read by rows: $T(n,k) = (n-k+1)^n, n \ge 1, 1 \le k \le n$.

c = 387420489 = Mat([3, 18]) b = 1516625 = [5, 6; 97, 1] a = 385904864 = [2, 5; 31, 1; 73, 3] radical = 658530 quality = 1.255527009976646 merit = 2.911981379002185 c = 134217728 Mat([2, 27]) b = 926757 = [3, 2; 13, 1; 89, 2] a = 13320971 = [11, 1; 59, 4] radical = 4505358 quality = 1.215420118507678 merit = 2.05248524891323 c = 40353607 = Mat([7, 9]) b = 42823 = [11, 1; 17, 1; 229, 1] a = 40310784 = [2, 11; 3, 9] radical = 1798566 quality = 1.155210286177332 merit = 0.25248524891323 c = 40353607 = Mat([2, 24]) b = 2197 = Mat([13, 3]) a = 16775019 = [3, 6; 23011, 1] radical = 1794858 quality = 1.155210286177332 merit = 0.292786288097512 c = 10077696 = [2, 9; 3, 9] b = 14125 = [5, 3; 113, 1] a = 10063571 = [7, 2; 59, 3] radical = 81040070 quality = 1.1390446029894408 merit = 0.294762868997512 c = 5764801 = Mat([7, 8]) b = 648 = [2, 3; 3, 4] a = 5764153 = [73, 1; 281, 2] radical = 815146 quality = 1.1390486029894408 merit = 0.646087145194057645 c = 829543 Mat([7, 7]) b = 371293 = Mat([13, 5]) a = 4452200 = [2, 1; 3, 3; 5, 3; 67, 1] radical = 182910 quality = 1.1104214132202936 merit = 0.346460002140961 c = 655536 = Mat([5, 8]) b = 117649 = Mat([7, 6]) a = 272976 = [2, 4; 3, 1; 11, 2; 47, 1] radical = 108570 quality = 1.010479546089314 c = 46655 = [2, 6; 3, 6] b = 4913 = Mat([17, 6]) a = 472976 = [2, 4; 3, 1; 11, 2; 47, 1] radical = 108570 quality = 1.010479546089314 c = 465536 = Mat([2, 16]) b = 709 = Mat([709, 1]) a = 64827 = [3, 3; 7, 4] radical = 229778 quality = 1.0765740735526552 merit = 0.140879546089314 c = 465536 = [2, 6; 3, 6] b = 4913 = Mat([17, 6]) a = 472976 = [5, 2; 67, 1] radical = 22197 quality = 1.0765740735526552 merit = 0.140879546089314 c = 2187 = Mat([3, 7]) b = 512 = Mat([2, 9]) a = 1675 = [5, 2; 67, 1] radical = 2010 quality = 1.01096139914133 merit = 0.00018025557361094 c = 729 = Mat([3, 6]) b = 200 = [2, 3; 5, 2] a = 529 = Mat([23, 2]) radical = 690 quality = 1.0084113092374762 merit = 0.000882588486994485



Figure 4.7: Increasing quality in sequence nr. 13



Figure 4.8: Increasing merit in sequence nr. 13

4.2.5 Sequence 5

A098293 Powers of 2 alternating with powers of 3.

c = 387420489 = Mat([3, 18]) b = 1516625 = [5, 6; 97, 1] a = 386904864 = [2, 5; 31, 1; 73, 3] radical = 6585330 quality = 1.252527009976646 merit = 2.9119813790027185 c = 129140163 = Mat([3, 16]) b = 1 = matrix(0, 2) a = 43046720 = [2, 6; 5, 1; 17, 1; 41, 1; 193, 1] radical = 4321290 quality = 1.2523529048295212 merit = 2.0596112874736856 c = 43046721 = Mat([3, 16]) b = 1 = matrix(0, 2) a = 43046720 = [2, 6; 5, 1; 17, 1; 41, 1; 193, 1] radical = 4036630 quality = 1.1556225447925967 merit = 1.0027234075306528 c = 43248907 = Mat([3, 15]) b = 619115 = [5, 1; 7, 3; 19, 2] a = 13729729 = [2, 15; 419, 1] radical = 663810 quality = 1.156027474364855 merit = 0.0586448729415037 c = 47722869 Mat([3, 14]) b = 2725 [5, 2; 109, 1] a = 4780244 = [2, 2; 7, 2; 29, 3] radical = 663810 quality = 1.14714607778128 merit = 0.0551200767869274 c = 1594323 = Mat([3, 13]) b = 90508 = [2, 2; 11, 3; 17, 1] a = 1503815 = [5, 1; 67, 3] radical = 375870 quality = 1.072971402163212 merit = 0.2823544581129452 c = 531441 = Mat([3, 12]) b = 240625 = [5, 5; 7, 1; 11, 1] a = 290816 = [2, 12; 71, 1] radical = 164010 quality = 1.0979971402163212 merit = 0.2823544581129452 c = 131072 = Mat([2, 11]) b = 32657 = [17, 2; 113, 1] a = 936135 = [3, 1; 7, 3; 353, 1] radical = 57630 quality = 1.072971402163212 merit = 0.2823544581129452 c = 131072 = Mat([2, 11]) b = 32657 = [17, 2; 113, 1] a = 98415 = [3, 9; 5, 1] radical = 376870 quality = 1.074960648269333 merit = 0.04845748856131326 c = 59049 = Mat([3, 10]) b = 485 = [5, 1; 97, 1] a = 58564 = [2, 2; 11, 4] radical = 32010 quality = 1.059055296303734 merit = 0.04845748856131326 c = 59049 = Mat([3, 10]) b = 485 = [5, 1; 97, 1] a = 58684 = [2, 2; 11, 4] radical = 32010 quality = 1.0590255296303734 merit = 0.04845748856131326 c = 59049 = Mat([2, 15]) b = 8378 = [3, 2; 7, 2; 19, 1] a = 24389 = Mat((28, 3]) radical = 23142 quality = 1.034609323206775 merit = 0.0014982332806691320 c = 8192 = Mat([3, 6]) b = 10 mat([11, 1]) a = 8181 = [3, 4; 101, 1] radical = 2010 quality = 1.0034130732735738 rit = 0.010498



Figure 4.9: Increasing quality in sequence nr. 14



Figure 4.10: Increasing merit in sequence nr. 14

4.2.6 Sequence 6

A075363 Triangle read by rows, in which the *n*-th row gives n smallest powers of n. or

A099615 Triangle read by rows, $2 \le k \le n$: T(n,k) = denominator of $(1+1/n)^k - (1+k/n)$ and of $(1-1/n)^k - (1-k/n)$. or A095896 Triangle read by rows: $T(n,k) = n^{(n-k+1)}, n \ge 1, 1 \le k \le n$.

c = 387420489 = Mat([3, 18]) b = 1515625 = [5, 6; 97, 1] a = 385904864 = [2, 5; 31, 1; 73, 3] radical = 6585330 quality = 1.259527009976646 marit = 2.9119813790027185 c = 43046721 = Mat([3, 16]) b = 1 = matrix(0,2) a = 43046720 = [2, 6; 5, 1; 17, 1; 41, 1; 193, 1] radical = 4035630 quality = 1.1556225447925967 merit = 1.0027234075306528 c = 16777216 = Mat([2, 24]) b = 2197 = Mat([13, 3]) a = 16775019 = [3, 6; 23011, 1] radical = 1794558 quality = 1.155210286177332 merit = 0.9252983022149146 c = 4782969 Mat([3, 14]) b = 2725 = [5, 2; 109, 1] a = 4780244 = [2, 2; 7, 2; 29, 3] radical = 663810 quality = 1.155210286177332 merit = 0.42724965386318153 c = 5037152 = Mat([2, 21]) b = 144027 = [3, 2; 13, 1; 1231, 1] a = 1953125 = Mat([5, 9]) radical = 480090 quality = 1.127039049372605 merit = 0.42724965386318153 c = 513441 = Mat([3, 12]) b = 240625 = [5, 5; 7, 1; 11, 1] a = 290816 = [2, 17, 1, 1] radical = 164010 quality = 1.0974093814132679 merit = 0.286128629987797 c = 117649 = Mat([7, 6]) b = 324 = [2, 2; 3, 4] a = 117325 = [5, 2; 13, 1; 19, 2] radical = 51670 quality = 1.07545548461615 merit = 0.147238658429088 c = 59049 = Mat([3, 10]) b = 10000 = [2, 4; 5, 4] a = 49049 = [7, 3; 11, 1; 13, 1] radical = 5194 quality = 1.065584273830386 merit = 0.147283658429088 c = 46656 = [2, 6; 3, 6] b = 4913 = Mat([17, 3]) a = 41743 = [13, 3; 19, 1] radical = 25149 quality = 1.0687694124516602 c = 32768 = Mat([2, 15]) b = 3379 = [3, 2; 7, 2; 19, 1] a = 24389 = Mat([29, 3]) radical = 23142 quality = 1.03460342605775 merit = 0.027776098986953608 c = 729 = Mat([3, 6]) b = 200 = [2, 3; 5, 2] a = 529 = Mat([23, 2]) radical = 680 quality = 1.0604113092374762 merit = 0.000662588466994485



Figure 4.11: Increasing quality in sequence nr. 15



Figure 4.12: Increasing merit in sequence nr. 15

4.2.7 Sequence 7

A246549 Prime powers p^e where p is a prime and $e \ge 3$ (prime powers without 1, the primes, or the squares of primes). or A100391 Prime powers which are local minima of A006530.

c = 130321 = Mat([19, 4]) b = 12672 = [2, 7; 3, 2; 11, 1] a = 117649 = Mat([7, 6]) radical = 8778 quality = 1.2971091295739554 merit = 1.7682278466791408
c = 117649 = Mat([7, 6]) b = 28561 = Mat([13, 4]) a = 89088 = [2, 10; 3, 1; 29, 1] radical = 18330 quality = 1.1074005954433029 merit = 0.9438019575269713
c = 103823 = Mat([47, 3]) b = 5408 = [2, 5; 13, 2] a = 98415 = [3, 9; 5, 1] radical = 18330 quality = 1.176660194171514 merit = 0.6997282156086191
c = 83521 = Mat([17, 4]) b = 1152 = [2, 7; 3, 2] a = 82369 = [7, 2; 41, 2] radical = 29274 quality = 1.1019400973819453 merit = 0.2490837312668208
c 79507 Mat([43, 3]) b = 4096 Mat([2, 12]) a = 75611 = [3, 4; 7, 2; 19, 1] radical = 34314 quality = 1.0084620371706 merit = 0.1686144415276661
c 68921 = Mat([41, 3]) b = 600 = Mat([2, 12]) a = 75611 = [3, 4; 29, 2] radical = 36570 quality = 1.0628359874003266 merit = 0.09724557089118555
c 665536 = Mat([2, 16]) b = 6487 = [13, 1; 499, 1] a = 56049 = Mat([3, 10]) radical = 36922 quality = 1.04297420915307 merit = 0.06666258521041194
c = 59049 = Mat([2, 16]) b = 6487 = [3, 2; 7, 2; 19, 1] a - 24389 = Mat([24, 10]) radical = 20142 quality = 1.0350372922734312 merit = 0.030779769126492332
c = 3122 Mat([2, 15]) b = 137 = [3, 2; 7, 2; 19, 1] a - 24389 = Mat([28, 3]) radical = 20142 quality = 1.03400732373573 merit = 0.0019900255575361094
c = 2187 = Mat([3, 7]) b = 512 = Mat([2, 9]) a = 1675 = [5, 2; 67, 1] radical = 2010 quality = 1.0084113092374762 merit = 0.00086825888466994485
c = 729 = Mat([3, 6]) b = 200 = [2, 3; 5, 2] a = 529 = Mat([23, 2]) radical = 600 quality = 1.0084113092374762 merit = 0.0008682588466994485



Figure 4.13: Increasing quality in sequence nr. 16



Figure 4.14: Increasing merit in sequence nr. 16

4.2.8 Sequence 8

A111261 Numbers of the form (m/n)(m-n) with $m \ge n$.

c = 43046721 = Mat([3, 16]) b = 83521 = Mat([17, 4]) a = 42963200 = [2, 8; 5, 2; 7, 2; 137, 1] radical = 489090 quality = 1.3417856234336722 merit = 3.9370152947185657 c = 33564432 = Mat([2, 25]) b = 9156027 = [3, 1; 1747, 2] a = 24338406 = [5, 1; 47, 4] radical = 2463270 quality = 1.1774600236737605 merit = 1.2462687493946097 c = 16777216 = Mat([2, 23]) b = 2197 = Mat([13, 3]) a = 16775019 = [3, 6; 23011, 1] radical = 1794586 quality = 1.155210268177332 merit = 0.9252883022149146 c = 8388608 = Mat([2, 23]) b = 30375 = [3, 5; 5, 3] a = 8358233 = [13, 2; 19, 2; 137, 1] radical = 1794586 quality = 1.155210268177332 merit = 0.9252883022149146 c = 4782969 = Mat([3, 14]) b = 2725 = [5, 2; 109, 1] a = 4780244 = [2, 2; 7, 2; 29, 3] radical = 663810 quality = 1.145210286177322 merit = 0.8470589366203329 c = 4782969 = Mat([2, 22]) b = 33492 = [3, 2; 3881, 1] a = 4158375 = [5, 5; 11, 3] radical = 69906 quality = 1.04435626691146151 merit = 0.2645435865657991 c = 2097152 Mat([2, 21]) b = 33492 = [3, 2; 3881, 1] a = 4158375 = [5, 5; 11, 3] radical = 1280730 quality = 1.0463563266116151 merit = 0.238165994256586 c = 531441 = Mat([3, 12]) b = 1 = matrix(0,2) a = 531440 = [2, 4; 5, 1; 7, 1; 13, 1; 73, 1] radical = 199290 quality = 1.0603794158656546 merit = 0.19722605156582415 c = 330625 Mat([5, 16]) b = 220 = Mat([7, 2]) a = 388416 = [2, 6; 3, 1; 7, 1; 17, 2] radical = 167790 quality = 1.06015749712233273626 merit = 0.09998420588198229 c = 117649 = Mat([7, 6]) b = 12500 = [2, 2; 5, 5] a = 105149 = [11, 3; 79, 1] radical = 66462 quality = 1.061574973269269 merit = 0.099786253276362 c = 65536 Mat([2, 16]) b = 6437 = [13, 1; 499, 1] a = 52049 = Mat([3, 10]) radical = 66862 quality = 1.0638794702015307 merit = 0.00947685253276362 c = 65536 Mat([2, 16]) b = 6437 = [13, 1; 499, 1] a = 5439 = [43, 10]) radical = 60820 quality = 1.0587974607123037722734312 merit = 0.03779769126492332 c = 32766 Mat([2, 16]) b = 6437 = [13, 1; 499, 1] a = 54389 = Mat([24, 3]) radical = 2342 quality = 1.0346037322734312 merit = 0.03077976912649233



Figure 4.15: Increasing quality in sequence nr. 18



Figure 4.16: Increasing merit in sequence nr. 18

4.2.9 Sequence 9

A120458 Triangle read by rows: row 0 is 1; for n > 0, row n gives 1^n , prime $(1)^n$, prime $(2)^n$, ..., prime $(n)^n$. or

A319075 Square array T(n, k) read by antidiagonal upwards in which row n lists the *n*-th powers of primes, hence column k lists the powers of the k-th prime, $n \ge 0, k \ge 1$.

c = 815730721 = Mat([13, 8]) b = 25 = Mat([5, 2]) a = 815730696 = [2, 3; 3, 3; 11, 2; 23, 2; 59, 1] radical = 5821530 quality = 1.3172945889635699 merit = 4.306060599870464
c = 410338673 = Mat([17, 7]) b = 566048 = [2, 5; 7, 2; 19, 2] a = 409772625 = [3, 1; 5, 3; 103, 3] radical = 6986490 quality = 1.2145769242411055 merit = 2.002646548075681
c = 214358881 = Mat([11, 8]) b = 3207361 = [7, 4; 31, 2] a = 212051520 = [2, 6; 3, 6; 5, 1; 101, 1] radical = 7232610 quality = 1.2145769242411055 merit = 2.002646548075681
c = 214358881 = Mat([11, 3]) b = 6276 = [2, 2; 3, 1; 523, 1] a = 63742241 = Mat([99, 4]) radical = 3630666 quality = 1.1827016523025258 merit = 1.4596831454643133
c = 19467171 = Mat([11, 7]) b = 561526 = [2, 10; 3, 2; 631, 1] a = 13761875 = [5, 9; 7, 1] radical = 1457610 quality = 1.1827016523025258 merit = 1.2566842267207559
c = 5764801 = Mat([13, 6]) b = 648 = [2, 3; 3, 4] a = 6764153 = [73, 1; 221, 2] radical = 861566 quality = 1.1320846208984408 merit = 0.6913177200951904
c = 4826809 = Mat([13, 6]) b = 55536 = Mat([2, 16]) a = 1706025 = [3, 1; 5, 2; 22, 2; 43, 1] radical = 328370 quality = 1.132498537786449 merit = 0.6511527941829186
c = 1771561 = Mat([11, 6]) b = 55536 = Mat([2, 16]) a = 1706025 = [3, 1; 5, 2; 22, 2; 43, 1] radical = 328370 quality = 1.132476994405735 merit = 0.67127679244037188
c = 323643 = Mat([7, 7]) b = 371293 = Mat([13, 5]) a = 452250 = [2, 1; 3, 3; 5, 7; 1] radical = 182910 quality = 1.124716994405735 merit = 0.67127679244037188
c = 329625 = Mat([5, 6]) b = 177649 = Mat([14, 5]) b = 15629 = [2, 6; 3, 3; 7, 1] a = 148955 = [5, 1; 1] radical = 108570 quality = 1.124716994405735 merit = 0.466087145911057645
c = 16051 = Mat([11, 6]) b = 5129 = (2, 6; 3, 3; 7, 1] a = 148955 = [5, 1; 31, 3] radical = 71610 quality = 1.102716522308 merit = 0.446800374513
c = 117649 = Mat([7, 6]) b = 31298 = [2, 6; 3, 3; 7, 1] a = 148955 = [5, 1; 31, 3] radical = 71610 quality = 1.072500857041634 merit = 0.14185106903747513
c = 116051 = Mat([17, 6]) b = 6377 e [13, 1; 23, 2] a = 8748 = [2, 2; 3,



Figure 4.17: Increasing quality in sequence nr. 22



Figure 4.18: Increasing merit in sequence nr. 22

4.2.10 Sequence 10

A076470 Perfect powers m^k where m is an integer and k > 5. or A122494 Set of numbers of the form a^b with $2 \le a \le b$, sorted.

c = 1594323 = Mat([3, 13]) b = 7424 = [2, 8; 29, 1] a = 1586899 = [17, 4; 19, 1] radical = 56202 quality = 1.305873782653318 merit = 2.4476817174888095 c = 1048576 = Mat([2, 20]) b = 50575 = [5, 2; 7, 1; 17, 2] a = 998001 = [3, 6; 37, 2] radical = 132090 quality = 1.4756986573787018 merit = 0.698107396532553 c = 823543 = Mat([7, 7]) b = 3072 = [2, 10; 3, 1] a = 820471 = [17, 3; 167, 1] radical = 119238 quality = 1.1653276277840765 merit = 0.7855216932161503 c = 531441 = Mat([3, 12]) b = 919 = Mat([191, 1]) a = 531250 = [2, 1; 5, 6; 17, 1] radical = 97410 quality = 1.147706998739997 merit = 0.6117837913212401 c = 524288 Mat([2, 13]) b = 4913 = Mat([17, 3]) a = 513757 = [3, 1; 5, 6; 177, 1] radical = 141270 quality = 1.110421413220236 merit = 0.5865359164897 c = 309625 = Mat([5, 8]) b = 117649 = Mat([7, 6]) a = 272976 = [2, 4; 3, 1; 11, 2; 47, 1] radical = 108570 quality = 1.110421413220236 merit = 0.368604377424422 c = 171147 Mat([3, 11]) b = 456 = [5, 1; 19, 1] a = 130977 = [3, 5; 7, 2; 11, 1] radical = 43890 quality = 1.10625512066344 merit = 0.2653014130816672 c = 131072 = Mat([2, 17]) b = 95 = [5, 1; 19, 1] a = 130977 = [3, 5; 7, 2; 11, 1] radical = 29778 quality = 1.065625512066344 merit = 0.2653014130816672 c = 55049 mat([3, 10]) b = 1000 = [2, 4; 5, 4] a = 49049 = [7, 3; 11, 1; 13, 1] radical = 29778 quality = 1.06562737830386 merit = 0.0364608314 c = 55049 mat([3, 10]) b = 1000 = [2, 4; 5, 4] a = 49049 = [7, 3; 11, 1; 13, 1] radical = 29778 quality = 1.065807373830386 merit = 0.0364648625849088 c = 46656 = [2, 6; 3, 6] b = 4913 = Mat([17, 3]) a = 41743 = [13, 3; 19, 1] radical = 25142 quality = 1.065802615104984 merit = 0.00876941245716602 c = 32768 = Mat([2, 15]) b = 8379 = [3, 2; 7, 2; 19, 1] a = 24389 = Mat([29, 3]) radical = 23142 quality = 1.0034260775 merit = 0.008769541245716602 c = 8192 = Mat([3, 7]) b = 512 = Mat([2, 9]) a = 1675 = [5, 2; 67, 1] radical = 2010 quality = 1.0024120783723573 merit = 0.010496233280691932 c = 2187 = Mat([3, 7]) b = 512 = Mat([2, 9]) a = 1675 = [5, 2; 67, 1] radi



Figure 4.19: Increasing quality in sequence nr. 24



Figure 4.20: Increasing merit in sequence nr. 24

4.2.11 Sequence 11

A128686 Least number of the form semiprime - 1 which is the product of exactly n primes.

c = 2418925581072 = [2, 41; 11, 1] b = 196367346709 = [19, 3; 31, 5] a = 2399288464363 = [3, 1; 7, 8; 13, 2; 8209, 1] radical = 29039616606 quality = 1.2791392693573205 merit = 5.973058523906151 c = 20890720927744 = [2, 40; 19, 1] b = 291356693369 = [13, 1; 59, 2; 6438373, 1] a = 20599365234376 = [3, 3; 5, 17] radical = 2814792291670 quality = 1.0699233564092074 merit = 0.4703211412429409 c = 1099511527776 = Mat([2, 40]) b = 239256535903 = [7, 11; 11, 2] a = 860255091873 = [3, 7; 251, 1; 311, 1] 5039, 1] radical = 181727413098 quality = 1.06943358062296973 merit = 0.470865410394979 c = 9977129359364 = [2, 35; 3, 3] b = 268091351 = [11, 4; 1311, 1] a = 92744844685 = [5, 1; 7, 1; 23, 6; 179, 1] radical = 17412553970 quality = 1.0645305002296973 merit = 0.3517705101038533 c = 343597383680 = [2, 36; 5, 1] b = 1744821 = [3, 4; 13, 1; 1657, 1] a = 4365638659 = [11, 3; 16067, 2] radical = 114212751510 quality = 1.0432580575191963 merit = 0.154237046642333 c = 17198691840 = [2, 36; 5, 1] b = 1744821 = [3, 4; 107, 1] a = 168410215519 = [7, 1; 13, 3; 29, 3; 449, 1] radical = 7267721890 quality = 1.034632785291926 merit = 0.096054882761119742 c = 115964116992 = [2, 32; 3, 3] b = 38679391031 = [13, 4; 1291271, 1] a = 79041256151 = [19, 6; 14, 2] radical = 78462008502 quality = 1.01557408220194 merit = 0.009008607582225685 c = 1207955552 = [2, 77; 3, 2] b = 471756427 = [23, 1; 29, 5] a = 736203125 = [5, 6; 7, 1; 53, 1; 127, 1] radical = 74260208502 quality = 1.0119971000350567 merit = 0.009000607582225685 c = 8386068 mat([2, 31]) b = 55049 = Mat([3, 10]) a = 8329559 = [7, 2; 169991, 1] radical = 2947494 quality = 1.0034634529673287648 c = 3145728 = [2, 20; 3, 1] b = 56451 = [11, 1; 71, 2] a = 3090277 = [17, 4; 37, 1] radical = 2947494 quality = 1.0034694943916004 merit = 0.00007682280132241918 c = 131072 = Mat([2, 17]) b = 65219 = [7, 2; 11, 3] a = 65853 = [3, 5; 271, 1] radical = 125202 quality = 1.003903525414824 merit = 0.0004048038050787064



Figure 4.21: Increasing quality in sequence nr. 26



Figure 4.22: Increasing merit in sequence nr. 26

4.2.12 Sequence 12

A132722 Numbers with k decimal digits that are also k-th powers.

c = 2541865828329 = Mat([3, 26]) b = 1012703329 = [11, 4; 263, 2] a = 2560853125000 = [2, 3; 5, 8; 23, 2; 29, 1; 53, 1] radical = 3068113290 quality = 1.307612605913739 merit = 6.374601882382574 c = 282423536481 = Mat([3, 24]) b = 47136567731 = [11, 5; 541, 2] a = 235292968750 = [2, 1; 5, 10; 7, 1; 1721, 1] radical = 2150750910 quality = 1.2269809315279852 merit = 3.3961555696623513 c = 31381059609 = Mat([3, 20]) b = 25 = Mat([5, 2]) a = 31381059584 = [2, 11; 7, 1; 173, 1; 12653, 1] radical = 456963490 quality = 1.217422683160224 merit = 2.676539456511415 c = 348674401 = Mat([2, 30]) b = 1369 = Mat([37, 2]) a = 1073740455 = [3, 8; 5, 1; 71, 1; 461, 1] radical = 46541826 quality = 1.20290012400325 merit = 2.17291603394677 c = 1073741824 = Mat([2, 30]) b = 1369 = Mat([37, 2]) a = 1073740455 = [3, 8; 5, 1; 71, 1; 461, 1] radical = 2020170 quality = 1.175596364691651 merit = 1.4818217103144523 c = 334720489 = Mat([3, 18]) b = 16540489 = [7, 4; 88, 2] a = 370880000 = [2, 9; 5, 4; 19, 1; 61, 1] radical = 2020170 quality = 1.1755983645891651 merit = 1.4640323465616403 c = 134217728 = Mat([2, 27]) b = 113643839 = [23, 4; 479, 1] radical = 9188178 quality = 1.155226473274 merit = 1.24438924495492897 c = 43046721 = Mat([2, 4]) b = 219 = Mat(13, 3]) a = 15775019 = [3, 6; 23011, 1] radical = 9188178 quality = 1.15522847325967 merit = 0.2528032214916 c = 16777216 = Mat([2, 4]) b = 2197 = Mat([13, 3]) a = 16775019 = [3, 6; 23011, 1] radical = 140458 quality = 1.15522847325967 merit = 0.252803221474 c = 4782969 = Mat([3, 14]) b = 2725 = [5, 2; 109, 1] a = 4780244 = [2, 2; 7, 2; 9, 3] radical = 663810 quality = 1.147731420776128 merit = 0.255810217446 c = 509478 = Mat([3, 12]) b = 240627 = [5, 5; 7, 1; 11, 1] a = 29816 = [2, 12; 71, 1] radical = 140010 quality = 1.07543848461615 merit = 0.27241503568611615 c = 51441 = Mat([3, 12]) b = 240627 = [5, 5; 7, 1; 11, 1] a = 29816 = [2, 12; 71, 1] radical = 164010 quality = 1.07543848461615 merit = 0.28610782629987797 c = 117649 = Mat([7, 6]) b = 337 = [2, 2; 3, 4] a = 117325 = [



Figure 4.23: Increasing quality in sequence nr. 28



Figure 4.24: Increasing merit in sequence nr. 28

List of Figures

Egyptian numeral system	5
Aristotle	6
Euclid	6
An excerpt from The Nine Chapters on the Mathematical Art	7
Al-Khwarizmi, the "father of algebra"	8
Pascal's triangle in the Jade Mirror of the Four Unknowns	8
Leonhard Euler (1707 - 1783)	9
Ramanujan (1887 - 1920)	10
David Hilbert (1862 - 1943)	10
\log_{10} of the number of triples below 10^N	18
First lines of c_no_duplicates.txt	19
First lines of Converted_OEIS.txt	20
First lines of Converted_OEIS0.txt	22
First lines of PARI-file.txt	24
How to input PARI-file.txt into PARI/GP	24
An extract of the abcFactors.txt file	25
An extract of the abcFinal.txt file	25
Increasing quality in sequence nr. 26	26
Increasing quality in sequence nr. 3	30
Increasing merit in sequence nr. 3	30
Increasing quality in sequence nr. 9	31
Increasing merit in sequence nr. 9	31
Increasing quality in sequence nr. 10	32
Increasing merit in sequence nr. 10	32
Increasing quality in sequence nr. 13	33
Increasing merit in sequence nr. 13	33
Increasing quality in sequence nr. 14	34
Increasing merit in sequence nr. 14	34
Increasing quality in sequence nr. 15	35
Increasing merit in sequence nr. 15	35
Increasing quality in sequence nr. 16	36
Increasing merit in sequence nr. 16	36
Increasing quality in sequence nr. 18	37
Increasing merit in sequence nr. 18	37
Increasing quality in sequence nr. 22	38
Increasing merit in sequence nr. 22	38
Increasing quality in sequence nr. 24	39
	Egyptian numeral system $\ Aristotle \ Ar$

Increasing merit in sequence nr. 24 .																	39
Increasing quality in sequence nr. 26																	40
Increasing merit in sequence nr. 26 .																	40
Increasing quality in sequence nr. 28																	41
Increasing merit in sequence nr. 28 .															•		41
	Increasing merit in sequence nr. 24 . Increasing quality in sequence nr. 26 . Increasing merit in sequence nr. 26 . Increasing quality in sequence nr. 28 . Increasing merit in sequence nr. 28 .	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28 Increasing merit in sequence nr. 28	Increasing merit in sequence nr. 24 Increasing quality in sequence nr. 26 Increasing merit in sequence nr. 26 Increasing quality in sequence nr. 28	Increasing merit in sequence nr. 24								

Bibliography

- Wikipedia contributors. History Of Mathematics. Wikipedia, The Free Encyclopedia. May 3, 2019, 18:52 UTC. Available at: https://en.wikipedia.org/wiki/History_of_mathematics
- [2] Jaakko J. HINTIKKA and Paul Vincent SPADE, "History Of Logic", *Encyclopaedia Britannica*, Available at: https://www.britannica.com/topic/history-of-logic
- [3] Wikipedia contributors. Sophist. Wikipedia, The Free En-2019,21:00 UTC. Available cyclopedia. May 13,at: https://en.wikipedia.org/wiki/Sophist_(dialogue)
- [4] Wikipedia contributors. History Of Logic. Wikipedia, The Free Encyclopedia. January 25, 2019, 08:40 UTC. Available at: https://en.wikipedia.org/wiki/History_of_logic
- [5] SMITH, Robin, "Aristotle's Logic, The Stanford Encyclopedia of Philosophy (Summer 2019 Edition), Available at: https://plato.stanford.edu/entries/aristotle-logic/#AriLogWorOrg
- [6] Wikipedia contributors. Stoic Logic. Wikipedia, The Free Encyclopedia.April 24, 2019, 09:39 UTC. Available at: https://en.wikipedia.org/wiki/Stoic_logic
- [7] Wikipedia contributors. Euclid's Elements. Wikipedia, The Free Encyclopedia. May 2, 2019, 17:24 UTC. Available at: https://en.wikipedia.org/wiki/Euclid#Elements
- [8] Wikiversity contributors. Euclidean geometry/Euclid's axioms. Wikiversity. 2019 Apr 20, 02:13 UTC Available at: https://en.wikiversity.org/wiki/Euclidean_geometry/Euclid%27s_axioms
- [9] Wikipedia contributors. Indian Philosophy. Wikipedia, The Free Encyclopedia. March 22, 2019, 10:07 UTC. Available at: https://en.wikipedia.org/wiki/Indian_philosophy
- [10] Wikipedia contributors. Nyaya Logic And Its Differences From Western Logic. Wikipedia, The Free Encyclopedia. May 17, 2019, 18:12 UTC. Available at: https://en.wikipedia.org/wiki/Nyaya
- [11] Wikipedia Pingala. contributors. Wikipedia, The Free Encvclopedia. January 30.2019.07:15UTC. Available at: https://en.wikipedia.org/wiki/Pingala

- [12] Wikipedia contributors. Binomial Theorem. Wikipedia, The Free Encyclopedia. May 18, 2019, 19:52 UTC. Available at: https://en.wikipedia.org/wiki/Binomial_theorem
- [13] Wikipedia contributors. Buddhist logico-epistemology. Wikipedia, The Free Encyclopedia. March 31, 2019, 15:17 UTC. Available at: https://en.wikipedia.org/wiki/Buddhist_logico-epistemology
- [14] Wikipedia contributors. The Nine Chapters On The Mathematical Art. Wikipedia, The Free Encyclopedia. March 8, 2019, 23:25 UTC. Available at: https://en.wikipedia.org/wiki/The_Nine_Chapters_on_the_Mathematical_Art
- [15] Wikipedia contributors. Logic In Islamic Philosophy. Wikipedia, The Free Encyclopedia. May 19, 2019, 22:55 UTC. Available at: https://en.wikipedia.org/wiki/Logic_in_Islamic_philosophy
- [16] MASTIN Luke, "Islamic Mathematics", The Story Of Mathematics, May 20, 2019. Available at: https://www.storyofmathematics.com/islamic.html
- [17] Wikipedia contributors. Mathematical Proof. Wikipedia, The Free Encyclopedia. May 23, 2019, 21:13 UTC. Available at: https://en.wikipedia.org/wiki/Mathematical_proof
- [18] MASTIN Luke, "Medieval Mathematics", *The Story Of Mathematics*, May 20, 2019. Available at: https://www.storyofmathematics.com/medieval.html
- [19] Wikipedia contributors. Chinese Mathematics. Wikipedia, The Free Encyclopedia. May 22, 2019, 00:43 UTC. Available at: https://en.wikipedia.org/wiki/Chinese_mathematics
- [20] Wikipedia contributors. Aryabhata. Wikipedia, The Free Encyclopedia. May 3, 2019, 18:42 UTC. Available at: https://en.wikipedia.org/wiki/Aryabhata
- [21] Wikipedia contributors. Brahmagupta. Wikipedia, The Free Encyclopedia. May 27, 2019, 19:06 UTC. Available at: https://en.wikipedia.org/wiki/Brahmagupta
- [22] Wikipedia contributors. Bhaskara II. Wikipedia, Free The Encyclopedia. May 22,2019,05:55UTC. Available at: https://en.wikipedia.org/wiki/Bh%C4%81skara_II
- [23] MASTIN Luke, "16th Century Mathematics", The Story Of Mathematics, May 21, 2019. Available at: https://www.storyofmathematics.com/16th.html
- [24] Wikipedia contributors. Leonhard Euler. Wikipedia, The Free Encyclopedia. April 25, 2019, 16:12 UTC. Available at: https://en.wikipedia.org/wiki/Leonhard_Euler
- [25] MASTIN Luke, "19th Century Mathematics", The Story Of Mathematics, May 21, 2019. Available at: https://www.storyofmathematics.com/19th.html

- [26] Wikipedia contributors. Hilbert's Program. Wikipedia, The Free Encyclopedia. May 3, 2019, 06:26 UTC. Available at: https://en.wikipedia.org/wiki/Hilbert%27s_program
- [27] Wikipedia contributors. Gödel's Incompleteness Theorems. Wikipedia, The Free Encyclopedia. May 24, 2019, 21:40 UTC. Available at: https://en.wikipedia.org/wiki/G%C3%B6del%27s_incompleteness_theorems
- [28] Wikipedia contributors. Hilbert's Problems. Wikipedia, The Free Encyclopedia. May 17, 2019, 15:51 UTC. Available at: https://en.wikipedia.org/wiki/Hilbert%27s_problems
- [29] Wikipedia contributors. Millenium Prize Problems. Wikipedia, The Free Encyclopedia. May 24, 2019, 01:01 UTC. Available at: https://en.wikipedia.org/wiki/Millennium_Prize_Problems
- [30] STEWART, Ian, The Great Mathematical Problems, (Great Britain: Profile Books Ltd., 2013), p. 291 - 293.
- [31] Wikipedia contributors. Andrew Wiles. Wikipedia, The Free Encyclopedia. May 9, 2019, 20:34 UTC. Available at: https://en.wikipedia.org/wiki/Andrew_Wiles
- [32] Wikipedia contributors. The ABC Conjecture. Wikipedia, The Free Encyclopedia. May 24, 2019, 15:23 UTC. Available at: https://en.wikipedia.org/wiki/Abc_conjecture
- [33] Numberphile, abc Conjecture Numberphile, (online video recording), Youtube, 12 October 2012, Available at: https://www.youtube.com/watch?v=RkB17WKzzRw
- [34] Wikipedia contributors. Mason-Stothers Theorem. Wikipedia, The Free Encyclopedia. October 21, 2018, 04:30 UTC. Available at: https://en.wikipedia.org/wiki/Mason%E2%80%93Stothers_theorem
- [35] SNYDER, Noah, "An Alternate Proof of Mason's Theorem", Elemente der Mathematik (Birkhäuser Verlag, 2000), edition 55, Available at: http://cr.yp.to/bib/2000/snyder.pdf
- [36] WOIT, Peter. "Proof Of The ABC Conjecture?" Not Even Wrong, May 26,2007.Available in at: http://www.math.columbia.edu/ woit/wordpress/?p=561
- [37] HAYES, Brian, "Easy as abc", bit-player.org, 24 May 2007. Available at: http://bit-player.org/2007/easy-as-abc
- [38] Wikipedia contributors. Inter-Universal Teichmüller Theory. Wikipedia, The Free Encyclopedia. April 14, 2019, 16:44 UTC. Available at https://en.wikipedia.org/wiki/Inter-universal_Teichm%C3%BCller_theory
- [39] KLARREICH, Erica, "Titans Of Mathematics Clash Over Epic Proof Of ABC Conjecture", Quantamagazine, 20 September 2018. Available at: https://www.quantamagazine.org/titans-of-mathematics-clash-overepic-proof-of-abc-conjecture-20180920/

- [40] Wikipedia contributors. Fermat's Last Theorem. Wikipedia, The Free Encyclopedia. May 15, 2019, 18:34 UTC. Available at: https://en.wikipedia.org/wiki/Fermat%27s_Last_Theorem
- "Does ABC [41] dalastboss in Imply Fermat's Last Theorem?", StackExchange, 20February 2015.Available at: https://math.stackexchange.com/questions/1157932/does-abc-impliesfermats-last-theorem
- [42] Wikipedia contributors. Hall's Conjecture. Wikipedia, The Free Encyclopedia. January 19, 2017, 00:23 UTC. Available at: https://en.wikipedia.org/wiki/Hall%27s_conjecture
- [43] Wikipedia contributors. Fermat-Catalan Conjecture. Wikipedia, The Free Encyclopedia. September 20, 2018, 18:05 UTC. Available at https://en.wikipedia.org/wiki/Fermat%E2%80%93Catalan_conjecture
- [44] DE SMIT, Bart, "ABC triples / by size". Available at: http://www.math.leidenuniv.nl/ desmit/abc/?set=1
- [45] Wikipedia contributors. The ABC@Home Project. Wikipedia, The Free Encyclopedia. April 21, 2018, 02:43 UTC. Available at: https://en.wikipedia.org/wiki/ABC@Home
- [46] The OEIS Foundation, "The On-Line Encyclopedia Of Integer Sequences. Available at: https://oeis.org/
- [47] Wikipedia contributors. Online Encyclopedia Of Integer Sequences. Wikipedia, The Free Encyclopedia. May 22, 2019, 17:26 UTC. Available at: https://en.wikipedia.org/wiki/On-Line_Encyclopedia_of_Integer_Sequences
- [48] PALENSTIJN, Willem Jan, "Radicals In Arithmetic", Universiteit Leiden, Chapter 7, 22 May 2014. Available at: https://openaccess.leidenuniv.nl/bitstream/handle/1887/25833/07.pdf?sequence=12
- [49] Enthought, Inc., ENTHOUGHT CANOPY, Available at: https://www.enthought.com/product/canopy/
- [50] Facts About Ancient Egyptians, "Ancient Egyptian Mathematics, Numerals, Geometry", ancientegyptianfacts.com, May 26, 2016. Available at: http://www.ancientegyptianfacts.com/ancient-egyptian-mathematics.html
- [51] KITSANTONIS Niki, "Greek Archaeologist Says He Has Found Aristotle's Tomb", The New York Times. Available at: https://www.nytimes.com/2016/05/27/world/europe/greece-aristotle-tomb.html
- [52] TAISBAK "Eu-Christian, VAN DER WAERDEN Bartel, clid", Encyclopaedia Britannica, 20May 2019.Available at: https://www.britannica.com/biography/Euclid-Greek-mathematician
- [53] MASTIN Luke, "Islamic Mathematics Al-Khwarizmi", The Story Of Mathematics, May 21, 2019. Available at: https://www.storyofmathematics.com/islamic_alkhwarizmi.html

- [54] ROGERS, Nick, "Pascal's Triangle Lesson for Kids: Definition & History", Study.com. Available at: https://study.com/academy/lesson/pascals-triangle-lesson-forkids-definition-history.html
- [55] BYRD, Deborah, "This date in science: Leonhard Euler's 306th birthday", Earthsky Communications Inc., 15 April 2013. Available at: https://earthsky.org/human-world/this-date-in-science-leonhardeulers-306th-birthday
- [56] GREGERSEN, Erik, "6 Interesting Facts about Srinivasa Ramanujan", Encyclopaedia Britannica, 15 April 2013. Available at: https://www.britannica.com/story/interesting-facts-about-srinivasa-ramanujan
- [57] "David Hilbert", The Famous People. Available at: https://www.thefamouspeople.com/profiles/david-hilbert-471.php