Spectral Theory of Random Matrices

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Abstract

In general, one in interested in the understanding of large-dimensional matrices, for they are an abstract illustration of most scientific problems. Moreover, it has been found out that the general idea works best for random matrices, which is a matrix with random variable entries. This results in random eigenvalues and eigenvectors and our goal is to understand their distribution and make some nice observations.
One of the first scientists to attack this problem was Eugene Wigner, whose theory will be our base for this paper. Although he was in charge of quantum mechanics, Wigner's random matrix also had a big impact on mathematical physics and probability theory. In particular, we will discuss Wigner's semi-circle distribution and a couple of peculiar cases.

1 Introduction

1.1 History

Eugene Paul Wigner [1] was a Hungarian-American theoretical physicist, engineer and mathematician, who moved to Germany to study in Berlin and later on, work at the University of Göttingen. At the age of 25, he discovered certain symmetries in quantum mechanics and introduced what we will discuss in our paper: Wigner matrices and their corresponding random matrix theory.
Originally, Wigner matrices helped in the understanding of group theory in quantum mechanics because they are unitary matrices, written in an irreducible way of the unitary and rotation group:

$$SU(2) = \left\{ \begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$$

$$SO(3) = \{3 \times 3 \text{ orthogonal matrices with } det = 1\}$$

Let J_x, J_y, J_z be generators of the Lie algebra of the above groups, i.e. we have a vector space g with a non-associative, alternating bilinear map: $g \times g \to g; (x, y) \mapsto [x, y]$, satisfying the Jacobi identity, which means the sum of all even permutations is zero. Then these three operators are the components of a vector operator, known as the angular momentum. In quantum mechanic, this representation is often used for the angular momentum of electrons in an atom or the rotation of an atom around itself, called "spin".
However, this theory does not only explain different energy levels in an atom, but those newly-discovered random matrices helped in the understanding of many important physical systems because they can be represented as a matrix problem.
Thus, a variety of applications of random matrices will appear.

1.2 Applications

We will focus on two applications of Random Matrix Theory, one in quantum mechanics [4] and one about image processing [1] [6].

1.2.1 Quantum Mechanics

In the mid 50s a large number of experiments with heavy nuclei was performed. These heavy atoms absorb and emit thousands of frequencies. An experiment of this kind offers us a great number of differences in the energy levels and it is difficult to find the set of levels behind the given differences. In fact, it was impossible to determine the energy levels exactly. To understand this problem, first one has to look at Schrödinger's equation:

$$\frac{\hbar^2}{2m} \nabla^2 \Psi + V(\mathbf{r}) \Psi = -i\hbar \frac{\partial \Psi}{\partial t}$$

where $\hbar$ is Planck's constant, V the potential energy and Ψ a wave function. Without going too much in details, we can say that linear algebra is the main tool to understand this equation. Indeed, if we set the following:

$$H\Psi := i\hbar\frac{\partial\Psi}{\partial t} \quad \text{and} \quad E\Psi := -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\mathbf{r})\Psi$$

then we can write the equation above as:

$$H\Psi = E\Psi$$

where H is a Hermitian operator on function-space, Ψ is an eigenvector and E is the corresponding eigenvalue. This operator H is also called Hamiltonian operator and E denotes the energy levels. Unfortunately, understanding this Hamiltonian operator is a hard problem as there are hundreds of nucleons involved.

Eugene Wigner then suggested that instead of dealing with the actual operator H, one can consider a family of random matrices and compute the distribution of the eigenvalues of these matrices. As H is defined on an infinite-dimensional vector space, one should look for the asymptotic behaviour as the size of these matrices goes to infinity. This view led Wigner to develop a theory based on random matrices for explaining the distribution of the energy levels.

1.2.2 Image Processing

Although Wigner's random matrices have had a huge impact on quantum mechanic problems, there are also concrete applications of random matrix theory in our everyday life, for example image processing.

There is a connection with random matrix theory and image denoising filters: Image noise is a random variation of brightness or color information in images, and it is an undesirable by-product of an image capture, which obscures the desired information. Originally "noise" denotes "unwanted signal" and there exist many different types of "noise" inter alia "Gaussian noise", which is caused by poor illumination, poor transmission or high temperature during a digital image acquisition. Gaussian noise is additive and independent of the signal at each pixel neighbourhood. We assume that each local matrix associated to these pixels is random and since the eigenvalue density of a random matrix is known, this provides a large threshold for removing the additive Gaussian random noise in the capture, while preserving the main information of the image.

To put it in a nutshell, the goal is to compute the eigenvalue density for each local random matrix and find out at which scale the eigenvalue changes from the original image to the noisy counterpart. Hence, by solving the inverse problem, one can free images from unwanted Gaussian noise.

Restoration of a noisy picture

Original Picture(L), Noisy Version(M), After Image Processing(R)

2 Recall

Definition 2.1. (Hermitian Matrix)

A square matrix $A \in M_n(\mathbb{C})$ is called hermitian if $A^* = A$, where A^* denotes the transjugate of A, i.e. if $(a_{i,j}) = (\overline{a_{j,i}})$.

One important property of these matrices is that every hermitian matrix is diagonalizable and its eigenvalues are real and its eigenvectors are two by two orthogonal.

Definition 2.2. (Probability density function)

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space, $X : \Omega \mapsto \mathbb{R}$ a random variable. The probability density function $f_X(x)$ of a continuous distribution is defined as the derivative of the cumulative distribution function $F_X(x)$ (absolutely continuous):

$$F_X(x) = \mathbb{P}\{X \leq x\} = \int_{-\infty}^{x} f_X(t)dt$$

Definition 2.3. (Empirical measure)

Let $X_1, X_2, ...$ be a sequence of independent identically distributed random variables with values in $\mathbb{R}$. We denote by P their probability distribution. The empirical measure

P_n of a measurable subset $A \subset \mathbb{R}$ is given by

$$P_n(A) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_A(X_i) = \frac{\#\{1 \leq i \leq n | X_i \in A\}}{n}$$

where $\mathbb{1}_A$ is the indicator function.

Note that if we choose $A =]-\infty, x]$, $\forall x \in \mathbb{R}$, then $P_n(A)$ is the empirical distribution function.

3 Wigner Matrices

Definition 3.1. [3]
A Wigner matrix $W_n \in M_n(\mathbb{C})$ is a hermitian matrix $(X_{i,j})_{i<j}$ such that

- $X_{i,j}$ are independent and identically distributed complex random variables for $i < j$
- $X_{i,i}$ are independent and identically distributed real random variables
- $\mathbb{E}[X_{i,j}] = 0$, $\forall i, j$
- $\mathbb{E}[|X_{i,j}|^2] = s^2$ if $i \neq j$
- $\mathbb{E}\left[X_{i,i}^2\right] < \infty$.

Remark 3.1. [1] [9] We call a Wigner ensemble the collection of all these matrices. Some iconic Wigner ensembles are the Gaussian Unitary Ensemble (GUE) and the Gaussian Orthogonal Ensemble (GOE), which are defined as follows:

$$(\text{GUE}) : (W_n)_{ij} := \begin{cases} X_{i,j} \equiv N(0,1)_{\mathbb{C}},\ i > j \\ X_{i,j} \equiv N(0,1)_{\mathbb{R}},\ i = j \end{cases}$$

Let $C \in \mathbb{C}^{n \times n}$ be unitary, then $CC^* = I$ and $C^* W_n C$ has same distribution than W_n, i.e. (GUE) is invariant under unitary conjugation.

$$(\text{GOE}) : (W_n)_{ij} := \begin{cases} X_{i,j} \equiv N(0,1)_{\mathbb{R}},\ i > j \\ X_{i,j} \equiv N(0,2)_{\mathbb{R}},\ i = j \end{cases}$$

Let $C \in \mathbb{R}^{n \times n}$ be orthogonal, then $CC^T = I$ and $C^T W_n C$ has same distribution than W_n, i.e. (GOE) is invariant under orthogonal conjugation.

In the next pages, we will put our focus on Gaussian Wigner Matrices, whose entries are Gaussian random variables with zero mean and variance s^2 if $i \neq j$ and $2s^2$ if $i = j$, but the theory that we will expose holds for general distributions too.

3.1 Theory

Definition 3.2. (Operator Norm)
Let $M \in Mat_{n\times n}(\mathbb{C})$ be a matrix. The operator matrix norm of M is defined as

$$\|M\|_{op} := \sup_{\|x\|\leq 1} \|Mx\|$$

where $x \in \mathbb{C}^n$ and $\|.\|$ is a vector norm on $\mathbb{C}^n$.

Theorem 3.1. *(Bai-Yin theorem, upper bound [5])*
Let W_n be a Wigner matrix. Then we have almost surely:

$$\limsup_{n\to\infty} \frac{\|W_n\|_{op}}{\sqrt{n}} \leq 2.$$

Hence, it is natural to deal with the normalized version $X_n := \frac{W_n}{\sqrt{n}}$.

Theorem 3.2. *(Wigner's semicircle law [9])*

Let $(W_n)_{n\geq 1}$ be a sequence of Wigner matrices, let μ_n be the probability measure

$$\mu_n(I) = \frac{\#\{i \in \{1,...,n\} : \lambda_i(X_n) \in I\}}{n}, \quad I \subset \mathbb{R}$$

where $\lambda_1(X_n) \leq ... \leq \lambda_n(X_n) \in \mathbb{R}$ are the eigenvalues of X_n.
Then μ_n converges weakly to the semicircle distribution,

$$\mu_{sc}(x)dx = \frac{1}{2\pi s^2}\sqrt{4s^2 - x^2}\mathbb{1}_{|x|\leq 2s}dx.$$

Proof. (Moment Method: Idea of proof)

One of the most iconic and direct proofs of the macroscopic scale of Wigner random matrices uses the moment method. This approach relies on the intuition that eigenvalues of Wigner matrices are distributed according to a limiting law - which, in our case, is the semicircle distribution μ_{sc}. The moments of the empirical distribution μ_n correspond to sample moments of the limiting distribution, where the number of samples is given by the size of the matrix.
We want to compute the k-th moment with law μ of a random variable X, which is the expectation $\mathbb{E}(X^k)$. We denote the eigenvalues of X_n by $\lambda_j(X_n)$ with order $\lambda_1(X_n) \leq \lambda_2(X_n) \leq ... \leq \lambda_n(X_n)$.
Note that we can diagonalize X_n as it is hermitian. Indeed, we have $X_n = U^t D_n U$ where $D_n = diag\left(\lambda_1(X_n), \lambda_2(X_n), ..., \lambda_n(X_n)\right)$. Therefore, we get for the k-th moment:

$$\mathbb{E}_{\mu_n}\left[X_n^k\right] = \frac{1}{n}\sum_{j=1}^{n} \lambda_j\left(X_n\right)^k = \frac{1}{n}\sum_{j=1}^{n} \left(D_n\right)_{jj}^k = \frac{1}{n}Tr\left(U^t D_n^k U\right) = \frac{1}{n}Tr\left(X_n^k\right).$$

The moments of the semicircle law are given by:

$$\mathbb{E}_{\mu_{sc}}[X_n^k] = \int x^k \mu_{sc}(x) = \begin{cases} s^k C_{\frac{k}{2}} & \text{, if } k \text{ is even} \\ 0 & \text{, if } k \text{ is odd} \end{cases}$$

where C_n is the nth Catalan number, which is given directly in terms of binomial coefficients by $C_n = \frac{1}{n+1}\binom{2n}{n}$.
Miraculously, it turns out that the semi-circle law is the unique distribution where the k-th moments are given by the Catalan numbers:

$$\mathbb{E}_{\mu_n}[X_n^k] \xrightarrow{n} \int x^k \mu_{sc}$$

We can conclude that $\forall k \geq 1$, $\mathbb{E}_{\mu_n}\left(X^k\right) \to \mathbb{E}_{\mu_{sc}}\left(X^k\right)$ as $n \to \infty$ which is equivalent to $\mu_n \to \mu_{sc}$ weakly as $n \to \infty$.

□

The following MatLab [7] code gives a formal verification of the convergence explained in the proof.

```
function C = moment(i,f,p,k,s)
x = (i:p:f);
v = ones(f+1,ceil(f/p));
C = zeros(k+1,ceil(f/p));

for j=i:p:(f+1)
    H = s*(randn(j) + 1i*randn(j));
    for l=1:length(H)
        H(l,l) = sqrt(2)*H(l,l);
    end
    W = (H+H')/2;
    w = eig(W);
    w(f+1) = 0;
    v(:,ceil(j/p)) = w;
end

v = v./sqrt(repmat(x,size(v,1),1));

for n=1:k
    S = sum(v.^n,1);
    S = S./repmat(x,size(S,1),1);
    C(n+1,:) = S;
end
end
```

In the code, we construct different Wigner matrices and store their eigenvalues in an array. As one requires to use normalized eigenvalues, we have to divide column-wise by a vector x which contains the different matrix sizes.
Indeed, the output of the code will be an array whose columns represent a sequence $(W_i)_{i\geq 1}$ of Wigner matrices and whose rows represent the k first moments of the empirical distribution of the eigenvalues of W_i. While running the program, one is required to choose the initial size i of the matrices, the final size f, the iteration step p,

the moments k one wishes to compute as well as the standard deviation s of the random variables. Indeed, the code allows us to convince ourselves that the moments of the empirical distribution of the eigenvalues of W_i converge to the Catalan numbers, which in their turn represent the moments of the semi-circle distribution. In order to get the moments, we use the fact that the trace of a symmetric matrix is given by the sum of its eigenvalues.
The following table will represent the convergence of the moments of the empirical distribution of the eigenvalues of a sequence $(W_i)_{i\geq 1}$ of Wigner matrices for different inputs. For simplicity, the authors chose $s = 1$.

size / k-th moment	100	500	1000	2000	5000	10000
1	-0.0029	-0.0006	0.0005	-0.0001	0.0005	-0.0002
2	1.0198	1.0054	1.0014	1.0004	1.0002	1.0001
3	0.0141	-0.0123	0.0018	-0.0006	0.0013	-0.0006
4	2.0722	2.0212	1.9998	2.0058	2.0013	2.0004
5	0.0694	-0.0486	0.0072	-0.0043	0.0044	-0.0018
6	5.2511	5.0768	4.9935	5.0213	5.0055	5.0014
7	0.2596	-0.1858	0.0283	-0.0218	0.0169	-0.0063
8	14.8977	14.2741	13.9624	14.0800	14.0229	14.0056
9	0.8952	-0.6908	0.1099	-0.0996	0.0664	-0.0224
10	45.2708	42.9646	41.8283	42.3046	42.0920	42.0240

Since the first Catalan numbers are given by $1, 2, 5, 14, 42$ if k is even and 0 if k is odd, we see that the convergence is verified.

3.2 Matlab-Code

3.2.1 Wigner's semi-circle law

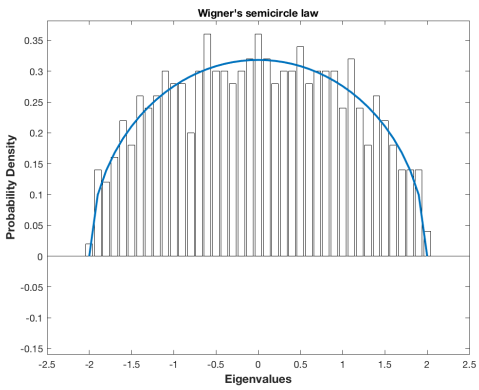
In order to have a better understanding of Wigner's semi-circle law, we use MatLab to construct a histogram of the eigenvalues of a set of Wigner matrices. This provides us a visualization of the semi-circle law. MatLab's built-in command "randn" generates a random matrix with independent and normally distributed entries. Next, we compute for various n-sized matrices their normalized eigenvalues, and using the minima and the maxima of these results, we can plot a nice histogram, visually supported by a semi-circular curve.

Using this code, we will plot several graphs in order to see when our histograms are shaped in a nearly perfect semi-circle. While running the program, one is required to choose the size n of the Wigner matrices, the standard deviation s of the random variables as well as the width dx of the bars in the histogram. As the variance of the random variables grows, the radius of the semi-circle increases too. Indeed, the program will not plot any histogram in case there is a surplus of bars. Therefore, one should be ready to adapt the width of the bars to the chosen s. The authors recommend picking $dx \in [0, 1]$.

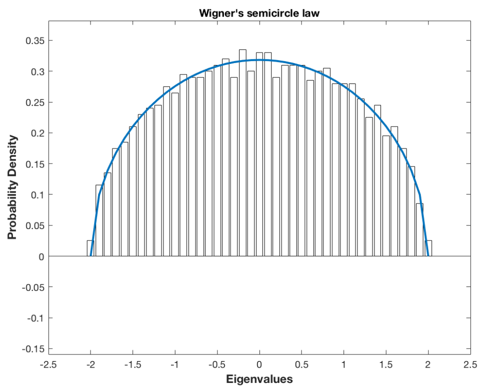
```matlab
function [h,N] = Wigner(n,s,dx)
v = [];
    for i=1:10
        H = s*(randn(n) + 1i*randn(n)); %random Gaussian matrix
        for j=1:length(H)
            H(j,j)=sqrt(2)*H(j,j);
        end
        W = (H+H')/2; %symmetrized matrix
        v = [v;eig(W)];
    end
v = v/sqrt(n); %normalized eigenvalues
[f,x] = hist(v,-2*s:dx:2*s);
cla reset
h = bar(x,f/(10*n*dx),'y'); %histogram
set(h,'FaceColor','none');
hold on;

ft = @(x) sqrt(4*s^2-x.^2)./(2*s^2*pi);
F = ft(x);
plot(x,F,'LineWidth',2); %semi-circle
axis([-2*s-0.5 2*s+0.5 -0.5/(s*pi) 1.2/(s*pi)]);
xlabel('Eigenvalues','FontWeight','bold','FontSize',12)
ylabel('Probability Density','FontWeight','bold','FontSize',12);
title('Wigner''s semicircle law');

N = norm(W)/(s*sqrt(n)); %operator norm
end
```

MatLab Code computing Wigner's semi-circle law

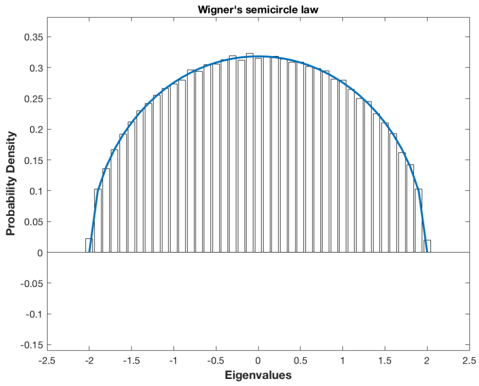
The following illustrations will represent Wigner's semi-circle law for different inputs.



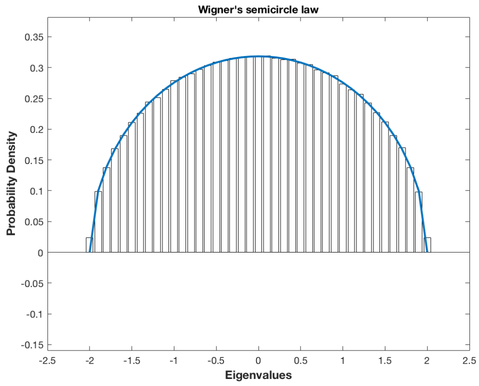
Histogram of the normalized eigenvalues from one realization of 50×50 random Wigner matrix with $s = 1$



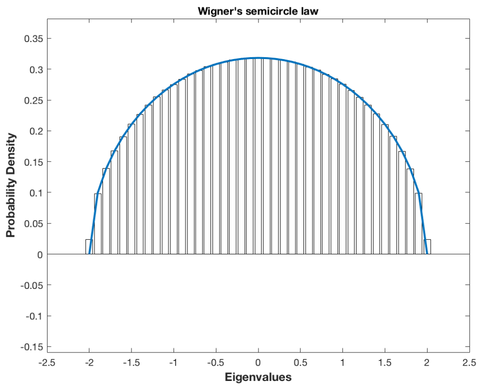
Histogram of the normalized eigenvalues from one realization of 200×200 random Wigner matrix with $s = 1$



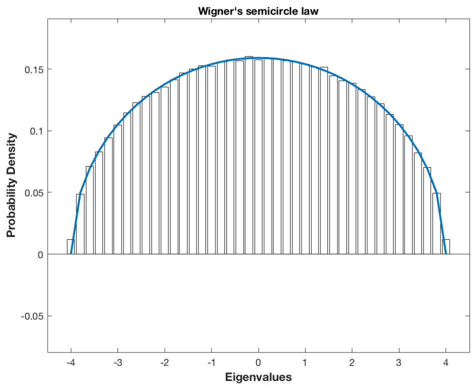
Histogram of the normalized eigenvalues from one realization of 1000×1000 random Wigner matrix with $s = 1$



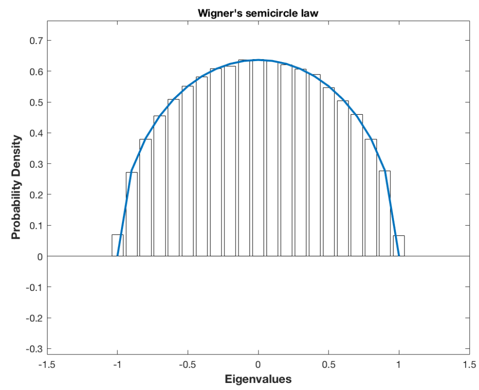
Histogram of the normalized eigenvalues from one realization of 2000×2000 random Wigner matrix with $s = 1$



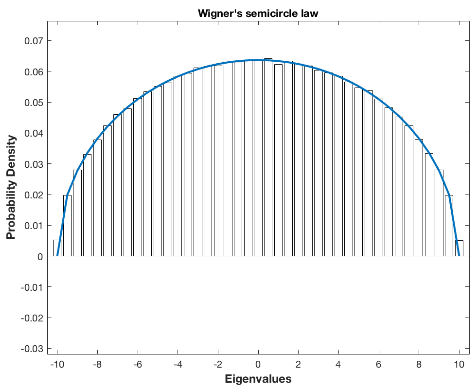
Histogram of the normalized eigenvalues from one realization of 10000×10000 random Wigner matrix with $s = 1$



Histogram of the normalized eigenvalues from one realization of 1000×1000 random Wigner matrix with $s = 2$



Histogram of the normalized eigenvalues from one realization of 1000×1000 random Wigner matrix with $s = \frac{1}{2}$



Histogram of the normalized eigenvalues from one realization of 1000×1000 random Wigner matrix with $s = 5$

The last line of our code above computes the norm of the Wigner matrices. Indeed, MatLab already possesses a built-in command which computes the operator norm of a given matrix. The following table gives some examples of computed values in order to check Bai-Yin's upper bound.

size of matrices	50	100	500	1000	2000	5000	10000
operator norm	1.9230	1.9409	1.9656	1.9835	1.9931	1.9957	1.9969

The table confirms Bai-Yin's theorem as the normalized operator norm of a Wigner matrix does not exceed 2 when n grows.

3.2.2 Error estimate of Wigner's semi-circle law

To estimate the error between the empirical distribution of the eigenvalues of the normalized Wigner matrices and the semicircle law we compute for given n the absolute error between these two and look for the biggest difference. To do this we use Matlab's built-in command "ecdf", which computes the empirical cumulative distribution function for a given sample.

```
function E = error(n,s)
v = [];
    for i=1:10
        H = s*(randn(n) + 1i*randn(n)); %random Gaussian matrix
        for j=1:length(H)
            H(j,j)=sqrt(2)*H(j,j);
        end
        W = (H+H')/2; %symmetrized matrix
        v = [v;eig(W)];
    end

v = v/sqrt(n); %normalized eigenvalues
[f,x] = ecdf(v); %empirical distribution function

ft = @(x) sqrt(4*s^2-x.^2)./(2*s^2*pi);
F = ft(x); %semi-circle law

E = max(abs(f-F));
end
```

Error Code

size of matrices (n)	50	100	500	1000	2000	5000	10000
error (%)	5.83	4.21	2.35	1.89	1.08	0.83	0.14

3.2.3 Observations

For $n = 50$, we compute an error percentage of $5,83$. Taking n up to 10000, we observe a convergence to the semi-circular curve with error 0.14.

3.2.4 Convergence at the extreme eigenvalues

Fix B, $E \in \mathbb{N}_{\geq 1}$ such that $E > B$.
We generate t matrices of size $B \times B$ and store the biggest and the smallest eigenvalues at the B-st place in an array. Then we repeat this process for $(B + 1) \times (B + 1)$, $(B + 2) \times (B + 2)$, and so on until we reach the size $E \times E$.

```
% B = Beginning size
% E = Ending size
% t = numner of matrices per size
function [Lmax,Lmin] = minmax(B,E,t)
for N = B:E
    for i = 1:t
        M = randn(N)+1i*randn(N); %random Gaussian matrix
        M = (M + M')*0.5; %symmetrized matrix
        EV = eig(M); % Eigenvalues of this matrix
        Ma(i) = max(EV); % storing biggest eigenvalue
        Mi(i) = min(EV); % storing smallest eigenvalue
    end
    Lmax(N)= max(Ma); % storing the biggest eigenvalue of all t matrices
    Lmin(N)= min(Mi); % storing the smallest eigenvalue of all t matrices
end
```

Code computing minima / maxima

The minmax function, is a simple function that returns two arrays one containing the biggest and one containing the smallest eigenvalue for each $N \times N$ matrix, from a beginning size B to a ending size E.

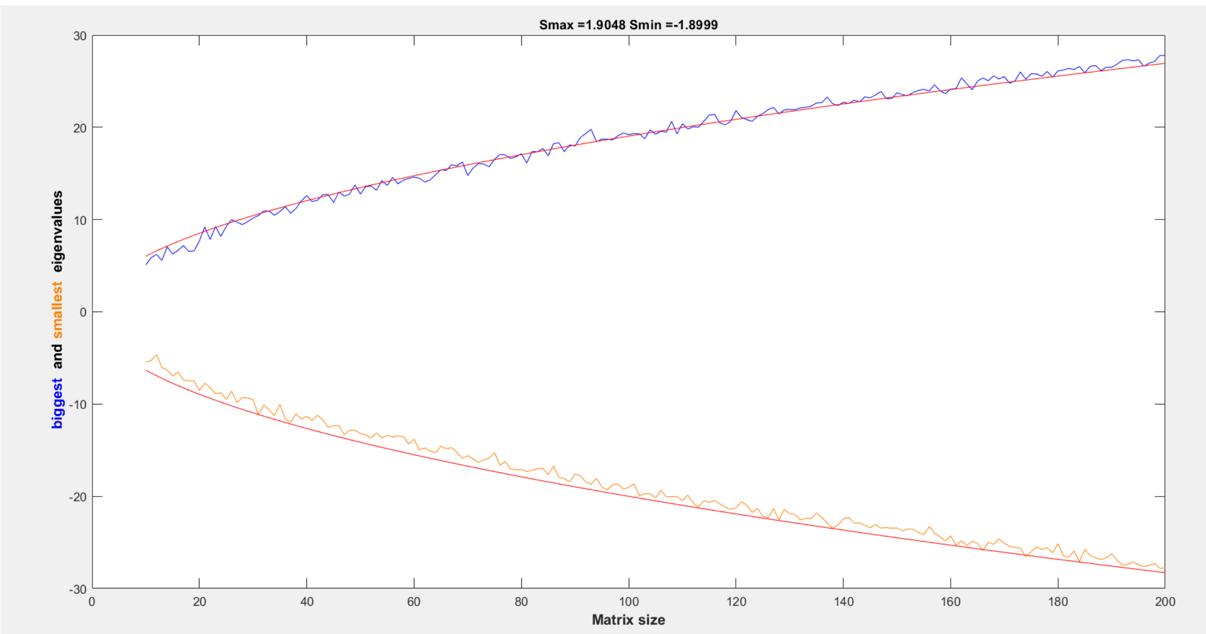
```
% Here we will use our minmax function to plot the rate growth of the biggest
% and smallest eigenvalues

b = 10; % smallest matrix size
e = 3200; % biggest matrix size

[Lmax,Lmin] = minmax(b,e,1);

% here we calculate the average multiplier S such that S*sqrt(n) fits the
% growth of the maximal/minimal eigenvalues
Smax = 0;
Smin = 0;
for i = b:e
    Smax = Smax+(Lmax(i)/sqrt(i));
    Smin = Smin+(Lmin(i)/sqrt(i));
end
Smax = Smax/(e-b);
Smin = Smin/(e-b);

```

```
20    %initializing the plot
21    figure();
22    plot(b:e,Lmax(b:e),'b'); % plotting max eigenvalues
23    hold on;
24    plot(b:e,Lmin(b:e),'color',[1 .5 0]); % plotting min eigenvalues
25    hold on;
26    plot(b:e,Smax*sqrt(b:e),'r'); % plotting 2*sqrt(n)
27    hold on;
28    plot(b:e,-2*sqrt(b:e),'r'); % plotting -2*sqrt(n)
29    % adding title and labels for x and y axis
30    title(strcat('Smax = ',num2str(Smax),' Smin = ',num2str(Smin)));
31    xlabel('Matrix size','FontWeight','bold','FontSize',12)
32    big = '\color{blue} biggest \color{black}';
33    sma = '\color{orange} smallest \color{black}';
34    text = strcat(big,' and ',sma,' eigenvalues');
35    ylabel(text,'FontWeight','bold','FontSize',12);
```

Here we are using our MinMax function and plot the result, after observing a growth-rate of order $\sqrt{N}$, we added the code lines $11-18$ to find a good approximation of the constant S such that $S \times \sqrt{N}$ fits the growth of the eigenvalues and we finally plot our results.

3.3 Observations

We observe that the rate of growth of the biggest eigenvalue of a Wigner matrix is of order $\sqrt{N}$, which we show for a matrix size up to 200×200



Wigner Matrix Eigenvalue of order $\sqrt{N}$

The following table will show us that the growth seems to converge to $2 \times \sqrt{N}$:

Beginning size	End size	Smax	Smin
10	200	1.9084	-1.8999
10	400	1.933	-1.9321
10	800	1.9549	-1.9552
10	1600	1.9685	-1.9696
10	3200	1.9797	-1.9797
10	6400	1.9863	-1.9867

We can guess from this data, that S_{max} and S_{min} are converging towards 2 or -2 respectively and indeed this is what the theorem suggests!

3.4 Other models

3.4.1 Circular law

It was conjectured in the early 1950's that the empirical distribution of the eigenvalues of an $n \times n$ matrix, of independent and identically distributed entries, normalized by a factor of $\frac{1}{\sqrt{n}}$, converges to the uniform distribution over the unit disc on the complex plane, which is called the circular law [1] [2]. Only a special case of the conjecture, where the entries of the matrix are standard complex Gaussian, was known. In 2010 Terence Tao and Van H. Vu proved the circular law under the minimal assumptions stated below :

Theorem 3.3. *(Circular law)*

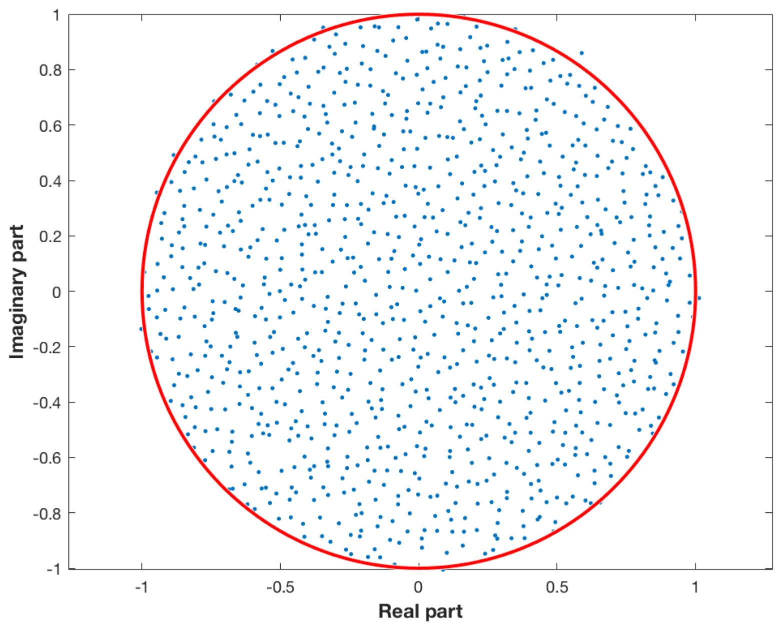
Let $(M_n)_{n\geq 1}$ be a sequence of $n \times n$ matrices whose entries are independent and identically distributed copies of a complex random variable with zero mean and variance 1.
Define $X_n := \frac{1}{\sqrt{n}} M_n$. Let $\lambda_1(X_n), \dots, \lambda_n(X_n)$ denote the eigenvalues of X_n. Then the probability measure

$$\mu_{X_n}(A) = \frac{\#\{j \leq n : \lambda_j \in A\}}{n} \quad , \quad A \in \mathcal{B}(\mathbb{C})$$

converges to the uniform measure on the unit disc.

A small example with a random Gaussian matrix of size 1000×1000:

```
function p = Circular(n)

H = 1/sqrt(2)*(randn(n) + 1i*randn(n)); %random complex Gaussian matrix
                                         %where entries have mean zero variance 1
v = eig(H); %eigenvalues

v = v/(sqrt(n)); %normalized eigenvalues
R = real(v);
I = imag(v);

p = plot(R,I,'.');
hold on
viscircles([0,0],1); % drawing a unit circle
xlabel('Real part','FontWeight','bold','FontSize',12)
ylabel('Imaginary part','FontWeight','bold','FontSize',12);
axis equal;
end
```



Plot of the eigenvalues on the unit disc

3.4.2 Marchenko-Pastur law

In this section we consider a particular case of random matrices with dependent entries and show that the limiting empirical spectral distribution is given by the Marchenko-Pastur law [8].

We are interested in $n \times m$ matrices $H_{n,m}$ where their entries are independent and normally distributed complex random variables with zero mean and variance s^2. Then the matrix $X_n \coloneqq \frac{1}{m} H_{n,m} H_{n,m}^T$ is an hermitian matrix, also called sample covariance matrix. Now we can proceed the same way as in Wigner's semicircle law and compute the eigenvalues $\lambda_1(X_n), ..., \lambda_n(X_n)$ of X_n which will be all real and distinct as X_n is hermitian. Using this kind of matrices we can formulate the following result:

Theorem 3.4. *(Marchenko-Pastur law)*

Let $(X_n)_{n\geq 1}$ be a sequence of matrices defined as above. Let $\lambda_1(X_n) \leq ... \leq \lambda_n(X_n)$ be the eigenvalues of X_n. Assume that $n, m \to \infty$ so that $\frac{n}{m} \to c \in]0, \infty[$. Then the probability measure

$$\mu_n(x) = \frac{\#\left\{i \in \{1, ..., n\} : \lambda_i(X_n) \leq x\right\}}{n}$$

converges to the Marchenko-Pastur law distribution

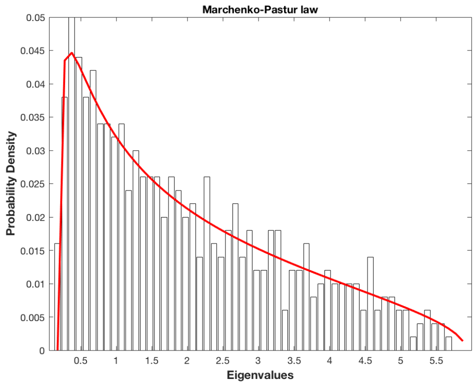
$$\mu_{mp} dx = \frac{1}{2\pi s^2} \frac{\sqrt{(x-a)(b-x)}}{cx} \mathbb{1}_{[a,b]}(x) dx$$

where $a = s^2(1 - \sqrt{c})^2$ and $b = s^2(1 + \sqrt{c})^2$.

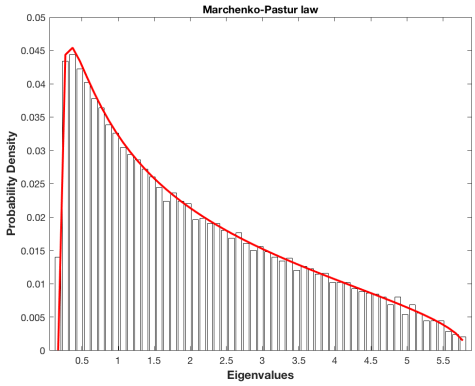
Similarly as we did with Wigner's semi-circle law, we use MatLab to construct a histogram of the eigenvalues of a set of matrices defined as above. This provides us a visualization of the behaviour of the eigenvalues of a random matrix with dependent entries.

```
function f = Marchenko(n,m)
c = n/m; %Ratio of matrix dimensions
v = []; %binsize
dx = .1;

for i=1:10
    H = randn(n,m) + 1i*randn(n,m); %random complex nxm Gaussian matrix
    s = std(H(:)); %standard deviation of the entries of H
    X = H*H'/m; %Hermitian matrix
    v = [v;eig(X)]; %eigenvalues
end

%Probability Density Function
%Boundaries
a = (s^2)*(1-sqrt(c))^2;
b = (s^2)*(1+sqrt(c))^2;

[f,x] = hist(v,a:dx:b);
%Normalization
f = f/sum(f);
%Theoretical probability distribution function
ft = @(x,a,b,c) (1./(2*pi*x*c*s^(2))).*sqrt((b-x).*(x-a));
F = ft(x,a,b,c);
F = F/sum(F);

%Plots
figure;
h = bar(x,f);
set(h,'FaceColor','none');
set(h,'LineWidth',dx);
xlabel('Eigenvalues','FontWeight','bold','FontSize',12);
ylabel('Probability Density','FontWeight','bold','FontSize',12);
title('Marchenko-Pastur law');

hold on;
plot(x,F,'r','LineWidth',2);
hold off;
end
```

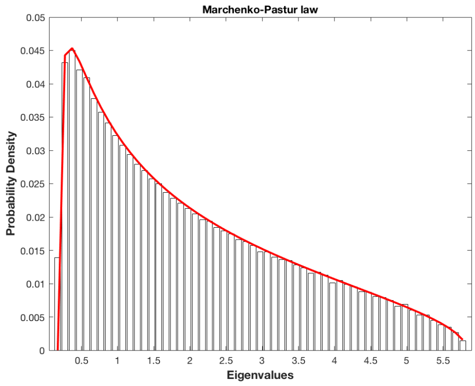
Marchenko-Pastur law computation



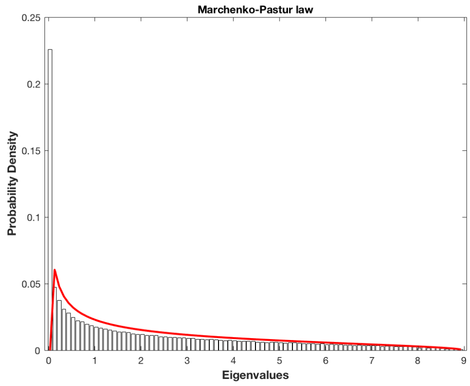
$n = 50,\ c = 2$



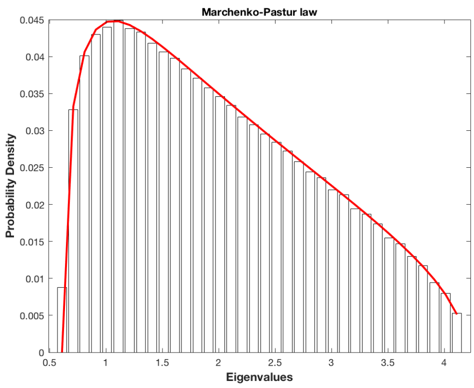
$n = 500,\ c = 2$



$n = 1000,\ c = 2$



$n = 1000,\ c = 0.8$



$n = 1000,\ c = 5$

The figures show the Marchenko-Pastur density for different inputs. We notice that as c becomes larger the length of the density's support increases. The convergence is again fast since at already $n = 500$ the empirical distribution comes very close to the representation of the Marchenko-Pastur density function.

4 Conclusion

The main purpose of the research project was studying the Wigner semi-circle law, one of the most famous and important theorems in the random matrix theory. The work was written to be self-contained and accordingly, there was an effort to include all the related computations along with graphical representations where appropriate.
This research project gives just an insight of the applications of random matrices. That being so, Wigner's semi-circle law was highly influential for future results.

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