How many zeros has a random polynomial?

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Contents

1 Introduction

The study of the number of zeros of a random polynomial has interested many scientists during the last century. The underlying formula of this theory is the so-called *Rice formula*, which can be stated in different ways. It is called after Stephen Oswald Rice (1907 - 1986), an American computer scientist who had a great influence in information theory and telecommunications. This formula has great influence, not only in mathematics. In the 1950s and 1960s one could find applications of Rice's formula to physical oceanography, for instance, whereas applications to other areas of random mechanics were developed some time later. Another important mathematician in this theory is Mark Kac (1914 - 1984), widely known in probability theory. He pointed out his formula known under the name *Kac's counting formula*, which, as we will see, helps to state Rice's formula.

One of the first class of studied random polynomials were algebraic polynomials of degree n, such as

$$F(x) = a_n x^n + \ldots + a_1 x + a_0 \ ,$$

where the coefficients a_k $(0 \leq k \leq n)$ are independent standard Gaussian random variables, i.e. with mean zero and variance one. Kac proved that in this case the asymptotic of the expected number of zeros N_0 is given by

$$\mathbb{E}\left[N_0\right] \sim \frac{2}{\pi} \log n \ .$$

Maslova established the known asymptotic for the variance of the number of real zeros, [11]

$$\mathrm{Var}(N_0) \sim \frac{4}{\pi}\left(1 - \frac{2}{\pi}\right) \log n \ .$$

Apart of the asymptotic for variance, Maslova also established a Central Limit Theorem for the number of zeros of this type of random polynomials, [8, 9]. An interesting fact arises when we take a look at the distribution of the zeros of such polynomials. It is shown that the real zeros concentrate near 1 and -1, whereas if one considers the complex plane, the complex zeros are distributed on the unit circle. The asymptotic expectation of zeros changes when we slightly modify the variance of the coefficients. Namely, setting the variance of a_k equal to $\binom{n}{k}$, we get an exact expected number of zeros, namely

$$\mathbb{E}\left[N_0\right] = \sqrt{n} \ ,$$

which is actually more than in the previous case[1]. For this type of polynomials a Central Limit Theorem has also been established, [4].

One more example of random polynomials are the trigonometric ones over $[0, 2\pi]$, considered by Dunnage,

$$F(x) = a_n \cos nx + b_n \sin nx + \ldots + a_1 \cos x + b_1 \sin x \ ,$$

for which he proves the estimate [6]

$$\mathbb{E}\left[N_0\right] \sim \frac{2}{\sqrt{3}} n \ .$$

For trigonometric polynomials, a Central Limit Theorem for the number of zeros has been established in the case where the coefficients a_k and b_k are independent standard Gaussian random variables. The proof of this theorem is based on Wiener Chaos expansion and Rice Formula.

[1] Indeed, $\forall x \in \mathbb{R}_+, \log x < x$, hence $\log x = \log \sqrt{x^2} = 2 \log \sqrt{x} < \sqrt{x}$.

Our main aim of this paper will be to take in review these previous results and provide simple arguments to prove some of them. At the end of this monograph, we will use Matlab to do some simulations on the distribution of zeros of random algebraic polynomials and on the Central Limit Theorem for random stationary trigonometric polynomials.

2 Basic ideas and defintions

In this first section we are interested in the main formula that will be useful all along this paper. Firstly we start by giving a rough idea of this topic and fix some notations, as well as provide some key definitions.

2.1 Random functions

In this paper we study the expected number of zeros of a random polynomial. The same can be done for the number of crossings through some given level u for a random function. Thus in order to study the distribution of the number of zeros of a random polynomial, we will mainly restrict to $u = 0$.

A random function viewed as a stochastic process. In our paper, a random polynomial F, that we will define in a next step, can be seen as a path of a real-valued smooth stochastic process $\mathcal{F}$ defined on some interval I, $\mathcal{F} = \{F(t) : t \in I\}$, which gives the evolution of the random variable F in terms of the time t.

Stationary stochastic process. Recall that a stochastic process $\{X(t) : t \geq 0\}$ is said to be stationary if the distribution of $X(s+t) - X(s)$ over the interval $(s, s+t)$ only depends on the length t of the interval, and not on the times t and s. In particular, if we assume that $X(0) = 0$, we have that $X(s+t) - X(s)$ has the same distribution as $X(t) = X(t) - X(0)$.

Notations. When we speak about the crossings of a stochastic process through some level u, we can have different situations, namely if $F : I \to \mathbb{R}$ is a real-valued differentiable function, then we denote

- $U_u(F, I) := \{t \in I : F(t) = u, F'(t) > 0\}$ the set of *up-crossings* of F
- $D_u(F, I) := \{t \in I : F(t) = u, F'(t) < 0\}$ the set of *down-crossings* of F
- $C_u(F, I) := \{t \in I : F(t) = u\}$ the set of *crossings* of F

The cardinality of this last set will be denoted by $N_u(F, I)$.

Let us now give the definition of a random function in general, taken from [12].

Definition 2.1. Let $I \subset \mathbb{R}$ be an interval and $f_0, f_1, \dots f_n : I \to \mathbb{R}$ some functions. Then a *random function* $F : I \to \mathbb{R}$ is given by the linear combination

$$F(t) = \sum_{k=0}^{n} a_k f_k(t) \ ,$$

where the coefficients a_k are random variables for $k = 0, 1, \dots, n$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

Remark 2.2. From this definition we see that the random funtion is a finite sum of random variables, and hence a random variable, too. For simplicity we will assume that the coefficients of the random function are i.i.d. random variables. Even stronger, we will for most of the time assume that the coefficients are Gaussian random variables, which makes it easier to handle with the formulas that we will meet.

2.2 Covariance function

Having defined a random function, we will now define a function, which we will use quite often in this paper, the *covariance function*. Let us start with the general definition.

Definition 2.3. Let $I \subset \mathbb{R}$. The covariance function[2] of the stochastic process $\mathcal{F} = \{F(t) : t \in I\}$ is given by the covariance of the values of the process at times x and y, i.e. it is the function $K : I \times I \to \mathbb{R}$, defined by

$$K(x,y) = \mathrm{Cov}(F(x), F(y)) = \mathbb{E}\left[(F(x) - \mathbb{E}\left[F(x)\right])((F(y) - \mathbb{E}\left[F(y)\right])\right] \ . \tag{2.1}$$

2.2.1 Covariance function for centered processes

Let us assume that the stochastic process $\mathcal{F} = \{F(t) : t \in I\}$ is centered, meaning that for all t in I, $\mathbb{E}\left[F(t)\right] = 0$. Then the covariance function as defined above in (2.1) takes the simpler form

$$K(x,y) = \mathbb{E}\left[F(x)F(y)\right] \ .$$

Thus, for a centered random function as in Definition 2.1, the above formula gives

$$K(x,y) = \mathbb{E}\left[\sum_{i=0}^{n} a_i f_i(x) \sum_{j=0}^{n} a_j f_j(y)\right] = \sum_{i=0}^{n}\sum_{j=0}^{n} f_i(x) f_j(y) \mathbb{E}\left[a_i a_j\right] \ , \tag{2.2}$$

where the last equality immediately follows from the linearity of the expectation.

Case of a random function with independent coefficients. Since for the remainder of this paper we are essentially interested in the case where the coefficients of the random function are independent Gaussian variables, we try to derive a simpler form of the covariance function.

Let us see what happens if we assume that the coefficients a_k of the random function are independent Gaussian variables with mean zero and variance v_k , i.e. $a_k \sim \mathcal{N}(0, v_k), 0 \leq k \leq n$. Since the random variables a_i and a_j are independent for $i \neq j$, we have that $\mathbb{E}\left[a_i a_j\right] = 0$ for $i \neq j$. Thus the second sum will only have an non zero contribution for $j = i$. Hence (2.2) gives

$$K(x,y) = \sum_{i=0}^{n} f_i(x) f_i(y) \mathbb{E}\left[{a_i}^2\right] \ ,$$

but since the coefficients have all mean zero, we have

$$v_i = \mathrm{Var}(a_i) = \mathbb{E}\left[{a_i}^2\right] - \mathbb{E}\left[a_i\right]^2 = \mathbb{E}\left[{a_i}^2\right] \ ,$$

so that

$$K(x,y) = \sum_{i=0}^{n} v_i f_i(x) f_i(y) \ . \tag{2.3}$$

In this last formula, we can see that assuming the coefficients to be independent Gaussian variables, we have simplified the definition of the covariance function, so that it is easier to deal with it.

[2] The covariance function of a stochastic process is also called the covariance kernel.

Covariance function of stationary processes. It often occurs that our stochastic process that we analyse is stationary. In this case, the covariance function $K(x, y)$, which is a priori a function of two variables, only depends on the difference $|x - y|$, thus it only needs one variable instead of two. Thus, instead of $K(x, y)$, we can write $K(|x - y|, 0)$. This notation is often made easier by simply writing $K(|x - y|)$ or $K(\tau)$ with $\tau = |x - y|$.

2.3 Kac's counting formula

In this section, our goal is to provide the so-called *Kac's counting formula*, that will help us to derive Rice's formula. Although we are interested in the expected number of zeros of a random function, we will prove the general formula for crossings through some level u. For this part, we follow Section 3.1 of [2] and Section 2 of [12].

Let us first make some assumptions on our random fuction F.

Definition 2.4. Let $F : [a, b] \to \mathbb{R}$ be a C^1-function. Then F is called *convenient* if the following conditions are satisfied:

- $F(a) \neq u$ and $F(b) \neq u$,
- if $F(t) = u$, then $F'(t) \neq 0$.

Remark 2.5. The second condition in the above definition means that the crossings of F through level u are all nondegenerate. It is equivalent to writing that the set $\{t \in [a, b] : F(t) = u, F'(t) = 0\}$ is the empty set.

Lemma 2.6 (Kac's counting formula)**.** *Let $F : [a, b] \to \mathbb{R}$ be a convenient C^1-function. Then the number of u-crossings of F in $[a, b]$ is given by*

$$N_u(F, [a, b]) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_a^b \mathbf{1}_{\{|F(t)-u|<\varepsilon\}} |F'(t)| \, dt \ . \tag{2.4}$$

Proof. The assumption that F is convenient implies that F crosses level u finitely many times, say $N_u(F, I) = n < \infty$. We will distinguish two cases, $n = 0$ and $n \geq 1$.

If $n = 0$, then equality (2.4) is satisfied, since the integrand in the r.h.s is equal to 0 for ε small enough, so that we have 0 on both sides.

Assume $n \geq 1$. Denote $C_u(F, I) = \{s_1, s_2, \ldots, s_n\}$. Since F is convenient, $F'(s_k) \neq 0$ for all $k = 1, 2, \ldots, n$. Then if $\varepsilon > 0$ is small enough, the inverse image of the open intervals $(u-\varepsilon, u+\varepsilon)$ is the disjoint union of n intervals $I_k = (a_k, b_k)$ such that $s_k \in I_k$ for $k = 1, 2, \ldots, n$. Observe that since a_k and b_k are the extremities of the intervals I_k, we have that

$$F(a_k) = u \pm \varepsilon \ \text{ and } \ F(b_k) = u \mp \varepsilon \ , \forall k = 1, 2, \ldots, n \ ,$$

and thus by the fundamental theorem of calculus, we obtain

$$\int_{I_k} |F'(t)| \, dt = \int_{a_k}^{b_k} |F'(t)| \, dt = |F(t)| \Big|_{a_k}^{b_k} = 2\varepsilon \ .$$

Finally, since ε is small enough, we get

$$\frac{1}{2\varepsilon} \int_a^b \mathbf{1}_{\{|F(t)-u|<\varepsilon\}} |F'(t)| \, dt = \frac{1}{2\varepsilon} \sum_{k=1}^{n} \int_{a_k}^{b_k} |F'(t)| \, dt = \frac{1}{2\varepsilon} \sum_{k=1}^{n} 2\varepsilon = n = N_u(F, [a, b]) \ ,$$

which concludes the proof. □

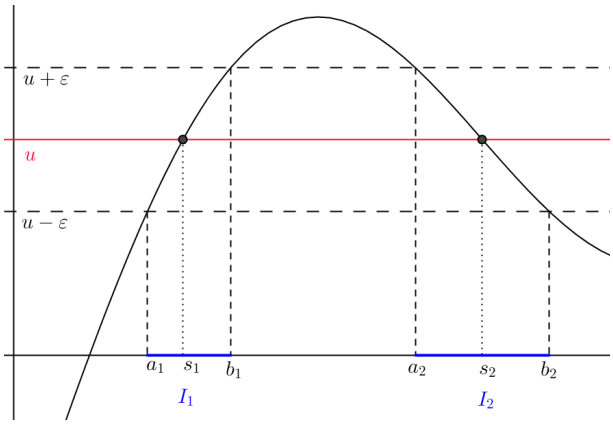

Figure 1: Kac's counting formula

Approximation of the Dirac function. We give here a rough idea of how one could derive Kac's formula in a different way. We follow the reasoning as in Section 2.1 of [5]. Let us consider the function

$$\eta_\varepsilon : \mathbb{R} \to \mathbb{R} \ , \ x \mapsto \frac{1}{2\varepsilon} \mathbf{1}_{\{|x|<\varepsilon\}} \ .$$

This function is an approximation of Dirac's function when ε tends to 0. Using the properties of Dirac's funtion, we can write

$$\int_{\mathbb{R}} \delta_u(x)\, dx = 1 \ ,$$

which, by the change of variables $x \mapsto F(t)$, gives

$$\int_{\mathbb{R}} \delta_u(F(t))|F'(t)|\, dt = \int_{I_k} \delta_u(F(t))|F'(t)|\, dt = 1 \ ,$$

for $k = 1, 2, \ldots, n$. Now summing over k on both sides of the last equality, and using the fact that F crosses level u in each of the intervals I_k at $t = s_k$, it follows

$$\int_{[a,b]} \delta_u(F(t))|F'(t)|\, dt = n \ .$$

To derive Kac's counting formula from here, we use the fact that the function η_ε approximates Dirac's function when ε is small enough, thus the last equality can be rewritten as

$$\lim_{\varepsilon\to 0} \int_{[a,b]} \eta_\varepsilon(F(t)-u)|F'(t)|\, dt = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{[a,b]} \mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\, dt = n \ ,$$

and hence we find back Kac's counting formula.

Remark 2.7. To simplify notations, we will denote

$$N_u^\varepsilon(F, I) := \frac{1}{2\varepsilon} \int_I \mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\, dt \ . \tag{2.5}$$

Using this notation, we can rewrite Kac's counting formula in (2.4) as

$$N_u(F, [a, b]) = \lim_{\varepsilon\to 0} N_u^\varepsilon(F, [a, b]) \ . \tag{2.6}$$

3 Kac-Rice formulas

3.1 Kac-Rice formulas for u-crossings

We are now in position to establish Kac-Rice's formula, which will cover the main part of this subsection. Firstly we state a lemma, which turns out to be useful for what comes.

Lemma 3.1. *Let $I \subset \mathbb{R}$ and $F : I \to \mathbb{R}$ be a convenient C^1-function. Assume that for $u \in \mathbb{R}$ the function $F(t) - u$ has r zeros and s critical points.*[3] *Then for any ε we have the following inequality*

$$N_u^\varepsilon(F, I) \leq r + 2s \ .$$

Proof. Denote $G(t) = F(t) - u$. Then G has the same critical points as $F - u$ and since the number of u-crossings of F coincides with the number of zeros of G, we have that $N_u^\varepsilon(F, I) = N_0^\varepsilon(G, I)$. Thus it is equivalent to prove $N_0^\varepsilon(G, I) \leq r + 2s$.

Since G has only finitely many critial points, the equation $G'(t) = 0$ has only finitely many solutions. Then Rolle's theorem[4] implies that for any real constant c the equation $G(t) = c$ has finitely many solutions. Let us now fix $\varepsilon > 0$. Using the above argument, the equation $|G(t)| = \varepsilon$ has finitely many solutions. Thus $\{|G(t)| < \varepsilon\}$ has finitely many components, say n, which are bounded intervals $I_k = (a_k, b_k)$ such that for all $k = 1, 2, \ldots, n$, $|G(a_k)| = |G(b_k)| = \varepsilon$.

Let us denote by j_k the number of turning points of $G(t)$ in I_k, that are the points where G' changes sign. If I_k contains no turning points, i.e. if $j_k = 0$, then the function G is either increasing or decreasing on I_k, and hence $G(a_k)G(b_k) < 0$. Thus, since G is continuous, it has a unique zero in I_k.

Let us introduce the two following sets

$$\tau_0 = \{k \in \{1, \ldots, n\} : I_k \ \text{ contains no turning points}\}$$

$$\tau_1 = \{k \in \{1, \ldots, n\} : I_k \ \text{ contains turning points}\} \ .$$

Let us look at the cardinalities of τ_0 and τ_1. Since I_k has no turning points implies that G has a unique zero in I_k , we clearly have that $|\tau_0| = r$. Now $k \in \tau_1$ means I_k contains at least one turning point, i.e. $G' = 0$ at least once in this interval. But by assumption G' has s critical points, therefore $|\tau_1| \leq s$.

Then we can write

$$\begin{aligned} N_0^\varepsilon(G, I) &= \frac{1}{2\varepsilon} \int_I \mathbf{1}_{\{|G(t)|<\varepsilon\}} |G'(t)| \, dt = \frac{1}{2\varepsilon} \sum_{k=1}^{n} \int_{a_k}^{b_k} |G'(t)| \, dt \\ &= \frac{1}{2\varepsilon} \sum_{k \in \tau_0} \int_{a_k}^{b_k} |G'(t)| \, dt + \frac{1}{2\varepsilon} \sum_{k \in \tau_1} \int_{a_k}^{b_k} |G'(t)| \, dt \ . \end{aligned} \tag{3.1}$$

But since

$$\int_{a_k}^{b_k} |G'(t)| \, dt = |G(t)| \Big|_{a_k}^{b_k} = 2\varepsilon \ ,$$

we have that the first sum in (3.1) is equal to $|\tau_0| = r$.

[3] Observe that the critical points of the function $F(t) - u$ are the same as those of $F(t)$ since $(F(t) - u)' = 0$ implies $F'(t) = 0$.

[4] Rolle's theorem states that if $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b) such that $f(a) = f(b)$, then there exists some constant $c \in (a, b)$ such that $f'(c) = 0$.

Now let $k \in \tau_1$. We denote $t_1 < t_2 < \ldots < t_{j_k}$ the turning points[5] in I_k. Then

$$\begin{aligned}\int_{a_k}^{b_k} |G'(t)|\, dt &= \int_{a_k}^{t_1} |G'(t)|\, dt + \int_{t_1}^{t_2} |G'(t)|\, dt + \ldots + \int_{t_{j_k}}^{b_k} |G'(t)|\, dt \\ &= |G(a_k) - G(t_1)| + |G(t_1) - G(t_2)| + \ldots + |G(t_{j_k}) - G(b_k)| \\ &\leq 2\varepsilon(j_k + 1)\ ,\end{aligned}$$

since all the $j_k + 1$ terms in the second equality are less or equal to 2ε. Thus (3.1) gives

$$N_0^\varepsilon(G, I) \leq r + \frac{1}{2\varepsilon} \sum_{k \in \tau_1} 2\varepsilon(j_k + 1) = r + \sum_{k \in \tau_1} j_k + |\tau_1| \leq r + 2s\ ,$$

where for the last inequality we noticed that the sum of the number of turning points is equal to s. □

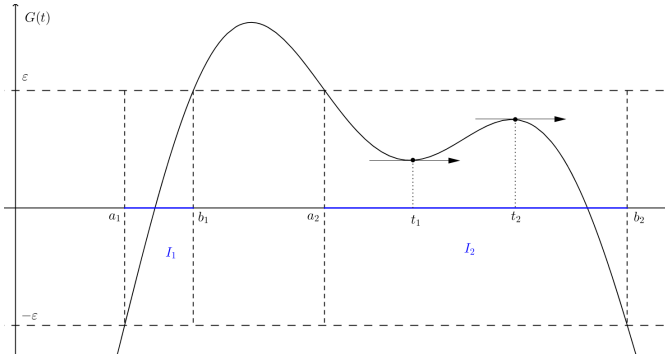

Figure 2: Lemma 3.1

In I_1 the function G is strictly increasing and has a unique zero, whereas in I_2, $t_1 < t_2$ are both turning points. In this case $j_1 = 0$, hence $1 \in \tau_0$ and $j_2 = 2$, hence $2 \in \tau_1$.

In the following part of this section, we will establish Kac-Rice's formula.Our development closely follows Section 2 of [12]. Up to now, we have seen Kac's counting formula, which gives us the number of crossings through some level u of a random function F. Here we are interested in the expected number of u-crossings of F. Hence the main idea is to start from Kac's counting formula (2.4) and take expectation on both sides of the equality. We will discuss details in the coming pages.

Let $I \subset \mathbb{R}$ and $F : I \to \mathbb{R}$ be a random function as defined in definition 2.1. Consider the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then the coefficients of F are random variables

$$a_k : \Omega \to \mathbb{R}\ , \omega \mapsto a_k(\omega)\ .$$

We may assume that the a_k are independent Gaussian random variables with mean zero and variance v_k. Let us start by making two assumptions on F, namely

$(A1)$ *The random function F is almost surely convenient.*

$(A2)$ *$\exists\ M > 0$ constant such that almost surely $N_u(F, I) + N_u(F', I) < M$.*

[5] Remember that j_k stands for the number of turning points in I_k.

By definition of the expectation and using (2.6), we can write

$$\mathbb{E}\left[N_u(F,I)\right] = \int_\Omega N_u(F,I)\,\mathbb{P}(d\omega) = \int_\Omega \lim_{\varepsilon\to 0} N_u^\varepsilon(F,I)\,\mathbb{P}(d\omega)\ .$$

Now in order to continue, we would like to interchange the limit and the integration operator. This is possible by assumption $(A2)$, Lemma 3.1 and using Lebesgue's dominated convergence theorem. Thus after switching both operators, this last equality gives

$$\lim_{\varepsilon\to 0}\int_\Omega N_u^\varepsilon(F,I)\,\mathbb{P}(d\omega) = \lim_{\varepsilon\to 0}\mathbb{E}\left[N_u^\varepsilon(F,I)\right]\ ,$$

and using the definition of $N_u^\varepsilon(F,I)$ in (2.5), we obtain

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[N_u^\varepsilon(F,I)\right] = \lim_{\varepsilon\to 0}\mathbb{E}\left[\frac{1}{2\varepsilon}\int_I \mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\,dt\right] = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_I \mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right]\,dt\ ,$$

where we have interchanged the expectation and the integration operator in the last equality. Hence we finally have

$$\mathbb{E}\left[N_u(F,I)\right] = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_I \mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right]\,dt\ . \tag{3.2}$$

Thus to know $\mathbb{E}\left[N_u(F,I)\right]$, (3.2) leads us to compute the expectation of $\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|$. In order to do so, we observe that the random vector $(F(t),F'(t))$ is Gaussian[6]. Indeed, $(F(t),F'(t))$ is a Gaussian random vector since for all $m_1,m_2\in\mathbb{R}$, the random variable $m_1F(t)+m_2F'(t)$ is Gaussian as a finite sum of Gaussian random variables. Its covariance matrix[7] is given by the symmetric matrix

$$M_t = \begin{pmatrix} A_t & B_t \\ B_t & C_t \end{pmatrix}\ ,$$

where, using the fact that $\mathbb{E}\left[F(t)\right]$ and $\mathbb{E}\left[F'(t)\right]$ are zero, we have[8]

$$\begin{aligned} A_t &= \mathrm{Cov}(F,F) = \mathbb{E}\left[F^2\right] - \mathbb{E}\left[F\right]^2 = \mathbb{E}\left[F^2\right]\ ,\\ B_t &= \mathrm{Cov}(F,F') = \mathbb{E}\left[FF'\right] - \mathbb{E}\left[F\right]\mathbb{E}\left[F'\right] = \mathbb{E}\left[FF'\right]\ ,\\ C_t &= \mathrm{Cov}(F',F') = \mathbb{E}\left[(F')^2\right] - \mathbb{E}\left[F'\right]^2 = \mathbb{E}\left[(F')^2\right]\ . \end{aligned}$$

Now using the covariance kernel of F, $K(x,y)=\mathbb{E}\left[F(x)F(y)\right]$, and denoting $K_x := \frac{\partial K}{\partial x}$, resp. $K_{xy} := \frac{\partial^2 K}{\partial x\partial y}$ the partial derivatives of K with respect to x resp. to x and y, we can write

$$A_t = K(t,t)\ , B_t = K_x(x,y)\big|_{x=y=t}\ , C_t = K_{xy}(x,y)\big|_{x=y=t}\ . \tag{3.3}$$

For A_t this is clear, however for B_t and C_t, we need some computations in order to see this, namely, using (2.3),

$$K_x(x,y) = \frac{\partial}{\partial x}\sum_{i=0}^n v_i f_i(x) f_i(y) = \sum_{i=0}^n v_i f_i'(x) f_i(y) = \mathbb{E}\left[F'(x)F(y)\right]\ ,$$

[6]Recall the definition of a Gaussian random verctor. Let $X=(X_1,\ldots,X_n)\in\mathbb{R}^n$ be a random vector. Then we say that X is a Gaussian random vector if for all reals $a_1,\ldots,a_n$, the random variable defined by the linear combination $a_1X_1+\ldots+a_nX_n$ is a Gaussian random variable.

[7]The covariance matrix of a Gaussian random X is given by $C=(\mathrm{Cov}(X_i,X_j))_{ij}$.

[8]We write F for $F(t)$ and F' for $F'(t)$.

and hence $B_t = K_x(x,y)\big|_{x=y=t}$. By the same reasoning,

$$K_{xy}(x,y) = \frac{\partial^2}{\partial x \partial y} \sum_{i=0}^{n} v_i f_i(x) f_i(y) = \frac{\partial}{\partial x} \sum_{i=0}^{n} v_i f_i(x) f_i{}'(y) = \mathbb{E}\left[F'(x)F'(y)\right] \ ,$$

and hence $C_t = K_{xy}(x,y)\big|_{x=y=t}$.

Denote by Δ_t the determinant of the covariance matrix M_t of (F, F'), then we make the following assumption

$$(A3) \quad \forall t \in I : \Delta_t := A_t C_t - {B_t}^2 > 0 \ .$$

Set $U_t = \Delta_t / A_t = C_t - {B_t}^2 / A_t$.

Now in order to compute $\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right]$, we observe that this is the expectation of a function depending on the random variables F and F' ,[9] namely

$$\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right] = \mathbb{E}\left[G(F(t), F'(t))\right] \ ,$$

where $G(x,y) = \mathbf{1}_{\{|x-u|<\varepsilon\}}|y|$. Then it follows

$$\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right] = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|x-u|<\varepsilon\}}|y| p_{(F,F')}(x,y)\, dx\, dy \ , \tag{3.4}$$

where $p_{(F,F')}$ is the density of the Gaussian random vector[10] $(F(t), F'(t))$. Since this vector has mean $(\mathbb{E}[F], \mathbb{E}[F']) = (0,0)$, its density is given by

$$p_{(F,F')}(\mathbf{x}) = \frac{1}{2\pi\sqrt{\Delta_t}} \exp\left(-1/2 \mathbf{x}^T {M_t}^{-1} \mathbf{x}\right) \text{, where } \ \mathbf{x} \in \mathbb{R}^2 \ . \tag{3.5}$$

Remark 3.2. Here $\mathbf{x}^T$ denotes a line vector and $\mathbf{x}$ denotes a column vector. Thus we write

$$\mathbf{x}^T = (x,y) \ \text{ and } \ \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \ .$$

We will now compute $\mathbf{x}^T {M_t}^{-1} \mathbf{x}$. By assumption $(A3)$, the matrix M_t is invertible and

$${M_t}^{-1} = \frac{1}{\Delta_t} \begin{pmatrix} C_t & -B_t \\ -B_t & A_t \end{pmatrix} \ ,$$

and thus $\mathbf{x}^T {M_t}^{-1} \mathbf{x}$ is equal to

$$\frac{1}{\Delta_t}(x,y) \begin{pmatrix} C_t & -B_t \\ -B_t & A_t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\Delta_t}(C_t x^2 - 2B_t xy + A_t y^2)$$

The next step is now to write this expression into a convenient way, namely

$$\begin{aligned} & C_t x^2 - 2B_t xy + A_t y^2 \\ = \ & A_t y^2 - 2B_t xy + \tfrac{{B_t}^2}{A_t} x^2 + \left(C_t - \tfrac{{B_t}^2}{A_t}\right) x^2 \\ = \ & A_t \left(y^2 - 2\tfrac{B_t}{A_t} xy + \left(\tfrac{B_t}{A_t} x\right)^2\right) + \left(C_t - \tfrac{{B_t}^2}{A_t}\right) x^2 \\ = \ & A_t \left(y - \tfrac{B_t}{A_t} x\right)^2 + \left(C_t - \tfrac{{B_t}^2}{A_t}\right) x^2 \\ = \ & A_t \left(y - \tfrac{B_t}{A_t} x\right)^2 + \tfrac{\Delta_t}{A_t} x^2 \ . \end{aligned}$$

[9] We recall that if X and Y are two random variables, $G : \mathbb{R}^2 \to \mathbb{R}$ a function, then $\mathbb{E}[G(X,Y)] = \int_{\mathbb{R}}\int_{\mathbb{R}} G(x,y) p_{(X,Y)}(x,y)\, dx\, dy$, where $p_{(X,Y)}$ is the density of the random vector (X,Y).

[10] If $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ is a Gaussian random vector of mean $\mathbb{E}[X] = (\mathbb{E}[X_1], \ldots, \mathbb{E}[X_n]) = \mu$ and covariance matrix C, then if C is invertible, its density is given by $p_X(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det C}} \exp\left(-1/2 (\mathbf{x}-\mu)^T C^{-1} (\mathbf{x}-\mu)\right)$, where $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

From this last equality, it follws that

$$-\frac{1}{2}\mathbf{x}^T {M_t}^{-1}\mathbf{x} = -\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2 - \frac{x^2}{2A_t} \ ,$$

and plugging this in (3.5), we finally obtain the desired density of $(F(t), F'(t))$,

$$p_{(F,F')}(x,y) = \frac{1}{2\pi\sqrt{\Delta_t}} \exp\left[-\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2 - \frac{x^2}{2A_t}\right] \ . \tag{3.6}$$

In order to compute $\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right]$, we continue with (3.4), where we replace $p_{(F,F')}(x,y)$ by the expression obtained in (3.6). This gives

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|x-u|<\varepsilon\}}|y| p_{(F,F')}(x,y)\,dx\,dy \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{1}_{\{|x-u|<\varepsilon\}}|y| \frac{1}{2\pi\sqrt{\Delta_t}} \exp\left[-\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2 - \frac{x^2}{2A_t}\right] dx\,dy \\
&= \int_{\mathbb{R}} \frac{1}{2\pi\sqrt{\Delta_t}} \mathbf{1}_{\{|x-u|<\varepsilon\}} \left(\int_{\mathbb{R}} |y| \exp\left[-\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2 - \frac{x^2}{2A_t}\right] dy\right) dx \\
&= \int_{u-\varepsilon}^{u+\varepsilon} \frac{1}{2\pi\sqrt{\Delta_t}} \left(\int_{\mathbb{R}} |y| \exp\left[-\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2\right] \exp\left[-\frac{x^2}{2A_t}\right] dy\right) dx \\
&= \int_{u-\varepsilon}^{u+\varepsilon} \frac{1}{2\pi\sqrt{\Delta_t}} \exp\left[-\frac{x^2}{2A_t}\right] \left(\int_{\mathbb{R}} |y| \exp\left[-\frac{A_t}{2\Delta_t}\left(y - \frac{B_t}{A_t}x\right)^2\right] dy\right) dx \ .
\end{aligned} \tag{3.7}$$

Now using that $U_t = \Delta_t/A_t$, we can write

$$\frac{1}{2\pi\sqrt{\Delta_t}} = \frac{1}{\sqrt{2\pi A_t}} \frac{1}{\sqrt{2\pi\Delta_t/A_t}} = \frac{1}{\sqrt{2\pi A_t}} \frac{1}{\sqrt{2\pi U_t}} \ ,$$

so that the last line in (3.7) gives

$$\begin{aligned}
&\int_{u-\varepsilon}^{u+\varepsilon} \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{x^2}{2A_t}\right] \left(\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi U_t}} |y| \exp\left[-\frac{1}{2U_t}\left(y - \frac{B_t}{A_t}x\right)^2\right] dy\right) dx \\
&= \int_{u-\varepsilon}^{u+\varepsilon} \Phi_t(x)\,dx \ ,
\end{aligned}$$

where $\Phi_t(x) := \dfrac{1}{\sqrt{2\pi A_t}} \exp\left[-\dfrac{x^2}{2A_t}\right] \left(\displaystyle\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi U_t}} |y| \exp\left[-\frac{1}{2U_t}\left(y - \frac{B_t}{A_t}x\right)^2\right] dy\right)$.

Let us now see what the integrand of the integral with respect to y in the expression of $\Phi_t(x)$ is. To do so, we observe that this integrand has a similar shape as the density of a Gaussian random variable[11], that is to say

$$\frac{1}{\sqrt{2\pi U_t}} |y| \exp\left[-\frac{1}{2U_t}\left(y - \frac{B_t}{A_t}x\right)^2\right] = |y| \frac{1}{\sqrt{2\pi}\sqrt{U_t}} \exp\left[-\frac{1}{2}\left(\frac{y - \frac{B_t}{A_t}x}{\sqrt{U_t}}\right)^2\right] = |y| \Psi_{\frac{B_t}{A_t}x, U_t}(y) \ ,$$

[11] Recall that if $X \sim \mathcal{N}(\mu, \sigma^2)$ is a Gaussian random variable of mean μ and variance σ^2, then its density is given by $p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$.

where

$$\Psi_{\frac{B_t}{A_t}x,U_t}(y) := \frac{1}{\sqrt{2\pi}\sqrt{U_t}} \exp\left[-\frac{1}{2}\left(\frac{y-\frac{B_t}{A_t}x}{\sqrt{U_t}}\right)^2\right] .$$

Here one clearly recognizes the density of a Gaussian random variable, say Y, of mean $\mathbb{E}[Y] = B_t x/A_t$ and variance $\mathrm{Var}(Y) = U_t$. Using this density, we can rewrite $\Phi_t(x)$ as

$$\Phi_t(x) = \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{x^2}{2A_t}\right] \left(\int_{\mathbb{R}} |y| \Psi_{\frac{B_t}{A_t}x,U_t}(y)\, dy\right) . \tag{3.8}$$

Remark 3.3. Having a closer look at the integral over $\mathbb{R}$ in (3.8), we observe that

$$\Phi_t(x) = \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{x^2}{2A_t}\right] \mathbb{E}[|Y|] .$$

Then by (3.7) it follows that

$$\mathbb{E}\left[\mathbf{1}_{\{|F(t)-u|<\varepsilon\}}|F'(t)|\right] = \int_{u-\varepsilon}^{u+\varepsilon} \Phi_t(x)\, dx ,$$

and thus, plugging this into (3.2), we have

$$\mathbb{E}[N_u(F, I)] = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_I \int_{u-\varepsilon}^{u+\varepsilon} \Phi_t(x)\, dx\, dt . \tag{3.9}$$

The idea is now to apply the limit on the integral with respect to x, this means we will have to interchange the limit with the integral with respect to t. As already done in a previous reasoning, we may use Lebesgue's dominated convergence theorem. Thus we have to see that there exists an integrable function $\mu(t)$ on I such that $|\Phi_t(x)| \leq \mu(t)$. In order to do this, we use Cauchy-Schwarz inequality[12] to observe that for any random variable X, we have the inequality[13]

$$\mathbb{E}[|X|] \leq \sqrt{\mathbb{E}[X^2]} = \sqrt{\mathrm{Var}(X) + \mathbb{E}[X]^2} . \tag{3.10}$$

Now using the fact that $|Y| \geq 0$ implies $\mathbb{E}[|Y|] \geq 0$, we have, by Remark 3.3, that $\Phi_t(x)$ is a positive function, so that $|\Phi_t(x)| = \Phi_t(x)$. Applying the inequality in (3.10) to the random variable Y, it follows that

$$\mathbb{E}[|Y|] \leq \sqrt{\mathrm{Var}(Y) + \mathbb{E}[Y]^2} = \sqrt{U_t + \left(\tfrac{B_t}{A_t}x\right)^2} \leq \sqrt{U_t} + \tfrac{|B_t x|}{A_t} , \tag{3.11}$$

where for the last inequality we used that for $a, b \geq 0$, $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. Hence using (3.11) in the expression of $\Phi_t(x)$ in remark 3.3, it comes

$$\Phi_t(x) \leq \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{x^2}{2A_t}\right] \left(\sqrt{U_t} + \frac{|B_t x|}{A_t}\right) .$$

Now we notice that the function $x \mapsto \exp(-x^2)$ has a maximum at $x = 0$ and $\exp(0) = 1$, thus for all $x \in \mathbb{R}$, $\exp(-x^2) \leq 1$. In particular, for $|x| < 1$, we can write

$$\Phi_t(x) \leq \frac{1}{\sqrt{2\pi A_t}} \left(\sqrt{U_t} + \frac{|B_t x|}{A_t}\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{\sqrt{\Delta_t}}{A_t} + \frac{|B_t|}{A_t^{3/2}}\right) := \mu(t) .$$

In order to use Lebesgue's dominated convergence theorem, we add a fourth assumption to our random function F,

[12] Cauchy-Schwarz inequality states that for any two random variables X and Y, $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$.

[13] Indeed, by Cauchy-Schwarz inequality, we obtain that $\mathbb{E}[|X|] = \mathbb{E}[1 \cdot |X|] \leq \sqrt{\mathbb{E}[1]\,\mathbb{E}[|X|^2]} = \sqrt{\mathbb{E}[X^2]}$.

$$(A4) \quad \textit{The function } \mu(t) \textit{ is integrable on } I, \textit{ i.e. } \int_I \mu(t)\,dt < \infty \ .$$

Hence assuming $(A4)$ and using dominated convergence, (3.9) gives

$$\mathbb{E}\left[N_u(F,I)\right] = \int_I \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{u-\varepsilon}^{u+\varepsilon} \Phi_t(x)\,dx\,dt = \int_I \Phi_t(u)\,dt \ , \tag{3.12}$$

where $\Phi_t(u)$ is obtained using (3.8),

$$\Phi_t(u) = \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{u^2}{2A_t}\right] \left(\int_{-\infty}^{+\infty} |y| \Psi_{\frac{B_t}{A_t}u, U_t}(y)\,dy\right) \ . \tag{3.13}$$

We are now in position to state Kac-Rice formula for u-crossings in the next theorem.

Theorem 3.4 (Kac-Rice theorem for u-crossings)**.** *Let $I \subset \mathbb{R}$ be an interval. For $k = 0, 1, \ldots, n$, consider $f_k : I \to \mathbb{R}$ smooth functions and a_k independent Gaussian random variables with mean zero and variance v_k defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, if the random function*

$$F : I \to \mathbb{R}, F(t) = \sum_{k=0}^{n} a_k f_k(t)$$

of covariance function $K(x,y) = \mathbb{E}\left[F(x)F(y)\right]$ satisfies assumptions $(A1)$, $(A2)$, $(A3)$, $(A4)$, the expected number of u-crossings of F on I is given by

$$\mathbb{E}\left[N_u(F,I)\right] = \int_I \frac{1}{\sqrt{2\pi A_t}} \exp\left[-\frac{u^2}{2A_t}\right] \left(\int_{-\infty}^{+\infty} |y| \Psi_{\frac{B_t}{A_t}u, U_t}(u)\,dy\right) dt \ , \tag{3.14}$$

where

$$\Psi_{\frac{B_t}{A_t}x, U_t}(y) = \tfrac{1}{\sqrt{2\pi}\sqrt{U_t}} \exp\left[-\tfrac{1}{2}\left(\tfrac{y - \frac{B_t}{A_t}x}{\sqrt{U_t}}\right)^2\right] ,$$

$$A_t = K(t,t) \ , B_t = K_x(x,y)\big|_{x=y=t} \ , C_t = K_{xy}(x,y)\big|_{x=y=t} \ , U_t = \tfrac{A_t C_t - {B_t}^2}{A_t} \ .$$

3.2 Kac-Rice formulas for 0-crossings

In this part of our paper, we will use the previous computations in order to derive Kac-Rice formula for 0-crossings. We will then state Kac-Rice theorem for 0-crossings and see that it is much simpler to handle with than the general case for u-crossings.

Replacing $u = 0$ in the expression of $\Phi_t(u)$ in (3.13), we obtain

$$\Phi_t(0) = \frac{1}{\sqrt{2\pi A_t}} \left(\int_{-\infty}^{+\infty} |y| \Psi_{0,U_t}(y)\,dy\right) \ , \tag{3.15}$$

where we obtain $\Psi_{0,U_t}(y)$ by (3.1) as

$$\Psi_{0,U_t}(y) = \frac{1}{\sqrt{2\pi}\sqrt{U_t}} \exp\left[-\frac{1}{2}\left(\frac{y}{\sqrt{U_t}}\right)^2\right] \ . \tag{3.16}$$

By an easy computation of the integral in (3.15), we obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} |y| \Psi_{0,U_t}(y)\,dy &= \frac{2}{\sqrt{2\pi}\sqrt{U_t}} \int_0^{+\infty} y \exp\left[-\frac{1}{2}\left(\frac{y}{\sqrt{U_t}}\right)^2\right] dy \\ &= \frac{2U_t}{\sqrt{2\pi}\sqrt{U_t}} = \frac{2\sqrt{U_t}}{\sqrt{2\pi}} \ , \end{aligned}$$

and consequently, using $U_t = \Delta_t / A_t$, it follows

$$\Phi_t(0) = \frac{1}{\sqrt{2\pi A_t}} \cdot \frac{2\sqrt{U_t}}{\sqrt{2\pi}} = \frac{\sqrt{U_t}}{\pi \sqrt{A_t}} = \frac{\sqrt{\Delta_t}}{\pi A_t} := \frac{1}{\pi} \rho_t \ , \tag{3.17}$$

where $\rho_t := \sqrt{\Delta_t}/A_t$, so that

$$\begin{aligned} {\rho_t}^2 = \frac{A_t C_t - {B_t}^2}{{A_t}^2} &= \frac{K(x,y)\big|_{x=y=t} K_{xy}(x,y)\big|_{x=y=t} - (K_x(x,y)\big|_{x=y=t})^2}{(K(x,y)\big|_{x=y=t})^2} \\ &= \frac{\partial^2}{\partial x \partial y} \log K(x,y)\Big|_{x=y=t} . \end{aligned}$$

Remark 3.5. Note that the last equality is non-trivial, but can be easily checked by calculating the expression of the last line. We use the fact $K_x(x,y)$ and $K_y(x,y)$ coincide when we evaluate them at $x = y = t$.

Hence, writing (3.12) for $u = 0$, we obtain using (3.17),

$$\mathbb{E}\left[N_0(F, I)\right] = \int_I \Phi_t(0)\, dt = \frac{1}{\pi} \int_I \rho_t \, dt \ .$$

Remark 3.6. The quantity $\frac{1}{\pi}\rho_t$ in the integral above represents the expected density of zeros of F at t.

We will now state Kac-Rice formula for 0-crossings in the next theorem.

Theorem 3.7 (Kac-Rice theorem for 0-crossings)**.** *Let $I \subset \mathbb{R}$ be an interval. For $k = 0, 1, \ldots, n$, consider $f_k : I \to \mathbb{R}$ smooth functions and a_k independent Gaussian random variables with mean zero and variance v_k defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then, if the random function*

$$F : I \to \mathbb{R}, F(t) = \sum_{k=0}^{n} a_k f_k(t)$$

of covariance function $K(x,y) = \mathbb{E}\left[F(x)F(y)\right]$ satisfies assumptions[14] *(A1), (A2), (A3), (A4), the expected number of zeros of F on I is given by*

$$\mathbb{E}\left[N_0(F, I)\right] = \frac{1}{\pi} \int_I \rho_t \, dt \ , \tag{3.18}$$

where ρ_t is given by

$$\rho_t = \left(\frac{K(x,y)\big|_{x=y=t} K_{xy}(x,y)\big|_{x=y=t} - (K_x(x,y)\big|_{x=y=t})^2}{(K(x,y)\big|_{x=y=t})^2} \right)^{1/2} , \tag{3.19}$$

or equivalently

$$\rho_t = \left(\frac{\partial^2}{\partial x \partial y} \log K(x,y)\Big|_{x=y=t} \right)^{1/2} . \tag{3.20}$$

Comparing this theorem to the general statement in Theorem 3.18, it seems to be easier to deal with this one, since we got rid of the integral involving the density Ψ. To prove the important results anounced in the introduction, we will use this theorem, as we are interested in the number of zeros of random polynomials.

[14] We obviously replace $u = 0$ in the assumptions, in particular in *(A1)* and *(A2)*.

3.3 Equivalent Rice formulas

In this section, we will see more general Rice formulas as those stated in the previous theorems. However, we are not going to use these formulas to prove the main results, but as Kac-Rice formula is mainly known under these formulas, we will just enounce them and make some comments. We mainly follow the arguments in [2].

To be as general as possible, we may assume that the coefficients a_k are independent random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$. As usual, we consider smooth functions $f_k : I \to \mathbb{R}$ for $k = 0, 1, \dots, n$ and form the random function on an interval I,

$$F(t) = \sum_{k=0}^{n} a_k f_k(t) \ .$$

As we already pointed out in the beginning, we may look at the random function as a stochastic process $\mathcal{F} = \{F(t) : t \in I\}$. In order to state the general Rice theorem rigorously, let us make some assumptions on our stochastic process $\mathcal{F}$.

(A1) *The paths of the process $\mathcal{F}$ are almost surely C^1.*

(A2) *The functions $(t, x) \mapsto p_{F(t)}(x)$ and $(t, x, y) \mapsto p_{(F(t),F'(t))}(x, y)$ are continuous for $t \in I$, x in a neighborhood of u and $y \in \mathbb{R}$.*

(A3) *For all $t \in I$ and x in a neighborhood of u, the conditional expectation $\mathbb{E}\left[|F'(t)| \middle| F(t) = x\right]$ has a continuous version.*

(A4) *If $\omega(F', h) = \sup_{s,t\in I, |s-t|\leq h} |F'(s) - F'(t)|$ denotes the modulus of continuity of F' on I, then $\mathbb{E}[\omega(F', h)] \to 0$ when $h \to 0$.*

Theorem 3.8. *Let $I \subset \mathbb{R}$ be an interval. If the stochastic process $\mathcal{F}$ satifies assumptions (A1), (A2), (A3), (A4), then the expected number of u-crossings on I of the random function F is given by*

$$\mathbb{E}[N_u(F, I)] = \int_I \mathbb{E}\left[|F'(t)| \middle| F(t) = u\right] p_F(u)\, dt \ , \tag{3.21}$$

where p_F denotes the probability density of the random variable F.

Remark 3.9. The proof of this theorem is a bit technical, hence we will not give the details. The idea is again to use Kac's counting formula.

Another form under which Kac-Rice formula is often written is the next one. Here we assume that all a_k have mean zero and constant variance v. We state it in the next corollary.

Corollary 3.10. *Let $I \subset \mathbb{R}$ be an interval. Assume that the coefficients of the random function have all mean zero and constant variance v. If the stochastic process $\mathcal{F}$ satifies assumptions (A1), (A2), (A3), (A4), then the expected number of u-crossings on I of the random function F is given by*

$$\mathbb{E}[N_u(F, I)] = \int_I \int_{\mathbb{R}} |y| p_{(F(t),F'(t))}(u, y)\, dy\, dt \ , \tag{3.22}$$

where $p_{(F,F')}$ denotes the probability density of the random vector (F, F').

Proof. Since for $k = 1, \dots, n$ the coefficients a_k are independent with mean zero and constant variance v, we have[15]

$$\mathbb{E}\left[F(t)^2\right] = \mathrm{Var}(F(t)) = \sum_{k=1}^{n} \mathrm{Var}(a_k) = nv \ ,$$

[15] Recall that if $X_1, \dots, X_n$ are random variables we have $\mathrm{Var}(\sum_{k=1}^n X_k) = \sum_{k=1}^n \mathrm{Var}(X_k) + 2\sum_{1\leq i\leq j\leq n} \mathrm{Cov}(X_i, X_j)$. Moreover, if the variables X_k are independent, $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$.

so that differentiating both sides with respect to t leads to[16]

$$\mathbb{E}\left[F(t)^2\right]' = \mathbb{E}\left[(F(t)^2)'\right] = 2\mathbb{E}\left[F(t)F'(t)\right] = 0\ ,$$

and therefore $\mathbb{E}\left[F(t)F'(t)\right] = 0$, which means that $F(t)$ and $F'(t)$ are independent. Now using the corresponding property of the conditional expectation[17], we can write the conditional expectation in (3.21) as an ordinary expectation, namely $\mathbb{E}\left[|F'(t)|\big|F(t) = u\right] = \mathbb{E}\left[|F'(t)|\right]$. Thus evaluating the integral in (3.21), $\mathbb{E}\left[N_u(F, I)\right]$ is equal to

$$\int_I \mathbb{E}\left[|F'(t)|\right] p_F(u)\, dt = \int_I \left(\int_{\mathbb{R}} |y| p_{F'}(y)\, dy\right) p_F(u)\, dt = \int_I \int_{\mathbb{R}} |y| p_{F'}(y) p_F(u)\, dy\, dt\ ,$$

and since F and F' are independent, $p_F(u)p_{F'}(y) = p_{(F,F')}(u, y)$. Plugging this into the last equality, we obtain the desired form (3.22). □

3.4 Factorial moments of the number of crossings

In this subsection we will provide a formula that permits to study the factorial moments of the random variable giving the number of crossings of a random polynomial. More precisely, we will state a theorem giving the k-th factorial moment of crossings. We will see in the next section how one can use this formula in order to obtain the variance of the number of zeros, for instance.

The following theorem is taken from [2].

Theorem 3.11. *Let $I \subset \mathbb{R}$ be an interval and $\mathcal{F} = \{F(t) : t \in I\}$ a Gaussian stochastic process with C^1-paths. Let $k \geq 1$ be an integer. Assume that for k pairwise different $t_1, \ldots, t_k$ in I, the random variables $F(t_1), \ldots, F(t_k)$ have a nondegenerate joint distribution. Then we have*

$$\mathbb{E}\left[N_u^{[k]}(F, I)\right] = \int_{I^k} \mathbb{E}\left[|F'(t_1)\ldots F'(t_k)|\big|F(t_1) = \ldots = F(t_k) = u\right] \cdot p_{(F(t_1),\ldots,F(t_k))}(u, \ldots, u)\, dt_1 \ldots dt_k\ , \quad (3.23)$$

where $M^{[k]} = M(M-1)\ldots(M-k+1)$ for positive integers M and k.

As in the section before, assuming that the coefficients of the stochastic process have mean zero and constant variance, we can give an equivalent formula to (3.23), namely

$$\mathbb{E}\left[N_u^{[k]}(F, I)\right] = \int_{I^k} \int_{\mathbb{R}^k} |y_1|\ldots|y_k| p_{(F(t_1),\ldots,F(t_k),F'(t_1),\ldots,F'(t_k))}(u, \ldots, u, y_1, \ldots, y_k)\, dy_1 \ldots dy_k\, dt_1 \ldots dt_k\ . \quad (3.24)$$

Remark 3.12. The one-dimensional formulas in Theorem 3.8 respectively in Corollary 3.10 are obtained if we replace $k = 1$ in (3.23) respectively (3.24).

Finiteness of the moments of crossings. For what comes, we follow the corresponding parts in Section 3.2 of [2] and Section 2.1 of [10]. In many applications, it is irrelevant to know the exact value of the expectation of the factorial moments of crossings given in (3.23). One may rather be interested if this number is finite or not. Moreover, in certain cases, the problem to compute the r.h.s. of (3.23) is a very challenging task, that is why it sometimes suffices to know finiteness. The question of efficient procedures to approximate the factorial moments is

[16] Note that one can pass the derivative inside the expectation.

[17] Let X be a random variable and $\mathcal{C}$ a conditional event. If X and $\mathcal{C}$ are independent, then $\mathbb{E}\left[X|\mathcal{C}\right] = \mathbb{E}\left[X\right]$.

still a subject of great interest. Finiteness of moments of crossings have been considered by, among others, Belayev (1966) and Cuzick (1975). Belayev proposed a sufficient condition for the finiteness of the k-th factorial moment for the number of zeros on the interval $[0,t]$. This condition is given by means of the covariance matrix Σ_k of the Gaussian vector $(F(t_1),\dots,F(t_k))$ and of the conditional variances $\sigma_i^2 := \mathrm{Var}(F'(t_i)|F(t_j)=0, 1\le j\le k)$, namely

Theorem 3.13 (Belayev,1967). *Let $I\subset\mathbb{R}$ be an interval. If*

$$\int_0^t dt_1 \dots \int_0^t dt_k \left(\frac{\prod_{i=1}^k \sigma_i^2}{\det \Sigma_k}\right)^{1/2} < \infty \ , \tag{3.25}$$

then $\mathbb{E}\left[N_u^{[k]}(F,I)\right] < \infty$.

In 1975 Cuzick proved that the condition for finiteness given by Belayev was not only sufficient, but actually also necessary, that is the k-th factorial moment of the number of crossings is finite if and only if condition (3.25) is satisfied. For factorial moments of order 2, Cramér and Leadbetter deduced a sufficient condition by means of the covariance function of a stationary stochastic process. This condition is nowadays known under the name of *Geman condition*. We state it in the following theorem.

Theorem 3.14 (Cramér and Leadbetter). *Let F be a stationary stochastic process with covariance function K. If $\exists\delta>0$ such that*

$$R(t) := \frac{K''(t)-K''(0)}{t} \in L^1([0,\delta],dx) \ , \tag{3.26}$$

then $\mathbb{E}\left[N_u^{[2]}(F,[0,\delta])\right] < \infty$.

In 1972, Geman proved that the condition (3.26) was also necessary, by showing that if $R(t)$ diverges on $(0,\delta)$, then so does the integral appearing in the r.h.s. of (3.24).

4 Random algebraic polynomials

For the following section, we closely follow [12]. Let us first define a random algebraic polynomial.

Definition 4.1. Let $I\subset\mathbb{R}$ be an interval. A random algebraic polynomial of degree n, $F: I\to\mathbb{R}$ is given by

$$F(t) = \sum_{k=0}^{n} a_k t^k \ ,$$

where a_k are random variables for $k=0,1,\dots,n$ defined on the same probability space $(\Omega,\mathcal{A},\mathbb{P})$.

Remark 4.2. Comparing this definition to defintion 2.1 of a random function, we see that we have to take $f_k(t)=t^k$ for all k, in order to obtain an algebraic polynomial.

For the rest of this section, we will assume that the coefficients a_k are Gaussian variables of mean zero and variance v_k.

In Kac-Rice theorem we derived a formula giving the expected number of 0-crossings of a random function on an interval I of $\mathbb{R}$. A question that one could ask is how does this expected number change if we change the lenght of the interval I. This is what we will state in the following lemma.

Lemma 4.3. *Let $F : I \to \mathbb{R}$ be a random algebraic polynomial. Assume that the coefficients a_k are independent Gaussian random variables with mean zero and variance v_k. Then we have the following results.*

(*i*) $\mathbb{E}\left[N_0(F, \mathbb{R}_+)\right] = \mathbb{E}\left[N_0(F, \mathbb{R}_-)\right]$.

(*ii*) *Assume additionally that the variances v_k verify the symmetry condition $v_k = v_{n-k}$ for all k. Then* $\mathbb{E}\left[N_0(F, (0,1))\right] = \mathbb{E}\left[N_0(F, (1, \infty))\right]$.

(*iii*) *Under the same assumptions as in (ii),*
$\mathbb{E}\left[N_0(F, \mathbb{R})\right] = 4\mathbb{E}\left[N_0(F, (0,1))\right] = 4\mathbb{E}\left[N_0(F, (1, \infty))\right]$.

Proof. (*i*) If we define $F^-(t) := F(-t)$ we have that

$$F^-(t) = \sum_{k=0}^{n} a_k (-t)^k = \sum_{k=0}^{n} (-1)^k a_k t^k \ .$$

The random variables $(-1)^k a_k$ are i.i.d. Gaussian random variables with mean zero and variance v_k, since $\mathbb{E}\left[(-1)^k a_k\right] = (-1)^k \mathbb{E}\left[a_k\right] = 0$ and $\mathrm{Var}((-1)^k a_k) = (-1)^{2k}\mathrm{Var}(a_k) = v_k$. So the random polynomials $F$and F^- have the same law and thus $\mathbb{E}\left[N_0(F, \mathbb{R}_+)\right] = \mathbb{E}\left[N_0(F, \mathbb{R}_-)\right]$.

(*ii*) Define $\tilde{F}(t) := t^n F(1/t)$. Then

$$\tilde{F}(t) = t^n \sum_{k=0}^{n} a_k t^{-k} = \sum_{k=0}^{n} a_k t^{n-k} = \sum_{k=0}^{n} a_{n-k} t^k \ .$$

The random variables a_{n-k} all have mean zero and variance v_k, since $\mathbb{E}\left[a_k\right] = 0$ for all k, and $\mathrm{Var}(a_{n-k}) = v_{n-k} = v_k$, by hypothesis. So, again, the random polynomials F and $\tilde{F}$ have the same law and thus $\mathbb{E}\left[N_0(F, (0,1))\right] = \mathbb{E}\left[N_0(F, (1, \infty))\right]$.

(*iii*) This is a direct consequence of (*i*) and (*ii*). Indeed by (*i*), we have

$$\mathbb{E}\left[N_0(F, \mathbb{R})\right] = \mathbb{E}\left[N_0(F, \mathbb{R}_-)\right] + \mathbb{E}\left[N_0(F, \mathbb{R}_+)\right] = 2\mathbb{E}\left[N_0(F, \mathbb{R}_+)\right] \ ,$$

and since $\mathbb{R}_+ = [0,1] \cup [1, \infty)$, it follows by (*ii*)[18]

$$\begin{aligned} \mathbb{E}\left[N_0(F, \mathbb{R}_+)\right] &= \mathbb{E}\left[N_0(F, (0,1))\right] + \mathbb{E}\left[N_0(F, (1, \infty))\right] \\ &= 2\mathbb{E}\left[N_0(F, (0,1))\right] \\ &= 2\mathbb{E}\left[N_0(F, (1, \infty))\right] \ , \end{aligned}$$

so that $\mathbb{E}\left[N_0(F, \mathbb{R})\right] = 4\mathbb{E}\left[N_0(F, (0,1))\right] = 4\mathbb{E}\left[N_0(F, (1, \infty))\right]$.

□

4.1 Expected number of zeros

Before we can apply Kac-Rice theorem to random algebraic polynomials we note that such polynomials verify assumptions (*A1*), (*A2*), (*A3*) and (*A4*) of Theorem 3.7.

Let us now pass to the two fundamental results concerning random algebraic polynomials with coefficients whose variances satisfy the symmetry condition enounced in Lemma 4.3: *Kac polynomials* and *Kostlan-Shub-Smale polynomials*.

[18] Notice that $\mathbb{E}\left[N_0(F, [0,1])\right] = \mathbb{E}\left[N_0(F, (0,1))\right]$, since the integral over a single point is 0, so adding a point to the interval does not change the expected number of zeros of a random function.

4.1.1 Kac polynomials

Consider the random Kac polynomial, that is a random algebraic polynomial as in definition 4.1,

$$F(t) = \sum_{k=0}^{n} a_k t^k \ ,$$

where the coefficients a_k are i.i.d. standard Gaussian variables, i.e. with mean zero and variance one for all $k = 0, 1, \ldots, n$.

Remark 4.4. Note that since the coefficients have all variance one, they fulfill the the symmetry condition.

Using the definition of the covariance function of the random polynomial F in (2.3), it follows

$$K(x,y) = \sum_{i=0}^{n} x^i y^i = \sum_{i=0}^{n} (xy)^i = \frac{1-(xy)^{n+1}}{1-xy} \ .$$

In order to calculate the expected number of zeros of the random Kac polynomial, we are going to use formula (3.20) to compute ρ_t. We have

$$\log K(x,y) = \log \frac{1-(xy)^{n+1}}{1-xy} = \log\left(1-(xy)^{n+1}\right) - \log\left(1-xy\right) \ ,$$

so that

$$\frac{\partial}{\partial x} \log K(x,y) = \frac{y}{1-xy} - \frac{(n+1)(xy)^n y}{1-(xy)^{n+1}} \ ,$$

$$\frac{\partial^2}{\partial x \partial y} \log K(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} \log K(x,y)\right) = \frac{1}{(1-xy)^2} - \frac{(n+1)^2 (xy)^n}{(1-(xy)^{n+1})^2} \ .$$

Evaluating this second partial derivative at $x = y = t$, we obtain

$$\left.\frac{\partial^2}{\partial x \partial y} \log K(x,y)\right|_{x=y=t} = \frac{1}{(1-t^2)^2} - \frac{(n+1)^2 t^{2n}}{(1-t^{2n+2})^2} =: H_n(t) \ , \tag{4.1}$$

and therefore, using Lemma 4.3, the expected number of zeros of F over $\mathbb{R}$ is given by

$$\mathbb{E}\left[N_0(F,\mathbb{R})\right] = \frac{4}{\pi} \int_1^{+\infty} \sqrt{H_n(t)}\, dt \ . \tag{4.2}$$

Remark 4.5.

- The function $\sqrt{H_n(t)}$ is the expected density of zeros of random Kac polynomials. Plotting its graph for different values of n, we see that if n increases, the graph has two remarkable peaks at $t = -1$ and $t = 1$. This is actually an interesting result, namely the real zeros of a random Kac polynomial tend to concentrate near -1 and 1. In Figure 3, we see the three different densities for random Kac polynomials of degrees 10, 20 and 30.

- We know that any polynomial of degree n has n complex roots. One can show that the complex roots of a random Kac polynomial of degree n are uniformly distributed on the unit circle. (see Section 6)

The following theorem gives the asymptotic behaviour of the expected number of zeros over $\mathbb{R}$.

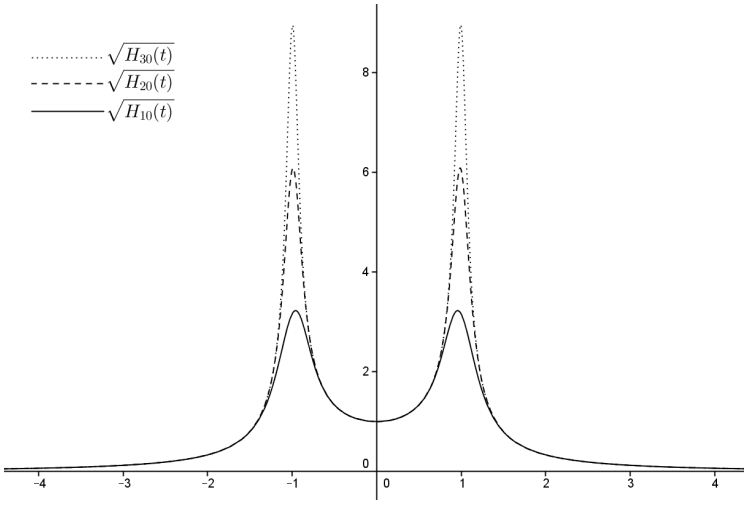

Figure 3: Density of real zeros of random Kac polynomials

Theorem 4.6 (Kac, Edelman-Kostlan). *Let F be the random Kac polynomial of degree n. As $n \to \infty$ we have*

$$\mathbb{E}\left[N_0(F,\mathbb{R})\right] = \frac{2}{\pi}(\log n + C) + o(1) \ ,$$

where the constant C is given by

$$C = \log 2 + \int_0^\infty \left\{ \left(\frac{1}{x^2} - \frac{1}{\sinh x^2} \right)^{1/2} - \frac{1}{x+1} \right\} dx \ .$$

Proof. Since we want the asymptotic behaviour of the expected number of zeros of the random polynomial F, we have to compute

$$\lim_{n\to\infty} \mathbb{E}\left[N_0(F,\mathbb{R})\right] = \lim_{n\to\infty} \frac{4}{\pi} \int_1^{+\infty} \sqrt{H_n(t)}\, dt \tag{4.3}$$

Making the change of variables $t = 1 + x/n$, the integral in (4.3) becomes

$$\int_1^{+\infty} \sqrt{H_n(t)}\, dt = \int_0^{+\infty} \sqrt{\frac{1}{n^2} H_n(1+\frac{x}{n})}\, dx = \int_0^{+\infty} \sqrt{G_n(x)}\, dx \ , \tag{4.4}$$

where $G_n(x) := \frac{1}{n^2} H_n(1+\frac{x}{n})$. Let us start by computing $G_n(x)$. Using (4.1),

$$G_n(x) = \underbrace{\left(\frac{1}{n(1-(1+\frac{x}{n})^2)} \right)^2}_{=:{A_n}^2(x)} - \underbrace{\left(\frac{\frac{n+1}{n}(1+\frac{x}{n})^n}{1-(1+\frac{x}{n})^{2n}(1+\frac{x}{n})^2} \right)^2}_{=:{B_n}^2(x)} \ .$$

To compute the limit of G_n when n tends to infinity, we compute limits of A_n and B_n. Using that $(1+x/n)^n \to e^x$, when $n \to \infty$, we obtain for $x > 0$,

$$A_n(x) = \frac{1}{n(1-(1+\frac{x}{n})^2)} = \frac{1}{n\frac{x}{n}(2-\frac{x}{n})} = \frac{1}{x(2-\frac{x}{n})} \overset{n\to\infty}{\longrightarrow} \frac{1}{2x} =: A(x) \ ,$$

$$B_n(x) = \frac{\frac{n+1}{n}(1+\frac{x}{n})^n}{1-(1+\frac{x}{n})^{2n}(1+\frac{x}{n})^2} \overset{n\to\infty}{\longrightarrow} \frac{e^x}{1-e^{2x}} = \frac{1}{e^{-x}-e^x} = -\frac{1}{2\sinh x} =: B(x) \ ,$$

so that

$$G(x) := \lim_{n\to\infty} G_n(x) = A^2(x) - B^2(x) = \frac{1}{4x^2} - \frac{1}{4\sinh^2 x} = \frac{\sinh^2 x - x^2}{4x^2\sinh^2 x} \ .$$

Note that $G(x)$ extends to 0 by continuity. Indeed the Taylor expansion[19] at $x = 0$ gives $\sinh^2 x = x^2 + x^4/3 + O(x^6)$, and then

$$\sinh^2 x - x^2 = \frac{x^4}{3} + O(x^6) \overset{x\to 0}{\longrightarrow} 0 \ .$$

In addition, $G_n(x)$ converges uniformly to $G(x)$ for $x \in [0,1]$. Thus the integral in (4.4) can be rewritten as[20]

$$\int_0^{+\infty} \sqrt{G_n(x)}\, dx = \int_0^1 \sqrt{G_n(x)}\, dx + \int_1^{+\infty} (\sqrt{G_n(x)} - A_n(x))\, dx + \int_1^{+\infty} A_n(x)\, dx \ .$$

Now letting n tend to infinity and using Lebesgue's dominated convergence theorem, we may pass the limit inside the integrals on both sides, that is

$$\int_0^{+\infty} \sqrt{G(x)}\, dx = \int_0^1 \sqrt{G(x)}\, dx + \int_1^{+\infty} (\sqrt{G(x)} - A(x))\, dx + \int_1^{+\infty} A_n(x)\, dx + o(1) \ . \quad (4.5)$$

Let us compute the two remainig integrals of the r.h.s. of (4.5). Since $A(x) = 1/(2x)$, we have

$$\int_1^{+\infty} (\sqrt{G(x)} - A(x))\, dx = \int_1^{+\infty} \left(\sqrt{G(x)} - \frac{1}{2x}\right) dx = \int_1^{+\infty} \sqrt{G(x)}\, dx - \frac{1}{2}\int_1^{+\infty} \frac{1}{x}\, dx \ .$$

The partial fraction decomposition of $A_n(x)$ gives

$$A_n(x) = \frac{1}{x(2 - \frac{x}{n})} = \frac{n}{x(2n+x)} = \frac{1}{2x} - \frac{1}{2(2n+x)} \ ,$$

and therefore, after computations, we obtain

$$\int_1^{+\infty} A_n(x)\, dx = \int_1^{+\infty} \left(\frac{1}{2x} - \frac{1}{2(2n+x)}\right) dx = \frac{1}{2}\log(2n+1) \ .$$

Now observe that

$$\int_1^{+\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx - \int_0^1 \frac{1}{x+1}\, dx = 0 \ ,$$

so that

$$\int_1^{+\infty} \frac{1}{x}\, dx = \int_0^1 \frac{1}{x+1}\, dx + \int_1^{+\infty} \frac{1}{x+1}\, dx = \int_0^{+\infty} \frac{1}{x+1}\, dx \ .$$

Thus, plugging this into (4.5), it comes

$$\begin{aligned}
& \int_0^1 \sqrt{G(x)}\, dx + \int_1^{+\infty} \sqrt{G(x)}\, dx - \frac{1}{2}\int_1^{+\infty} \frac{1}{x}\, dx + \frac{1}{2}\log(2n+1) + o(1) \\
= \ & \int_0^{+\infty} \sqrt{G(x)}\, dx - \frac{1}{2}\int_0^{+\infty} \frac{1}{x+1}\, dx + \frac{1}{2}\log(2n+1) + o(1) \\
= \ & \int_0^{+\infty} \left(\sqrt{G(x)} - \frac{1}{2(x+1)}\right) dx + \frac{1}{2}\log(2n+1) + o(1) \\
= \ & \frac{1}{2}\int_0^{+\infty} \left\{ \left(\frac{1}{x^2} - \frac{1}{\sinh^2 x}\right)^{1/2} - \frac{1}{x+1} \right\} dx + \frac{1}{2}\log(2n+1) + o(1) \ .
\end{aligned}$$

[19] Recall that the Taylor expansion of sinh at $x = 0$ is given by $\sinh x = \sum_{k=0}^{\infty} \frac{x^{1+2k}}{(1+2k)!}$, so expanding this up to order 5, we get $\sinh x = x + x^3/6 + O(x^5)$, and therefore $\sinh^2 x = x^2 + x^4/3 + O(x^6)$.

[20] One checks that the functions $x \mapsto \sqrt{G_n(x)} - A_n(x)$ are integrable on $[1,\infty)$.

Writing $\log(2n+1) = \log 2 + \log n + o(1)$, it follows by (4.2) that

$$\begin{aligned} & \frac{4}{\pi}\int_1^{+\infty}\sqrt{H_n(t)}\,dt = \frac{4}{\pi}\int_0^{+\infty}\sqrt{G_n(x)}\,dx \\ = \; & \frac{2}{\pi}\left(\log n + \log 2 + \int_0^{+\infty}\left\{\left(\frac{1}{x^2}-\frac{1}{\sinh^2 x}\right)^{1/2} - \frac{1}{x+1}\right\}dx\right) + o(1)\ , \end{aligned}$$

which is what we wanted to prove. □

By this theorem we proved our first result, namely that the expected number of zeros of random Kac polynomials of degree n is asymptotic to $\frac{2}{\pi}\log n$.

4.1.2 Kostlan-Shub-Smale polynomials

The Kostlan-Shub-Smale polynomial (KSS for short) is defined as the random algebraic polynomial, where the coefficients a_k are independent Gaussian variables with mean zero and variance

$$\mathrm{Var}(a_k) = \binom{n}{k}\ .$$

The KSS polynomials are also called binomial polynomials, due to the binomial coefficient appering in the above definition.

Remark 4.7. Also here, we see that the variances of the coefficients of the KSS polynomials verify the symmetry condition of Lemma 4.3, since for all $0 \le k \le n$ the binomial coefficients satisfy $\binom{n}{k} = \binom{n-k}{k}$.

Our goal of this section is to compute the expected number of zeros of the KSS polynomial over $\mathbb{R}$.

The KSS polynomials have covariance function

$$K(x,y) = \sum_{i=0}^{n}\binom{n}{i}x^i y^i = \sum_{i=0}^{n}\binom{n}{i}(xy)^i = (1+xy)^n\ .$$

We are going to use formula (3.20) of Kac-Rice theorem in order to compute ρ_t. Since $\log K(x,y) = \log(1+xy)^n = n\log(1+xy)$, after computations, we obtain for the partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x}\log K(x,y) &= n\frac{\partial}{\partial x}\log(1+xy) = \frac{ny}{1+xy}\ , \\ \frac{\partial^2}{\partial x\partial y}\log K(x,y) &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\log K(x,y)\right) = \frac{n}{(1+xy)^2}\ . \end{aligned}$$

To obtain ρ_t, we evaluate the second partial derivative at $x = y = t$ and take the square root,

$$\rho_t = \sqrt{\frac{n}{(1+t^2)^2}} = \frac{\sqrt{n}}{1+t^2}\ .$$

Thus the expected number of zeros of the KSS polynomial over $\mathbb{R}$ is given by

$$\mathbb{E}\left[N_0(F,\mathbb{R})\right] = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sqrt{n}}{1+t^2}\,dt = \frac{1}{\pi}\sqrt{n}\pi = \sqrt{n}\ .$$

Thus, we have shown that for random KSS polynomials of degree n, the expected number of zeros is equal to $\sqrt{n}$.

Remark 4.8. Comparing this result to the one obtained previously for the Kac polynomial, we see that the KSS random polynomials have on average more zeros over $\mathbb{R}$ than the Kac random polynomials. Moreover for the KSS polynomials we have an exact value for the expected number of zeros.

Application: Expected number of fixed points of a rational function. This application is taken from [7]. Let us see how one can use this result in order to compute the expected number of fixed points of a rational mapping. Consider two independent random polynomials $P(t)$ and $Q(t)$ of degree n, with coefficients as in the Kostlan statistics. Our goal is to determine the expected number of fixed points of the rational function $f(t) := P(t)/Q(t)$.

The equation

$$P(t) - tQ(t) = 0 \tag{4.6}$$

gives zeros of the polynomial $P(t) - tQ(t)$ of degree $n+1$. But since this polynomial is a linear combination of Kostlan polynomials, it is itself a Kostlan polynomial, for which we have seen that the expected number of zeros is exactly the square root of the degree, i.e. $\sqrt{n+1}$ in our case. On the other hand, the equation in (4.6) is equivalent to

$$f(t) = \frac{P(t)}{Q(t)} = t \ ,$$

and thus, we obtain that the expected number of fixed points of the rational function f is $\sqrt{n+1}$.

4.2 Expected number of critical points

We point out two more results for random algebraic polynomials, namely the asymptotic for the expected number of critical points of Kac and Kostlan polynomials, i.e. the number of zeros of the derivative of these polynomials.

Intuitively, we expect that that a given polynomial has more critical points than zeros. M. Das has shown that the expected number of critical points of a random algebraic Kac polynomial of degree n is asymptotic to $\frac{1+\sqrt{3}}{\pi}\log n$, for large n, whereas for random KSS polynomials of degree n, it can be shown that the expected number of critical points is asymptotic to $\sqrt{3n-2}$. Thus, comparing these results to the corresponding ones for the number of zeros established previously, they confirm our feeling that a polynomial has more critical points that zeros, but they also tell us that the number of critical points in both cases is of the same order than the corresponding number of zeros.

4.3 Variance of the number of zeros

In this section, we will show how one can use Rice formula giving the second moment in order to compute the variance of the number of zeros. For this, we may assume that $\{F(t) : t \in I\}$ is a stationary centered Gaussian process[21] meaning that its covariance function K only depends on one variable, i.e., for $t > s$,

$$K(t-s) = \mathbb{E}\left[F(s)F(t)\right] \ .$$

In order to simplify the coming computations, we suppose that the covariance function of F at 0 is 1, $K(0) = 1$. For the following part, we write N_0 for $N_0(F, I)$. We have

$$\mathrm{Var}(N_0) = \mathbb{E}\left[{N_0}^2\right] - \mathbb{E}\left[N_0\right]^2 \ .$$

[21] Note that a non stationary Gaussian process can be made stationary by homogenising it. For instance, homogenising the random algebraic polynomial $F(t) = \sum_{k=0}^{n} a_k t^k$, yields the stationary Gaussian process $F_0(s,t) := \sum_{k=0}^{n} a_k t^k s^{n-k}$. The polynomial F_0 is homogeneous, i.e. $F_0(as, at) = a^n F_0(s,t)$, for any $a \in \mathbb{R}$. One may thus think of F_0 as acting on the unit circle. One easily checks that $F_0(s,t)$ is indeed stationary. Identifynig a given point $(x,y) \in S^1$ with $(\cos t, \sin t)$ for some $t \in \mathbb{R}$, this process depends only on the single variable t, namely $F_0(t) = \sum_{k=0}^{n} a_k \cos^k t \sin^{n-k} t$. Moreover, if t is a real root of F, then the radial projection of $(1,t)$ on the unit circle is a root of F_0.

Now rewriting the first term as $\mathbb{E}\left[{N_0}^2\right] = \mathbb{E}\left[N_0(N_0-1)\right] + \mathbb{E}\left[N_0\right]$, we obtain

$$\mathrm{Var}(N_0) = \mathbb{E}\left[N_0(N_0-1)\right] + \mathbb{E}\left[N_0\right] - \mathbb{E}\left[N_0\right]^2 \ .$$

Notice that all terms in the r.h.s. are known except the first one. In order to get this term, the idea is to use Theorem 3.11. By (3.23), we have

$$\mathbb{E}\left[N_0(N_0-1)\right] = \int_I \int_I \mathbb{E}\left[|F'(s)F'(t)| \big| F(s) = F(t) = 0\right] p_{(F(s),F(t))}(0,0)\, ds\, dt \ . \tag{4.7}$$

The double integral above is hard to compute due to the conditional expectation, which is in general a very demanding problem. But as we have already seen before, the condition can be removed if it is ndependent from the left part in the conditional expectation. That is what we will try to do now, namely find some coefficients $\alpha, \beta, \gamma, \delta$ in order that the random variables

$$\zeta(s) := F'(s) - \alpha F(s) - \beta F(t) \quad \text{and} \quad \zeta(t) := F'(t) - \gamma F(t) - \delta F(s)$$

are independent from the conditions, that is from $F(s)$ and $F(t)$. This leads to two systems of two equations with unknowns α, β, resp. γ, δ,

$$\begin{cases} \mathbb{E}\left[\zeta(s)F(s)\right] = 0 \\ \mathbb{E}\left[\zeta(s)F(t)\right] = 0 \end{cases} , \qquad \begin{cases} \mathbb{E}\left[\zeta(t)F(s)\right] = 0 \\ \mathbb{E}\left[\zeta(t)F(t)\right] = 0. \end{cases}$$

Note that, since $\zeta(s)$ and $\zeta(t)$ are linear combinations of Gaussian variables, they are still Gaussian random variables.

Resolution of the first system. The first equation of this system is equivalent to

$$\mathbb{E}\left[F'(s)F(s)\right] - \alpha \mathbb{E}\left[F(s)^2\right] - \beta \mathbb{E}\left[F(t)F(s)\right] = 0 \ . \tag{4.8}$$

Now, $\mathbb{E}\left[F(s)^2\right] = K(0) = 1$ by assumption, and $\mathbb{E}\left[F(t)F(s)\right] = K(t-s)$. It remains to see what the first term is. Since $\mathbb{E}\left[F(s)^2\right] = K(s) = 1$, we have

$$\mathbb{E}\left[F'(s)F(s)\right] = \frac{1}{2}\mathbb{E}\left[F(s)^2\right]' = 0 \ ,$$

so that (4.8) can be written as

$$\alpha + \beta K(t-s) = 0 \ . \tag{4.9}$$

Consider now the second equation, that is

$$\mathbb{E}\left[F'(s)F(t)\right] - \alpha \mathbb{E}\left[F(s)F(t)\right] - \beta \mathbb{E}\left[F(t)^2\right] = 0 \ . \tag{4.10}$$

As before, we have that $\mathbb{E}\left[F(s)F(t)\right] = K(t-s)$ and $\mathbb{E}\left[F(t)^2\right] = K(0) = 1$. The first term is

$$\mathbb{E}\left[F'(s)F(t)\right] = -K'(t-s) \ .$$

Hence the equation in (4.10) is equivalent to

$$\alpha K(t-s) + \beta = -K'(t-s) \ . \tag{4.11}$$

Thus the first system consisting in equations (4.9) and (4.11) can be written in matrix form as

$$\begin{pmatrix} 1 & K(t-s) \\ K(t-s) & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ -K'(t-s) \end{pmatrix} \ ,$$

so that, when inverting the first matrix, we obtain

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{1-K(t-s)^2} \begin{pmatrix} 1 & -K(t-s) \\ -K(t-s) & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -K'(t-s) \end{pmatrix} \ .$$

So we have for α and β the expressions

$$\alpha = \frac{K(t-s)K'(t-s)}{1-K(t-s)^2} \quad , \quad \beta = \frac{-K'(t-s)}{1-K(t-s)^2} \ . \tag{4.12}$$

Resolution of the second system. Proceeding exactly in the same way as for the first system, the second system leads to

$$\begin{pmatrix} K(t-s) & 1 \\ 1 & K(t-s) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} K'(t-s) \\ 0 \end{pmatrix} .$$

Thus, after inverting the coefficent matrix, we obtain

$$\begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \frac{1}{K(t-s)^2-1} \begin{pmatrix} K(t-s) & -1 \\ -1 & K(t-s) \end{pmatrix} \begin{pmatrix} K'(t-s) \\ 0 \end{pmatrix} ,$$

that is

$$\gamma = \frac{K(t-s)K'(t-s)}{K(t-s)^2-1} \quad , \quad \delta = \frac{-K'(t-s)}{K(t-s)^2-1} . \tag{4.13}$$

Now plugging in the expressions (4.12) and (4.13) in those of $\zeta(s)$ and $\zeta(t)$, we can say that the two Gaussian random variables

$$\zeta(s) = F'(s) - \frac{K(t-s)K'(t-s)}{1-K(t-s)^2}F(s) - \frac{-K'(t-s)}{1-K(t-s)^2}F(t) ,$$

$$\zeta(t) = F'(t) - \frac{K(t-s)K'(t-s)}{K(t-s)^2-1}F(t) - \frac{-K'(t-s)}{K(t-s)^2-1}F(s)$$

are independent from $F(s)$ and $F(t)$. Thus, we can replace the conditional expectation by an ordinary expectation, that is

$$\mathbb{E}\left[|F'(s)F'(t)|\big|F(s)=F(t)=0\right] = \mathbb{E}\left[|\zeta(s)\zeta(t)|\right] ,$$

where $\zeta(s)$ and $\zeta(t)$ are the random variables obtained above. Thus (4.7) gives

$$\mathbb{E}\left[N_0(N_0-1)\right] = \int_I \int_I \mathbb{E}\left[|\zeta(s)\zeta(t)|\right] p_{(F(s),F(t))}(0,0)\, ds\, dt .$$

The next step would now be to compute the ordinary expectation $\mathbb{E}\left[|\zeta(s)\zeta(t)|\right]$, but this is not so easy since we have the modulus of a product inside the expectation[22]. We will leave this part open since the computations are quite far-reaching. We give the main result that is obtained by following what we have done above, namely considering the random Kac polynomial of degree n, i.e. the coefficients being i.i.d. centered Gaussian variables having variance equal to 1, then one can show that the asymtotic for the variance of the number of zeros when n tends to infinity is given by

$$\mathrm{Var}(N_0) \sim \frac{4}{\pi}\left(1-\frac{2}{\pi}\right)\log n .$$

5 Random trigonometric polynomials

Let us now see how we can obtain the result for trigonometric polynomials. Our computations follow [12].

Definition 5.1. A random trigonometric polynomial $F : [0, 2\pi] \to \mathbb{R}$ is of the form

$$F(t) = \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt)\right) ,$$

where a_k and b_k are random variables for $k = 1, \ldots, n$ defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

[22] One way to solve this problem is to bound the modulus of the product as $|\zeta(s)\zeta(t)| \leq \frac{1}{2}(\zeta(s)^2 + \zeta(t)^2)$. Alternatively, one can expand $|\zeta(s)\zeta(t)|$ into a Taylor series.

Remark 5.2.

- The way to define random trigonometric polynomials is not unique. For instance, we could also define them using the definition of a random function, by taking $f_{2k}(t) = \cos(kt)$ and $f_{2k+1}(t) = \sin(kt)$.
- The polynomials in Definition 5.1 are called *stationary trigonometric* polynomials. Examples of different classes of random trigonometric polynomials are for instance *classic trigonometric* polynomials, $\sum_{k=1}^{n} a_k \cos(kt)$ or $\sum_{k=1}^{n} a_k \sin(kt)$. The advantage of stationary trigonometric random polynomials is that, as the name tells us, they are stationary with respect to t, which simplifies the analysis of level crossings.

5.1 Expected number of zeros

We may assume that the coefficients a_k and b_k are i.i.d. standard normal variables having mean zero and variance one. Let us give the asymptotic behaviour of the expected number of zeros of the stationary trigonometric polynomials. One checks that these polynomials vertify assumptions $(A1)$, $(A2)$, $(A3)$, $(A4)$ of Theorem 3.7.

The stationary trigonometric polynomials have as covariance function

$$K(x,y) = \sum_{k=1}^{n} \left(\cos(kx)\cos(ky) + \sin(kx)\sin(ky)\right) = \sum_{k=1}^{n} \cos(k(x-y)) \ . \tag{5.1}$$

We see that the result only depends on the difference $x-y$. We could also write $K(x-y)$ instead of $K(x,y)$. In order to compute ρ_t, we will this time use formula (3.19). We have

$$K_x(x,y) = -\sum_{k=1}^{n} k\sin(k(x-y)), \quad K_{xy}(x,y) = \sum_{k=1}^{n} k^2\cos(k(x-y)) \ .$$

Evaluating $K(x,y)$ and these two partial derivatives at $x=y=t$, we obtain

$$K(t,t) = n, \quad K_x(x,y)\big|_{x=y=t} = 0, \quad K_{xy}(x,y)\big|_{x=y=t} = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \ .$$

Thus using (3.19), it follows that

$$\rho_t = \left(\frac{n^2(n+1)(2n+1)}{6n^2}\right)^{1/2} = \left(\frac{(n+1)(2n+1)}{6}\right)^{1/2} \ ,$$

and therefore the expected number of zeros on $I = [0, 2\pi]$ is

$$\mathbb{E}\left[N_0(F,I)\right] = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{(n+1)(2n+1)}{6}\right)^{1/2} dt = \frac{2}{\sqrt{6}}\sqrt{(n+1)(2n+1)} \ ,$$

so that when $n \to \infty$, we have

$$\mathbb{E}\left[N_0(F,I)\right] = \frac{2}{\sqrt{6}}\sqrt{2n^2+3n+1} \sim \frac{2}{\sqrt{3}}n \ .$$

Thus, we have established the asymptotic behaviour of the expected numbers of zeros of stationary trigonometric polynomials.

Let us see this result on a concrete example. Consider the random trigonometric polynomial of degree 5

$$\begin{aligned} &-2.0644\sin t + 0.7574\cos t - 1.2338\sin 2t - 0.2712\cos 2t - 0.16\sin 3t - 0.5795\cos 3t \\ &\qquad - 0.3516\sin 4t + 1.8002\cos 4t - 0.2441\sin 5t + 0.0748\cos 5t \ , \end{aligned}$$

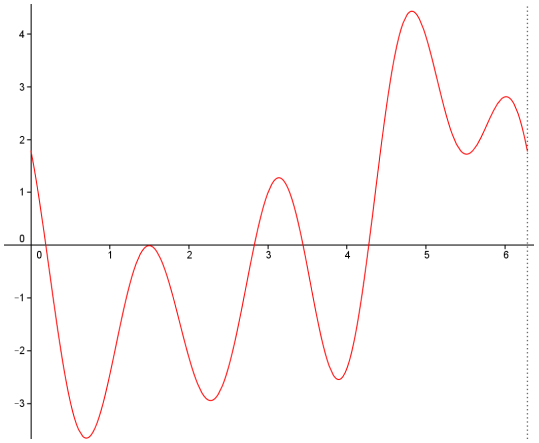

Figure 4: A random trigonometric polynomial of degree 5 on $[0, 2\pi]$

that is depicted in Figure 4. We see that this polynomial has 5 roots in $[0, 2\pi]$, which is quite close to the theoretical value, namely $\frac{2}{\sqrt{3}} \cdot 5 \simeq 5.7735$. Moreover, denoting T_n the number of critical points of a random trigonometric polynomial of degree n, we have the following asymptotic when n tends to infinity, (see Section 4 of [13])

$$\mathbb{E}[T_n] \sim 2n\sqrt{\frac{3}{5}} \, .$$

Looking at Figure 4, we count 8 critical points on $[0, 2\pi]$, whereas the above estimate gives $2 \cdot 5\sqrt{3/5} \simeq 7.7460$, which compared to 8 is quite reasonable.

5.2 Central Limit Theorem

In this section we will state a Central Limit Theorem (CLT) for the number of zeros of the random trigonometric polynomials and briefly present the main points of the corresponding theory that goes with it. For this section, ideas have been taken from Chapter 5 in [5] and Sections 1 and 2 of [1].

Consider the stationary random trigonometric polynomials of degree n for $t \in [0, \pi]$,

$$X_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(a_k \cos(kt) + b_k \sin(kt) \right) \, , \tag{5.2}$$

where the coefficients a_k and b_k are i.i.d. Gaussian variables with mean zero and variance one.

We formulate the main theorem here below.

Theorem 5.3 (Central Limit Theorem)**.** *The normalized number of zeros of X_n on $I = [0, \pi]$ converges in distribution to a Gaussian random variable,*

$$\frac{N_0(X_n, I) - \mathbb{E}[N_0(X_n, I)]}{\sqrt{n\pi}} \to \mathcal{N}(0, V^2) \, ,$$

when $n \to \infty$, where $0 < V < \infty$ is a constant.

General idea. We will see that for a convenient scale, the process X_n converges in a certain sense to a stationary process X with covariance function $\sin(t)/t$, for which a CLT can be obtained. Thus the CLT for the zeros of X_n turns out to be a consequence of the one for the

zeros of X. The idea is to define the processes X_n and X in the same probability space, which allows us to compute the covariance between these processes. Next we will write the Hermite expansions for both processes giving the normalized number of zeros of X_n and X. Using these representations, we can compute the L^2 distance between the number of zeros of X_n and X.

5.2.1 Scaling the process

We will see that replacing t by t/n, the process X_n converges to the stationary process X with covariance function[23] $\sin t/t$. The advantage of this is that we represent our initial processes X_n by one single limit process X, for which a CLT is known. We state in the following theorem.

Theorem 5.4. *Let X be a stationary centered Gaussian process with covariance function $K(t) = \sin t/t$. Then, the normalized number of zeros of X on $I = [0, n\pi]$ converges in distribution to a Gaussian random variable,*

$$\frac{N_0(X, I) - \mathbb{E}\left[N_0(X, I)\right]}{\sqrt{n\pi}} \to \mathcal{N}(0, V^2) \ ,$$

where $0 < V < \infty$ is a constant.

The proof of this theorem is based on the approximation of the covariance function K by covariance functions which have a compact support. This will gives us an m-dependent stationary Gaussian process, that is a sequence of stationary Gaussian processes $(X_i)_{i\geq 1}$, for which X_s and X_t are independent whenever $|s - t| > m$, for which we can obtain a CLT.

Indeed, if we replace t by t/n in (5.2), we obtain the scaled process on $[0, n\pi]$, given by

$$Y_n(t) := X_n(t/n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left(a_k \cos\left(\frac{k}{n} t\right) + b_k \sin\left(\frac{k}{n} t\right)\right) \ . \tag{5.3}$$

This permits us to find a limit for the process X_n. Note that, as defined above, the scaled process Y_n is still Gaussian. Following the computations done in (5.1), we obtain the covariance function K_{X_n} of the stationary process X_n,

$$K_{X_n}(t) = \frac{1}{n} \sum_{k=1}^{n} \cos\left(\frac{k}{n} t\right) \ , \tag{5.4}$$

so that the covariance function of the scaled process Y_n is then $K_{Y_n}(t) = K_{X_n}(t/n)$. Using the convergence of Riemann sums to the integral, we obtain that $K_{Y_n}(t) \to K(t) := \sin t/t$ when n tends to infinity. Thus we have a stationary Gaussian process having as covariance function the cardinal sine function, for which we have a CLT by Theorem 5.4.

5.2.2 Stochastic integrals with respect to a Brownian motion

Consider a standard Brownian motion[24] $B = \{B_t : t \in [0, 1]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a step function f defined by

$$f(t) = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}(t) \ ,$$

[23] The function $f(x) = \sin x/x$ is called the *cardinal sine function*.

[24] A standard Brownian motion (or Wiener Process) on an interval $I = [0, T]$ is a random variable B_t that depends continuously on $t \in I$ and satisfies $B_0 = 0$, for $0 \leq s < t < T \leq K$, we have $B_t - B_s \sim \mathcal{N}(0, t - s)$ and for $0 \leq s < t < u < v \leq T$, $B_t - B_s$ and $B_v - B_u$ are independent.

for $n \in \mathbb{N}, a_k \in \mathbb{R}$ and $A_k = (t_{k-1}, t_k]$ two by two disjoint intervals in $[0,1]$, with $0 \leq t_0 < t_1 < \ldots < t_n$. We define the stochastic integral with respect to B of the simple function f by

$$I_1^B(f) = B(f) = \int_0^1 f(t)\, dB_t := \sum_{k=1}^n a_k (B_{t_k} - B_{t_{k-1}}) \ .$$

Remark 5.5. One can extend the above definition of the stochastic integral with respect to a Brownian motion to every square integrable function on $[0,1]$.

Denote by $\mathbb{H}$ the Hilbert space $L^2([0,1], \mathcal{B}, d\lambda)$, where $\mathcal{B}$ denotes the Borel σ-algebra and $d\lambda$ the Lebesgue measure. Then the map $f \mapsto B(f)$ defines an isometry between $\mathbb{H}$ and $L^2(\Omega, \mathcal{F}, \mathbb{P})$. In this case, the Brownian motion B is called an *isonormal process* associated to $\mathbb{H}$.

The idea is now to define the q-fold multiple stochastic integral I_q^B with respect to the Brownian motion B. This construction is done in an analogous way as above in the one-dimensional case. Instead of using indicator functions of intervals, we may consider indicator functions of rectangles, i.e. products of pairwise disjoint intervals.

5.2.3 Wiener chaos expansion

We are now going to present the key ideas of the Wiener chaos expansion. Let us start by recalling some facts about Hermite polynomials.

Hermite polynomials. The Hermite polynomial of degree m is defined by

$$H_m(x) = (-1)^m e^{x^2/2} \frac{d^m}{dx^m} e^{-x^2/2} \ .$$

Thus, for instance the Hermite polynomials of degree $m \leq 3$ are given by

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 1, \quad H_3(x) = x^3 - 3x \ .$$

The scalar product is given by

$$\langle H_m, H_n \rangle = \int_{-\infty}^{+\infty} H_n(x) H_m(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \ .$$

Hermite polynomials are orthogonal with respect to the Gaussian density $\varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$, so that we have the property

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx = m! \delta_{n,m} \ .$$

Note that this formula can be rewritten as

$$\mathbb{E}\left[H_n(x) H_m(x)\right] = m! \delta_{n,m} \ .$$

Thus the polynomials $\overline{H}_m(x) = (m!)^{-1/2} H_m(x)$ satisfy $\langle \overline{H}_m, \overline{H}_n \rangle = \delta_{n,m}$.

Expansion into the chaos. We will now give the formula that gives the Wiener chaos expansion, also called Hermite expansion, for a square integrable function. Using the definition of Hermite polynomials and the properties of multiple stochastic integration, one can show that for $f \in L^2([0,1])$ having norm 1, we have

$$I_q^B(f^{\otimes q}) = H_q(B(f)) \ ,$$

where $f^{\otimes q} = f^{\otimes q}(x_1, \dots, x_q) = f(x_1) \dots f(x_q)$ stands for the tensor product of f with itself q times.

Recalling that $B(f) = I_q^B(f)$, we observe that this formula relates the one dimensional integral with respect to a Brownian motion to the multiple one by means of the Hermite polynomial of degree q. Using this, one can deduce the fundamental Hermite expansion for any square integrable function, more precisely, for $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, of the Brownian motion B, there exists a unique sequence of functionals $(f_k)_{k \geq 1}$ with $f_k \in L^2([0,1]^k, d\lambda)$, such that

$$F - \mathbb{E}[F] = \sum_{k=1}^{\infty} I_k^B(f_k) \ .$$

Thus, as already pointed out, the idea is to write the Hermite expansion of the number of crossings of the two processes X_n and X.

6 Simulations with Matlab

In this last part of our paper, we will do some simulations with Matlab[25]. The codes can be found in Appendix A.

6.1 Zeros of random Kac polynomials

6.1.1 Expectation and variance of the number of zeros

We know that, given a random Kac polynomial of degree n, we have the following asymptotics when n tends to infinity for the expectation and the variance of the number of zeros over $\mathbb{R}$,

$$\mathbb{E}[N_0] \sim \frac{2}{\pi} \log n \quad , \quad \mathrm{Var}(N_0) \sim \frac{4}{\pi} \left(1 - \frac{2}{\pi}\right) \log n \ . \tag{6.1}$$

Using Matlab, we can convince ourselves that these results hold true. We implement an algorithm (see Algorithm 1) that computes the expectation and the variance of the number of zeros for 1500 realizations[26]. We compute the quotients of the simulated values by the theoretical expressions in (6.1). We state some runnings of the algorithm for the expectation in Table 1 and for the variace in Table 2. We denote by n the degree, $\mathbb{E}[N_0]_s$ resp. $\mathrm{Var}(N_0)_s$ the simulated expectation resp. variance of the number of zeros. For comparison, we also put the values of $\mathbb{E}[N_0]$ and $\mathrm{Var}(N_0)$. We express the error of the simuated value and the theoretical value in %.

n	$\mathbb{E}[N_0]_s$	$\mathbb{E}[N_0]$	$\mathbb{E}[N_0]_s / \mathbb{E}[N_0]$	error (%)
200	3.9600	3.3730	1.1740	17.40
400	4.4920	3.8143	1.1777	17.77
700	4.8120	4.1705	1.1538	15.38
1500	5.2209	4.6557	1.1214	12.14

Table 1: Expected number of zeros of random Kac polynomials

[25] Matlab is not a free software. For this paper, we used the licence for student projects provided by the University of Luxembourg.

[26] By realizations we mean the followong: we generate 1500 random polynomials of same degree and then take the mean of the data that we want to observe. This gives much more accuracy than implementing only one single polynomial, since in that case the randomness (of the coefficients of the polynomial) can have remarkable repercussions on our results.

n	$\mathrm{Var}(N_0)_s$	$\mathrm{Var}(N_0)$	$\mathrm{Var}(N_0)_s/\mathrm{Var}(N_0)$	error (%)
200	2.7149	2.4514	1.1075	10.75
400	2.9893	2.7721	1.0784	7.84
700	3.2708	3.0310	1.0791	7.91
1500	3.5352	3.3836	1.0448	4.48

Table 2: Variance of the number of zeros of random Kac polynomials

We see that, comparing the errors in both tables, they are smaller in the second, i.e. for the variance. However, we also note, that the more we increase the degree, the smaller this error becomes. In Table 1, for instance, we started with an error of 17.40% for degree 200 and ended with an error of 12.14% for a much larger degree, 1500. This suggests that the error decreases only very slowly as the degree increases. For the variances, we started with an error of 10.75% and ended with an error of 4.48%.

6.1.2 Distribution of zeros

There are very interesting results concerning the distribution of real and complex zeros of random Kac polynomials.

Distribution of complex zeros. Consider a random Kac polynomial of degree n. It can be shown that the complex roots of this polynomial are distributed around the unit circle. We may do some simulations (see Algorithm 2) in order to visualize this fact. Figure 5 shows what we obtained for three different degrees, $50, 100$ and 200. One point on the figure represents a

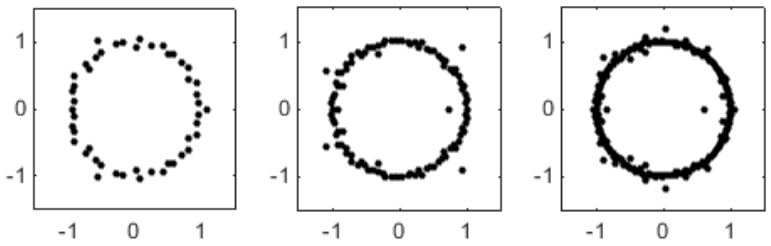

Figure 5: Distribution of complex zeros of a random Kac polynomial of degree $50, 100$ and 200

complex root of the polynomial. We immediately see that the larger the degree of the polynomial is, the better one recongnizes the unit circle, due to the density of the zeros, which increases with the degree. Indeed, comparing the first and the last plot in Figure 5, the black density of points representing the roots is much intenser in the last one since the number of roots is multiplied by 4.

General ideas of proofs of this result can be found in [3]. The authors consider, among others, random polynomials with independent complex coefficients having all mean zero and equal variance. It is shown that their roots tend to concentrate in the annulus near the unit circle of the complex plane, and that the width of the annulus tends to zero as the degree of the polynomials tends to infinity. In order to illustrate the result, they plot the roots of 200 trials of polynomials of degree 48. That is more or less what we have in the first plot of Figure 5, where we consider polynomials of degree 50.

Distribution of real zeros. Let us now come to the distribution of the real zeros of a random Kac polynomial. As already seen in Figure 3, the density of real zeros of a random Kac polynomial

n	$\mathbb{E}[N_0]$	N_0	$N_0^{\pm 1}$	$N_0^{\pm 1}/N_0$ (%)
200	3.3730	3.9853	2.3310	58.49
500	3.9563	4.3773	2.9170	66.64
1000	4.3976	4.7320	3.3890	71.62
1500	4.6557	4.8560	3.6420	75

Table 3: Distribution of real roots of random Kac polynomials

has two peaks at 1 and -1, which suggests that the real zeros are concentrated near these points. Using Matlab, we can do the same type of simulations as the one above for the complex zeros, however keeping the same degrees, we cannot really convince ourselves, since the logarithm increases very slowly. Indeed, the expected number of real zeros of a Kac polynomial of degree n being $2/\pi \log n$, we get very few real roots for degrees $50, 100$ and 200. For instance, for $n = 200$, we obtain in average 3.3730 real roots, which is clearly too low to be able to do good simulations. Sensibly increasing the degree does not change this aspect much, since for instance for $n = 200000$ (we multiply by 10^4 the previous degree !) we have on average 7.7706 real roots.

For completeness, we implement an algorithm (see Algorithm 3) that computes the real roots of a random Kac polynomial of degree n. We denote by N_0 the average number of real roots and by $N_0^{\pm 1}$ the average number of real roots that are close to ± 1. In order to decide whether a root is near to ± 1, we fix a tolerance at 15%, that is, all roots whose absolute value lies in the interval $]0.85; 1.15[$ are considered close to ± 1. Expressing in % the number of roots that are close to ± 1, we obtain Table 3. At each time, we run the algorithm for 1000 realizations. For comparision, we put the theoretical expected number of real roots in the second column, as we already did in Table 1. We observe that the percentage of the real roots close to ± 1 increases with the degree. For degree 200 we have a quite low percentage, but increasing the degree up to 1500, we obtain in average 75% of the real roots that are concentrated at ± 1. Increasing the degree would augment the percentage still more and hence confirm our result.

6.2 Zeros of random stationary trigonometric polynomials

In this part we focus on random stationary trigonometric polynomials. More precisely, we are going to simulate the expected number of zeros and the CLT. (see Algorithm 4)

6.2.1 Expected number of zeros

Consider a random stationary trigonometric polynomial of degree n. As wee have seen in the previous section, the expercted number of zeros over $[0, 2\pi]$ is $\frac{2}{\sqrt{3}}n$. We denote the simulated expected number of zeros by $\mathbb{E}[N_0]_s$ and the theoretical value by $\mathbb{E}[N_0] = \frac{2}{\sqrt{3}}n$. In Table 4 we state the results that we obtained for different degrees n, for 1500 realizations.

n	$\mathbb{E}[N_0]_s$	$\mathbb{E}[N_0]$	$\mathbb{E}[N_0]_s / \mathbb{E}[N_0]$	error (%)
100	116.4800	115.4701	1.0087	0.87
300	346.9253	346.9253	1.0015	0.15
500	578.4340	577.3503	1.0019	0.19
700	809.4587	808.2904	1.0014	0.14
1200	1385.8	1385.6	1.0001	0.01

Table 4: Expected number of zeros of random trigonometric polynomials

We observe that increasing the degree of our polynomial the quotient $\mathbb{E}[N_0]_s / \mathbb{E}[N_0]$ gets always closer to 1, i.e. the error, that we expressed in %, becomes smaller.

6.2.2 Central Limit Theorem for random trigonometric polynomials

We will simulate the CLT (see Algorithm 4) for stationary trigonometric polynomials. Denote X_n the random trigonmometric polynomial of degree n over $I = [0, 2\pi]$. We draw the histogram of the random variable

$$\frac{N_0(X_n, I) - \mathbb{E}\left[N_0(X_n, I)\right]}{\sqrt{n\pi}} , \tag{6.2}$$

for different degrees and see the evolution of the plot. If we take n very large[27], the histogram is supposed to look like a real Gaussian. We put here some plots of the histogram obtained

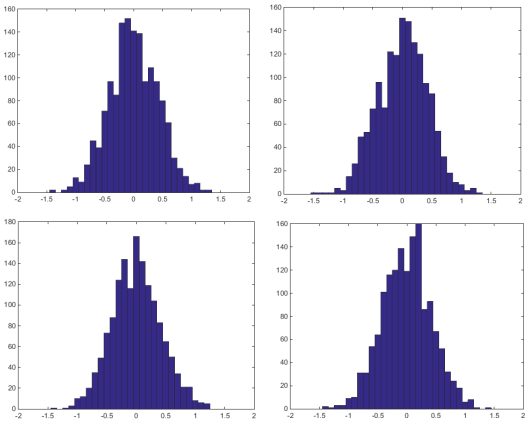

Figure 6: Histogram of the normalized number of zeros for 1500 polynomials

by always increasing the degrees. More precisely, Figure 6 shows what we obtained for random trigonometric polynomials of degree 1000, 2000 (in the first row) resp. 3000 and 4000 (second row).

For each of the four plots in Figure 6, we considered 1500 realizations. We observe that the histogram of the normalized number of zeros roughly has the shape of a Gaussian, but on the other hand, there are some fluctuations, in the sense that the obtained curve is not as smooth as it is supposed to be. This may have more reasons. A first one consists in too small degrees. As the CLT holds for the degree tending to infinity, taking as degree 4000 might not be large enough.

Another point that could explain the fluctuations is the following. By increasing the number of realizations we may obtain a more accurate result, since we take the mean of more values, which might contribute to more precision. For instance , let us have a look how Figure 6 changes when we double the number of realizations, i.e. consider 3000 polynomials. Comparing this to Figure 6, we notice that there are less fluctuations than before. Especially in the last three plots, i.e. for degrees $2000, 3000$ and 4000, the Gaussian curve appears neater.

Unkown velocity of convergence. An important remark to the CLT for random trigonometric polynomials arises when talking about the velocity of convergence. The only known fact is that, for a large degree, the distribution of the normalized number of zeros behaves as the distribution of a normal random variable, but we we do not have any information about the velocity of convergence. This may also explain our, so to speak, rough Gaussian curves obtained in Figures 6 and 7. Increasing the degree of the polynomials would probably give better results. However, one has to note that our algorithm turns out to be quite slow for degrees larger than

[27] Note that the theorem states the result for n tending to infinity, so we have to choose large but still reasonable values for the degree n in our simulations, in order to have an efficient result.

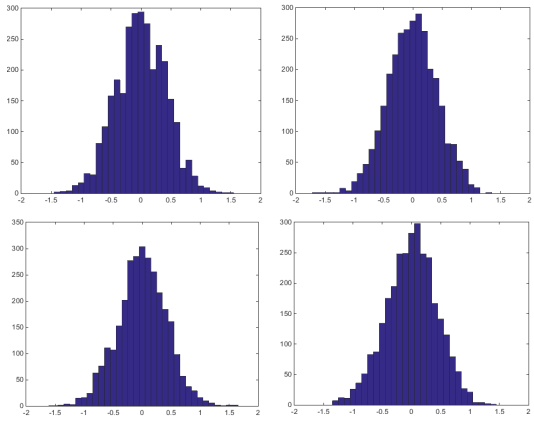

Figure 7: Histogram of the normalized number of zeros for 3000 polynomials

4000 and also for a number of realizations larger than 3000. These types of simulations take enormous time to a personal computer for a very high degree, that is why we limit our results to reasonable degrees, otherwise we would have to use a cluster of computers in order to do heavier simulations.

Kolmogorov-Smirnov hypotesis test. In order to convince ourselves somehow more than by only plotting the histograms, we may check normality of the normalized number of zeros given in (6.2) using Kolmogorov-Smirnov's hypotesis test. For this we formulate our null hypothesis as follows: The distribution of the random variable in (6.2) comes from a Gaussian distribution. We fix the significance level at 5%. At each time, the algorithm returns two numbers h, either 0 or 1, and p, the p value[28]. If $h = 0$ the Kolmogorov-Smirnov test accepts the null hypothesis at the 5% significance level, otherwise, if $h = 0$, we reject the null hypothesis, i.e. we accept the alternative hypothesis, that is the random variable is not normal. If the p value is larger than 5%, the test accepts the null hypothesis, otherwise the test rejects the null hypothesis in favour of the alternative hypothesis. In Table 5, we put some results obtained for different degrees, always obtained for 1500 realizations. Interpreting Table 5, we see that the null hypothesis is rejected in the first two cases, certainly due to the small degrees. Note that we want to see normality as the degree tends to infinity, so increasing the degree changes the result as expected. Thus, from this test, it becomes clear (as before with the plots) that increasing the degree, the normalized number of zeros behaves like a Gaussian random variable.

n	h	p
300	1	0.0064
900	1	0.0457
1200	0	0.1349
1500	0	0.3405
2500	0	0.2863

Table 5: Kolmogorov-Smirnov test for CLT

[28]Recall the notion of p value, that is the probability of obtaining the observed sample results, or more extreme results, assuming that the null hypothesis is true.

A Matlab Codes

A.1 Zeros of random Kac polynomials

A.1.1 Expectation and variance of the number of zeros

- Lines $1-3$: Generate `W` random Kac polynomials of degree `n` using Matlab's predeinfined command `randn`.
- Lines $4-8$: For each of the `W` polynomials, compute the complex roots `r` and only keep the real ones, `rr`. Count the number `l` of real roots, using the command `length`.
- Lines $9-10$: Compute the average expected number `m` of real roots, and the quotient of `m` by the theroretical expression of the expected number of zeros.
- Lines $11-12$: Do the same as in lines $9-10$ for the variance.

```
1  n = 1500;
2  W = 1500;
3  p = randn(W,n+1);
4  for i=1:W
5      r = roots(p(i,:));
6      rr = r(r == real(r));
7      l(i) = length(rr);
8  end
9  m = mean(l)
10 mquotient = m/((2/pi)*log(n))
11 v = var(rr)
12 vquotient = v/((4/pi)*(1-2/pi)*log(n))
```

Algorithm 1: Expectation and variance of the number of zeros

A.1.2 Distribution of zeros

Distribution of complex zeros.

- Line 1: Define the degree `n`. We want three simulations, one for each component of the vector `n`.
- Lines $5-6$: Generate a random Kac polynomial of degree `n` and compute its complex roots.
- Lines $7-9$: Plot the roots for each degree in the vector `n`.
- Lines $2, 4, 7-9$: Commands to define the plot.

```
1  n = [50 100 200];
2  MS = 'markersize'; ms = 12;
3  for i = 1:3
4      subplot(1,3,i)
5      a = randn(1,n(i)+1);
6      r = roots(a);
7      plot(r,'.k',MS,ms)
8      set(gcf,'color','w');
9      axis(1.5*[-1 1 -1 1]), axis square
10 end
```

Algorithm 2: Distribution of complex zeros of random Kac polynomials

Distribution of real zeros.

- Lines $1-3$: Generate `W` random Kac polynomials of degree `n`.
- Line 4: Define the fixed tolerance `tol` in order to have a condition when a real root is close to ± 1.
- Lines $5-10$: For each of the `W` polynomials, compute the complex roots `r` (line 6), then extract of this vector all the real roots `rr` and count them (lines $7-8$). Count the number of real roots close to ± 1, `goodrr`, using Matlab's built-in command `sum`, by putting a conditon on the absolute value of the roots.
- Lines $11-13$: Compute the average number of real roots `m` and the average number of real roots close to ± 1, `M`. Then compute the quotient of both.

```
1  n = 3000;
2  W = 1000;
3  p = randn(W,n+1);
4  tol = 0.15;
5  for i=1:W
6      r = roots(p(i,:));
7      rr = r(r == real(r));
8      l(i) = length(rr);
9      goodrr(i) = sum(abs(rr)<1+tol &abs(rr)>1-tol);
10 end
11 m = mean(l)
12 M = mean(goodrr)
13 quotient =  M/m
```

Algorithm 3: Distribution of real roots of random Kac polynomials

A.2 Zeros of random stationary trigonometric polynomials

- Lines $1-10$: Generate `W` random stationary trigonometric polynomials of degree `n` that we evaluate in different times `t` in $[0, 2\pi]$.
- Lines $11-12$: Compute the expected number of zeros `NO` of these `W` polynomials. Taking the mean of this vector conatining the numbers of zeros, we have the average number of zeros `ENO`.
- Lines $13-15$: Define `X` to be the normalized number of zeros. Compute its expectation and its variance.
- Lines $16-17$: `ENOt` is the theroetical expected number of zeros. We compute the quotient of the simulated one by the theoretical one.
- Lines $18-21$: Plot the histogram of the normalized number of zeros.
- Lines $22-23$: Execute the Kolmogorov-Smirnov test using Matlab's built-in command `kstest`. Note that this command compares the entered random variable to a standard Gaussian, thus we apply it to the normalized random variable `normX`.

```
1  n = 2500;
2  W = 1500;
3  t = 0:.0001:2*pi;
4  l = length(t);
```

```
A = (1:n)'*t;
cosine = cos(A);
sine = sin(A);
A = randn(2*W,n);
Poly = A(1:W,:)*cosine + A((W+1):(2*W),:)*sine;
Poly = Poly(:,1:(l-1)).*Poly(:,2:l);
N0 = sum(Poly'<0);
EN0 = mean(N0);
X = (N0-EN0)/(sqrt(n*pi));
EX = mean(X);
VarX = var(X);
EN0t = (2*n/sqrt(3))
quotient = EN0/EN0t
x = -1.5:.1:1.5;
figure;
set(gcf,'color','w');
hist(X,x);
normX = (X-EX)/sqrt(VarX);
[h,p] = kstest(normX)
```

Algorithm 4: Expected number of zeros and CLT for random trigonometric polynomials

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