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# Visualization of Bianchi Fundamental Polyhedra

Master Thesis

Author: Kelly Jost
Supervisor: Alexander Rahm
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# Contents

# Acknowledgment

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Among many others, a special thanks goes to my mother, who always stands behind me, supported me unconditionally and encouraged me to continue through all these years of study. Words can not express how grateful I am for her support without which I could not complete my studies.

# Introduction

In this Master thesis, we consider an imaginary quadratic number field $\mathbb{Q}(\sqrt{-m})$, with $m$ a squarefree positive integer, and its associated ring of integers $\mathcal{O}_{-m}$, which we also just denote by $O$. The groups $\mathrm{SL}_2(\mathcal{O})$ and $\mathrm{PSL}_2(\mathcal{O})$ are the so-called *Bianchi groups*. These groups act in a natural way on the 3-dimensional hyperbolic space. Luigi Bianchi (see [4]), an Italian mathematician who is primarily known for his contributions to differential geometry, computed in 1892 fundamental domains for this group action for the values of

$$m \in \{1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 11,\ 13,\ 15,\ 19\}\ .$$

As such a fundamental domain has the shape of a hyperbolic polyhedron (up to some missing vertices), we call it the *Bianchi fundamental polyhedron*.

The aim of this thesis is to develop a comprehensible description of the construction of a Bianchi fundamental polyhedron for a less advanced audience in mathematics. More precisely, the aim of chapter 2 is to describe a relevant algorithm that is used to compute the Bianchi fundamental polyhedron (see [1]). The second part of this thesis consists of computations of the Bianchi fundamental polyhedron for $m = 2$ and for $m = 5$.

Another part of this thesis was to visualize the Bianchi fundamental polyhedron. For this, I collected screenshots of the fundamental polyhedron for the Bianchi group of discriminant $-427$, computed with Bianchi.gp and visualized with the program "Geomview". Then using these screenshots, I established an animated "GIF"-file (see 'Appendix'). Furthermore, via the program "GeoGebra", I established constructions (mainly for the subsections 2.2.2 and 2.2.3) to clarify the ideas of definitions and notations, and to follow the development of the computation of the corresponding Bianchi fundamental polyhedron.

# Chapter 1

# Preliminaries

The aim of this chapter is to collect several definitions, explanations and results of hyperbolic Geometry to rise the comprehension for the next chapter.

## 1.1 Definitions, Notations and Examples

This section recalls some definitions of Geometry and Algebra.

Let us start with the definition of a group action, because in this thesis we will treat the action of $\mathrm{SL}_2(\mathcal{O})$ on the upper-half space model $\mathcal{H}$ (of which the details will follow later).

**Definition 1.** Let $E$ be a set with $E \neq \varnothing$ and let $G$ be a group. Then the map

$$\begin{aligned} \varphi : \ & G \times E \to E \\ & (g, x) \mapsto \varphi(g, x) := g \cdot x \end{aligned}$$

is called *(left) group action* of $G$ on $E$ if the following conditions are satisfied:

i) $e \cdot x = x$, for all $x \in E$, and $e \in G$ denotes the identity element of $G$,

ii) $g \cdot (g' \cdot x) = (gg') \cdot x$, for all $(g, g') \in G^2$, for all $x \in E$.

**Example 1.** The action

$$\begin{aligned} \cdot : G \times E &\to E \\ (g, x) &\mapsto x \end{aligned}$$

is called the *trivial action.*

**Example 2.** Let $G = \mathbb{Z}/2\mathbb{Z} = (\{1, -1\}, \cdot)$ and let $E = \mathbb{R}$. Then we consider the map

$$\begin{aligned} \varphi : \ \mathbb{Z}/2\mathbb{Z} \times \mathbb{R} &\to \mathbb{R} \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

which is a group action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{R}$. Indeed, $\varphi$ satisfies both conditions defined in Definition 1:

i) $1 \cdot x = x$, for all $x \in \mathbb{R}$, and $e = 1$ is the identity element of $\mathbb{Z}/2\mathbb{Z}$,

ii) By associativity of $\cdot$ , we clearly have $g \cdot (g' \cdot x) = (gg') \cdot x$, for all $(g, g') \in (\mathbb{Z}/2\mathbb{Z})^2$, and for all $x \in \mathbb{R}$.

**Definition 2.** Let $E$ be a set with $E \neq \varnothing$, $G$ be a group and $\varphi$ be a group action.

1. We call the *orbit* the subset $G \cdot x = \{g \cdot x \mid g \in G\}$, where $x \in E$.
2. We call the *stabilizer* of $x \in E$ the subgroup $G_x = \{g \in G \mid g \cdot x = x\}$ of $G$.
3. We call the *kernel* of the group action the subset $\ker(\varphi) = \{g \in G \mid g \cdot x = x \ , \ \forall x \in E\} = \bigcap_{x \in E} G_x$.

The stabilizer contains all group elements fixing the point $x$, i.e. that send $x$ to itself. The orbit of $x \in E$ is in fact the set of elements in $E$ to which $x$ can be moved by the elements of $G$.

**Example 3.** Let $G = \mathbb{Z}/2\mathbb{Z}$, $E = \mathbb{R}$ and let us consider the group action $\varphi$ as defined in Example 2.

1. The orbit is $\mathbb{Z}/2\mathbb{Z} \cdot x = \{g \cdot x \mid g \in \mathbb{Z}/2\mathbb{Z}\} = \{(x, -x)\}$, where $x \in \mathbb{R}$, as

$$\begin{aligned} \varphi : \ \mathbb{Z}/2\mathbb{Z} \times \mathbb{R} &\to \mathbb{R} \\ (1, x) &\mapsto 1 \cdot x = x \\ (-1, x) &\mapsto (-1) \cdot x = -x \ . \end{aligned}$$

   In other words, the orbits are pairs of points of opposite sign. Note that the orbit of 0 consists of just one point, namely $0 = -0$.

2. The stabilizer of $x \in \mathbb{R}$ equals to
$(\mathbb{Z}/2\mathbb{Z})_x = \{g \in \mathbb{Z}/2\mathbb{Z} \mid g \cdot x = x\} = \begin{cases} \{1\} , & \text{if } x \neq 0 \\ \{1, -1\} , & \text{if } x = 0 . \end{cases}$

3. The kernel of the group action is
$\ker(\varphi) = \{g \in \mathbb{Z}/2\mathbb{Z} \mid g \cdot x = x \ , \ \forall x \in \mathbb{R}\} = \bigcap_{x\in\mathbb{R}} (\mathbb{Z}/2\mathbb{Z})_x = \{1\}$.

**Definition 3.** Let $m \in \mathbb{N}$ be a positive integer. We call $m$ *squarefree* if $m = p_1 \cdot p_2 \cdot \ldots \cdot p_n$ with $p_i$, $p_j$ pairwise different prime numbers.

In other words, the prime decomposition of $m$ does not contain repeated factors.

**Example 4.**
- All prime numbers are obviously squarefree.
- Since $165 = 3 \cdot 5 \cdot 11$, we have that 165 is a squarefree positive integer.

Let us now recall some groups:

**Definition 4.** 1. The group

$$\mathrm{SL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{C} \text{ such that } ad - bc = 1 \right\}$$

is called the *special linear group* and contains the $2 \times 2$-matrices with entries in $\mathbb{C}$ and determinant equal to 1.

2. The quotient group

$$\mathrm{PSL}_2(\mathbb{C}) := {}^{\mathrm{SL}_2(\mathbb{C})}\big/_{\langle I, -I\rangle} \ ,$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ denotes the identity matrix, is called the *projective special linear group* over $\mathbb{C}$. So this group contains the $2 \times 2$-matrices with determinant one modulo its center $\{\pm I\}$. This means that we are identifying $I$ with $-I$; i.e. $I = -I$ in $\mathrm{PSL}_2(\mathbb{C})$.

Note that the elements of $\mathrm{PSL}_2(\mathbb{C})$ are actually $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \right\}$. But for the whole of this thesis we will use the notation $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for the elements of $\mathrm{SL}_2(\mathbb{C})$ and for their classes in $\mathrm{PSL}_2(\mathbb{C})$.

Let us now recall the definition of the ring of integers:

**Definition 5.** Let $K$ be an algebraic number field. Then its ring of integers is denoted by $\mathcal{O}_K$ and it is defined to be

$$\begin{aligned}\mathcal{O}_K &:= \{x \in K \mid \exists n \in \mathbb{N},\ \exists a_0, a_1, ..., a_n \in \mathbb{Z} \text{ and } a_n = 1:\ a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = 0\} \\ &= \{x \in K \mid \exists n \in \mathbb{N},\ \exists a_0, a_1, ..., a_{n-1} \in \mathbb{Z}:\ a_0 + a_1 x + a_2 x^2 + ... + a_{n-1} x^{n-1} + x^n = 0\}\ .\end{aligned}$$

Now consider $K := \mathbb{Q}(\sqrt{-m}\,) = \{q + r\sqrt{-m} \mid q, r \in \mathbb{Q}\} \subset \mathbb{C}$ the imaginary quadratic number field where $m$ is a fixed squarefree positive integer.
Then for $\mathcal{O}_{-m} \subset K$ the ring of imaginary quadratic integers, constructed according to Definition 5, it turns out that $\mathcal{O}_{-m} = \mathbb{Z} \oplus \omega \cdot \mathbb{Z}$ with

$$\omega = \begin{cases} \sqrt{-m}\ , & \text{if } m \equiv 1 \text{ or } 2 \mod 4 \\ \frac{\sqrt{-m}\ +1}{2}\ , & \text{if } m \equiv 3 \mod 4 \end{cases}$$

**Remark 1.** For $x = a + b\sqrt{-m} \in \mathbb{Z}[\sqrt{-m}]$, with $a, b \in \mathbb{Z}$ and $m$ a positive squarefree integer, we can use

$$\begin{aligned}x^2 &= a^2 + 2ab\sqrt{-m} - mb^2 \\ &= -a^2 + 2a^2 + 2ab\sqrt{-m} - mb^2\end{aligned}$$

Thus, we get $x^2 - 2a^2 - 2ab\sqrt{-m} = -a^2 - mb^2$, or equivalently, $x^2 - 2ax = -a^2 - mb^2$. Hence, if we set

$$-2a =: a_1 \in \mathbb{Z} \ \text{ and }\ - a^2 - mb^2 =: -a_0 \in \mathbb{Z}\ ,$$

we finally obtain $x^2 + a_1 x + a_0 = 0$; i.e. $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$. Thus, we have shown that $\mathbb{Z}[\sqrt{-m}] \subset \mathcal{O}_{\mathbb{Q}(\sqrt{-m})}$. Note that the inverse inclusion does not always exist. For example, for $m = 3$, we have that $\frac{1+\sqrt{-3}}{2} \in \mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$, but $\frac{1+\sqrt{-3}}{2} \notin \mathbb{Z}[\sqrt{-3}]$. Thus, we clearly have $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})} \not\subset \mathbb{Z}[\sqrt{-3}]$. Actually, $\mathcal{O}_{\mathbb{Q}(\sqrt{-3})}$ has twice as much points as $\mathbb{Z}[\sqrt{-3}]$.

To simplify the following reading, we set $\mathcal{O}_{-m} =: \mathcal{O}$.

**Example 5.** For $m = 1$, we have $\mathcal{O} = \mathbb{Z} \oplus \sqrt{-1} \cdot \mathbb{Z}$ where $\sqrt{-1} =: i$. Then the ring $\mathcal{O}$ is called the ring of *Gaussian integers*.

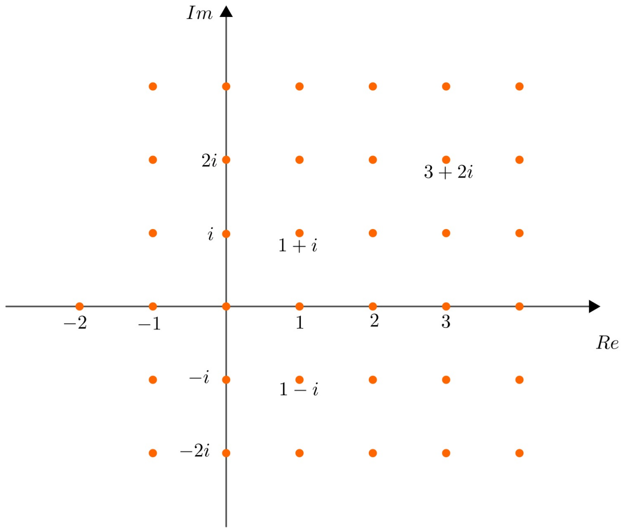



Figure 1.1: $\mathcal{O}_{-1}$

Note that $K = \mathbb{Q}(\sqrt{-1}\,)$ is then called the field of *Gaussian rationals* and that the set of units of $\mathcal{O}_{-1}$ is $(\mathcal{O}_{-1})^{\times} = \{\pm 1; \pm i\}$. The points in the grid $\mathcal{O}_{-1}$, displayed in Figure 1.1, are the possible entries for the matrices in $\mathrm{SL}_2(\mathcal{O}_{-1})$.

**Remark 2.**

- The groups $\mathrm{SL}_2(\mathcal{O}_{-m})$ and $\mathrm{PSL}_2(\mathcal{O}_{-m})$ play an important role in the next chapter.
- The groups of the form $\mathrm{SL}_2(\mathcal{O}_{-m})$ and $\mathrm{PSL}_2(\mathcal{O}_{-m})$, where $m$ is a squarefree positive integer, are called *Bianchi groups.*

**Note 1.** Let $K := \mathbb{Q}(\sqrt{-m}\,)$ be an imaginary quadratic number field where $m$ is a squarefree positive integer. Then the *discriminant* of $K$, denoted by $d_K$, takes the following values:

$$d_K = \begin{cases} -4m\ , & \text{if} \quad -m \equiv 2 \text{ or } 3 \mod 4 \\ -m\ , & \text{if} \quad -m \equiv 1 \mod 4\ . \end{cases}$$

Note that this is not the definition of the discriminant. The discriminant is defined more generally for algebraic number fields, and its definition can be found in books on algebraic number theory.

**Example 6.** As in the previous example, let $m = 1$. Then the discriminant $d_K$ of $K = \mathbb{Q}(\sqrt{-1})$ is equal to $-4$. Indeed, since $-1 \equiv 3 \mod 4$, then we have $d_K = -4$.

**Definition 6.** A pair of elements $(\mu, \lambda) \in \mathcal{O}^2$ is called *unimodular* if the ideal sum $\mu\mathcal{O} + \lambda\mathcal{O} = \{\lambda \cdot \alpha + \mu \cdot \beta \mid \alpha, \beta \in \mathcal{O}\}$ equals $\mathcal{O}$.

**Example 7.**

1. Let us check if $(2, 3)$ is a unimodular pair:
   Choose $(\alpha, \beta) = (1, -1) \in \mathcal{O}^2$, then $2\mathcal{O} + 3\mathcal{O} \ni 2 \cdot (-1) + 3 \cdot 1 = 1$. Thus, as $1 \in 2\mathcal{O} + 3\mathcal{O}$, we can conclude that $2\mathcal{O} + 3\mathcal{O} = \mathcal{O}$ ; or equivalently $(2, 3)$ is a unimodular pair.

2. Does $i\mathcal{O} + 2\mathcal{O} = \mathcal{O}$ ?
   Choose $(\alpha, \beta) = (0, -i) \in \mathcal{O}^2$, then $i\mathcal{O} + 2\mathcal{O} \ni i \cdot (-i) + 2 \cdot 0 = 1$. Thus, as $1 \in i\mathcal{O} + 2\mathcal{O}$, we can conclude that $i\mathcal{O} + 2\mathcal{O} = \mathcal{O}$ ; or equivalently $(i, 2)$ is a unimodular pair. Moreover, as we chose $\alpha = 0$, we also proved $i\mathcal{O} = \mathcal{O}$.

3. Let us check if $(2i, 3)$ is a unimodular pair:
   For $(\alpha, \beta) = (1, i) \in \mathcal{O}^2$, then we have $2i\mathcal{O} + 3\mathcal{O} \ni 2i \cdot i + 3 \cdot 1 = 1$. Thus, as $1 \in 2i\mathcal{O} + 3\mathcal{O}$, we can conclude that $2i\mathcal{O} + 3\mathcal{O} = \mathcal{O}$ ; or equivalently $(2i, 3)$ is a unimodular pair.

**Remark 3.**

i) Let us recall the definition of an ideal of a ring:
   Let $(R, +, \cdot)$ be a ring. Then $I \subseteq R$ is called a *(left) ideal* of $R$ if

   1. $(I, +)$ is a subgroup of $(R, +)$,
   2. for every $x \in I$, and for every $r \in R$, we have $rx \in I$.

ii) To draw the conclusion "if $1 \in \mu\mathcal{O} + \lambda\mathcal{O}$, then $\mu\mathcal{O} + \lambda\mathcal{O} = \mathcal{O}$" in Example 7, we use the following property of Algebra:
   Let $R$ be a ring, and let $I \subseteq R$ be an (left or right) ideal. If $1 \in I$, then $I = R$.
   Indeed, let $r \in R$. If $1 \in I$, then we have $r = r \cdot 1 \in I$ by the second property of ideals (see i)). Hence, $R \subseteq I$. Finally, we proved that, if $1 \in I$, then $I = R$.

iii) Let us check that $\mu\mathcal{O} + \lambda\mathcal{O} = \{\lambda \cdot \alpha + \mu \cdot \beta \mid \alpha, \beta \in \mathcal{O}\}$ is indeed an ideal:
   Let us consider the ideals $\mu\mathcal{O} = \{\mu \cdot \beta \mid \beta \in \mathcal{O}\}$ and $\lambda\mathcal{O} = \{\lambda \cdot \alpha \mid \alpha \in \mathcal{O}\}$ of $\mathcal{O}$.

   1. Let $\lambda \cdot \alpha_1 + \mu \cdot \beta_1 \in \mu\mathcal{O} + \lambda\mathcal{O}$, for $(\alpha_1, \beta_1) \in \mathcal{O}^2$, and let $\lambda \cdot \alpha_2 + \mu \cdot \beta_2 \in \mu\mathcal{O} + \lambda\mathcal{O}$, for $(\alpha_2, \beta_2) \in \mathcal{O}^2$. Then we have

$$(\lambda \cdot \alpha_1 + \mu \cdot \beta_1) - (\lambda \cdot \alpha_2 + \mu \cdot \beta_2) = \lambda \cdot \underbrace{(\alpha_1 - \alpha_2)}_{\in \mathcal{O}} + \mu \cdot \underbrace{(\beta_1 - \beta_2)}_{\in \mathcal{O}} \in \mu\mathcal{O} + \lambda\mathcal{O} \ .$$

2. Let $\lambda \cdot \alpha + \mu \cdot \beta \in \mu\mathcal{O} + \lambda\mathcal{O}$, for $(\alpha, \beta) \in \mathcal{O}^2$, and let $\nu \in \mathcal{O}$. Then we have

$$\begin{aligned} \nu(\lambda \cdot \alpha + \mu \cdot \beta) &= \nu(\lambda \cdot \alpha) + \nu(\mu \cdot \beta) \\ &= \lambda \cdot \underbrace{(\nu \cdot \alpha)}_{\in\, \mathcal{O}} + \mu \cdot \underbrace{(\nu \cdot \beta)}_{\in\, \mathcal{O}} \in \mu\mathcal{O} + \lambda\mathcal{O} \ . \end{aligned}$$

Hence, $\mu\mathcal{O} + \lambda\mathcal{O}$ is an ideal of $\mathcal{O}$.

## 1.2 Upper half-space Model

In this section, we want to describe the upper half-space model of the 3-dimensional hyperbolic space. More details can be found in [3]. Nowadays, there exist many convenient models (for example: Poincaré ball model, Kleinian model) to describe the 3-dimensional hyperbolic space, but we will use the upper half-space model because it is computationally convenient. In two dimensions, it is the upper half-plane model for the hyperbolic plane.

We define the upper half-space as a set

$$\mathcal{H} := \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\} = \mathbb{C} \times ]0; +\infty[ \ .$$

Moreover, the space $\mathcal{H}$ can be equipped with the *hyperbolic metric* coming from the line element:

$$ds^2 = \frac{dx^2 + dy^2 + d\zeta^2}{\zeta^2} \ ,$$

where $z := x + iy \in \mathbb{C}$, with $x, y \in \mathbb{R}$. Equipped with this metric, we have that the 3-dimensional hyperbolic space is the unique 3-dimensional connected and simply connected Riemannian manifold with constant sectional curvature equal to $-1$.

Let us start with some recalls of the plane hyperbolic Geometry.

We have that the special linear group

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{Z} \text{ such that } ad - bc = 1 \right\}$$

acts on the 2-dimensional upper half-plane

$$\mathbb{H}^2 = \{x + iy \in \mathbb{C} \mid (x; y) \in \mathbb{R}^2, \ y > 0, \ i^2 = -1\} \ \subset \mathbb{C}$$

via the so called *Möbius transformation*; i.e. let $z = x + iy \in \mathbb{H}^2$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{a \cdot z + b}{c \cdot z + d} \in \mathbb{H}^2 \ .$$

**Notation 1.** We can denote " $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}^2$ " as follows: $\mathrm{SL}_2(\mathbb{Z}) \circlearrowright \mathbb{H}^2$.

**Remark 4.** In the previous section we defined the group $\mathrm{PSL}_2(\mathbb{C})$. Actually, this group has a natural action on $\mathcal{H}$ which can be described as follows:
Let $M \in \mathrm{PSL}_2(\mathbb{C})$, then $M$ induces a biholomorphism of the complex projective line $\mathbb{P}^1\mathbb{C} := \mathbb{C} \cup \{\infty\}$, called the *Riemann sphere*.
Now, as

$$-I \cdot z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \cdot z = \frac{-1 \cdot z + 0}{0 \cdot z + (-1)} = z \ ,$$

where $-I \in \mathrm{SL}_2(\mathbb{C})$ and $z \in \mathbb{H}^2$, we say that $-I$ acts trivially. We have in fact that the subset $\{I, -I\}$ of $\mathrm{SL}_2(\mathbb{C})$ is the kernel of the group action described above. Moreover, the fact that $-I$ acts trivially implies that we really have an action on the quotient group $\mathrm{PSL}_2(\mathbb{C})$.

**Notation 2** (See [12], [13], [14])**.** For a given group action of a group $G$ on a set $E$, we have defined the orbit of a point as the collection of its images under the group action. Then, a subset of the set $E$, which contains exactly one point of each orbit, is called a strict *fundamental domain*. Thus, a fundamental domain is used as a realization for the set of representatives of the orbits (in a geometrical way if $E$ comes with a geometry).
Normally, it is requested that a fundamental domain is a connected subset with some restrictions. One example of such a restriction is on its boundary, as it will be the case in the next chapter. Note that a fundamental domain for a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ that is also a polyhedron is also called a *fundamental polyhedron*.
There are various versions of constructing a fundamental domain. But, once a fundamental domain is chosen, the images of it under the group action then "tile" the space. This is called a *tessellation*, and an individual image of the fundamental domain is called a *tile*.

**Example 8.** (See [7]) In the picture below, we have the **Dedekind** - tessellation of the upper half-plane in hyperbolic triangles drawn in black and white.

Any pair consisting of a black and a white hyperbolic triangle in this tessellation forms a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$.
More precisely, let us denote the fundamental domain for this action by $\mathcal{F}$. So we have

$$\mathcal{F} = \{1 \text{ black and } 1 \text{ white hyperbolic triangle}\}$$

and, as we have that these black and white triangles tile the the upper half-plan, we can write

$$\mathrm{SL}_2(\mathbb{Z}) \cdot \mathcal{F} = \mathbb{H}^2 \ .$$

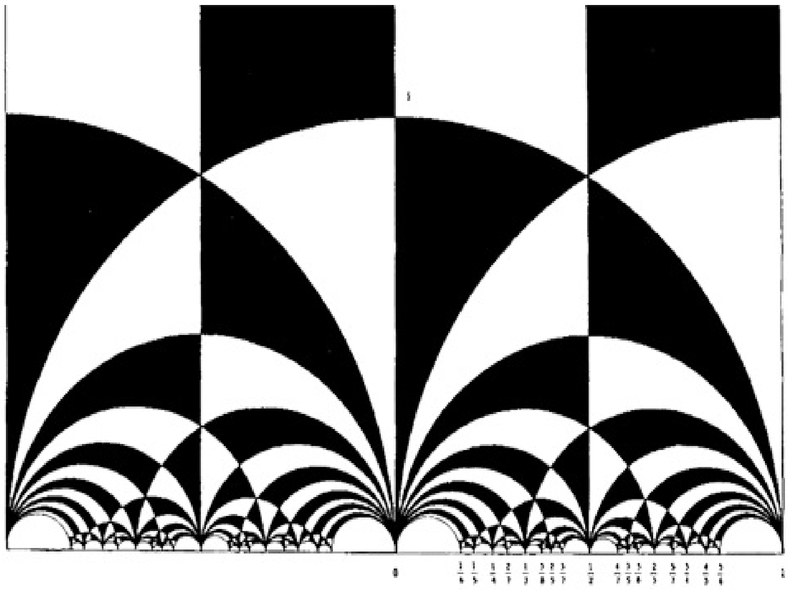


Figure 1.2: See [7]: Tessellation of the upper half-plane in hyperbolic triangles

We have that $\mathrm{SL}_2(\mathbb{Z})$ is generated by two rotations $\alpha$ and $\beta$, satisfying

$$\alpha^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I = \beta^3 \ .$$

So we obtain $\mathrm{SL}_2(\mathbb{Z}) \cong \langle\alpha\rangle *_{\{\pm I\}} \langle\beta\rangle$.
An example for $\alpha$ and $\beta$ would be

$$\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} .$$

In Figure 1.2, at some points where two black and two white triangles are coming together, we can see the following:

If we choose one white or black triangle, and if we rotate it twice by $\alpha$, then the triangle is again at its initial position; i.e. $\alpha^2 = -I$.
Now, at some points where three black and three white triangles are coming together, we can recognize the comparable following:
If we rotate around the fixed point of $\beta$ three times, then everything is again at its initial position; i.e. $\beta^3 = -I$ acts trivially.

In Figure 1.2, the half-circles and vertical lines are the geodesics which are sometimes called *hyperbolic lines*. Note that a geodesic is actually a "straight line" in a curved space; it describes the shortest path between two points on a curved surface.

**Note:** If we return to the 3-dimensional hyperbolic space, then these hyperbolic lines are orthogonal to the boundary plane $\mathbb{C}$ in the Euclidean sense. Moreover, the geodesic surfaces are Euclidean half-planes or hemispheres which are again orthogonal to the boundary $\mathbb{C}$.

Let us state explicitly the action of $\mathrm{SL}_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{C})$ on the upper-half space model $\mathcal{H}$, in the form in which we will use it:

**Lemma 1** (See [10]). *If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) = \{\sigma \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid det(\sigma) \neq 0\}$, then the action of $\gamma$ on $\mathcal{H}$ is given by $\gamma \cdot (z, \zeta) = (z', \zeta')$, where*

$$\zeta' = \frac{|\det\gamma|\zeta}{|cz - d|^2 + \zeta^2|c|^2} \ ,$$

$$z' = \frac{\left(\overline{d - cz}\right)(az - b) - \zeta^2\bar{c}a}{|cz - d|^2 + \zeta^2|c|^2} \ .$$

**Example 9.** Taking the point $(z, \zeta) = \left(\frac{1}{2} + \frac{\sqrt{-2}}{2}, \frac{1}{2}\right) \in \mathcal{H}$ computed in Section 2.2.2 and $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ (as $\det\gamma = 1$, it is also in $\mathrm{SL}_2(\mathbb{C})$), we obtain $\gamma \cdot (z, \zeta) = (z', \zeta')$, where

$$\begin{aligned} \zeta' &= \frac{|\det\gamma|\zeta}{|cz - d|^2 + \zeta^2|c|^2} \\ &= \frac{1 \cdot \frac{1}{2}}{|1 \cdot (\frac{1}{2} + i\frac{\sqrt{2}}{2}) - i|^2 + \frac{1}{4} \cdot 1} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\frac{1}{2}}{|\frac{1}{2} + i(\frac{\sqrt{2}}{2} - 1)|^2 + \frac{1}{4}} \\
&= \frac{\frac{1}{2}}{(\frac{1}{2})^2 + (\frac{\sqrt{2}}{2} - 1)^2 + \frac{1}{4}} \\
&= \frac{\frac{1}{2}}{\frac{1}{4} + \frac{3}{2} - \sqrt{2} + \frac{1}{4}} \\
&= \frac{\frac{1}{2}}{2 - \sqrt{2}} \\
&= \frac{1}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}} \\
&= \frac{4 + 2\sqrt{2}}{8} \\
&= \frac{1}{2} + \frac{\sqrt{2}}{4}
\end{aligned}
$$

and

$$
\begin{aligned}
z' &= \frac{\left(\overline{d - cz}\right)(az - b) - \zeta^2 \bar{c} a}{|cz - d|^2 + \zeta^2 |c|^2} \\
&= \frac{\left(\overline{i - 1 \cdot (\frac{1}{2} + i\frac{\sqrt{2}}{2})}\right)(0 \cdot z + 1) - \frac{1}{4} \cdot 1 \cdot 0}{2 - \sqrt{2}} \\
&= \frac{\left(\overline{-\frac{1}{2} + i(1 - \frac{\sqrt{2}}{2})}\right)}{2 - \sqrt{2}} \\
&= \frac{-\frac{1}{2} - i(1 - \frac{\sqrt{2}}{2})}{2 - \sqrt{2}} \cdot \frac{2 + \sqrt{2}}{2 + \sqrt{2}} \\
&= -\frac{2 + \sqrt{2}}{4} - i\frac{1}{2} \\
&= -\frac{2 + \sqrt{2}}{4} - \frac{\sqrt{-1}}{2} .
\end{aligned}
$$

**Remark 5.** By solving the equation

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, \zeta) = (z, \zeta)
$$

for $a, b, c$ and $d$, we obtain the stabilizer for $(z, \zeta)$.

**Lemma 2** (See [6]; [2], Lemma 3.3 and Lemma 3.4)**.** *The hyperbolic metric is invariant under the action of* $\mathrm{GL}_2(\mathbb{C})$*. The set of geodesics for the hyperbolic metric is stable under the action of* $\mathrm{GL}_2(\mathbb{C})$*.*

**Proposition 1** (See [3])**.** *The stabilizer of* $j = (0,0,1) \in \mathcal{H}$ *with respect to the action of* $\mathrm{SL}_2(\mathbb{C})$ *on* $\mathcal{H}$ *is equal to the special unitary group*

$$\mathrm{SU}_2 = \{V \mid V \in \mathrm{U}_2,\ \det(V) = 1\}\ ,$$

*where* $\mathrm{U}_2 = \{M \in \mathrm{GL}_2(\mathbb{C}) \mid \overline{M}^T M = I\}$ *is the unitary group of* $2\times 2$*-unitary matrices.*

*Proof.* Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. We have that $M$ belongs to the stabilizer of $j$ if and only if $|c|^2 + |d|^2 = 1$ and $a\bar{c} + b\bar{d} = 0$. Assume that $M$ belongs to the stabilizer of $j$. To accomplish the proof, we have to show that $M \in \mathrm{SU}_2$.
Since $M \in \mathrm{SL}_2(\mathbb{C})$, we have that $ad-bc = 1$. But this implies that the conditions above are equivalent to $M = \begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$. Indeed, $\det(M) = \bar{d}d + c\bar{c} = |d|^2 + |c|^2 = 1$. Moreover, we also have that $M \in \mathrm{U}_2$:

$$\overline{M}^T M = \begin{pmatrix} d & \bar{c} \\ -c & \bar{d} \end{pmatrix}\begin{pmatrix} \bar{d} & -\bar{c} \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d}d + \bar{c}c & -\bar{c}d + \bar{c}d \\ -c\bar{d} + c\bar{d} & \bar{c}c + \bar{d}d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I\ .$$

Hence, $M \in \mathrm{SU}_2$. □

**Note:** $\mathrm{SL}_2(\mathbb{C})$ has a simple set of generators. This set will be described in the following proposition:

**Proposition 2** (See [3])**.** *The group* $\mathrm{SL}_2(\mathbb{C})$ *is generated by the elements*

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

*with* $a \in \mathbb{C}$*. These generators operate on* $\mathcal{H}$ *as follows:*

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}(z,\zeta) = (z+a,\zeta) \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(z,\zeta) = \left(\frac{-\bar{z}}{|z|^2+\zeta^2}, \frac{\zeta}{|z|^2+\zeta^2}\right)\ ,$$

*where* $z = x+iy \in \mathbb{C}$ *(with* $x,y \in \mathbb{R}$*), and* $(z,\zeta)$ *denotes a point in* $\mathcal{H}$*.*

*Proof.* Suppose $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$.

- If $c \neq 0$, then we can factorize $M$ as follows
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} 1 & dc^{-1} \\ 0 & 1 \end{pmatrix} .$$
- If $c = 0$ then we obtain the factorization
$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} .$$

So we can conclude that the matrices defined in the statement together with the matrices
$$D_\beta := \begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} , \text{ with } 0 \neq \beta \in \mathbb{C} ,$$
generate $\mathrm{SL}_2(\mathbb{C})$. But $D_\beta$ may be factorized as a product of matrices defined in the statement. This becomes clear by the following:
$$\begin{pmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \beta^2 - \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-\beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} ,$$
$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} ,$$
$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$
This finishes the proof. □

The next Theorem gives us an important relation between the groups $\mathrm{PSL}_2(\mathbb{C})$and $\mathrm{PSL}_2(\mathcal{O})$.

**Theorem 1** (See [3]). *Let $K$ be an imaginary quadratic field of discriminant $d_K < 0$, and let $\mathcal{O}$ be its ring of integers. Then the group $\mathrm{PSL}_2(\mathcal{O})$ has the following properties:*

1. $\mathrm{PSL}_2(\mathcal{O})$ *is a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$.
2. $\mathrm{PSL}_2(\mathcal{O})$ *has a fundamental domain $\mathcal{F}_K$ bounded by finitely many geodesic surfaces.*
3. *The covering of $\mathcal{H}$ by the $\sigma\mathcal{F}_K$, $\sigma \in \mathrm{PSL}_2(\mathcal{O})$, is locally finite.*
4. *The set $\{\sigma \in \mathrm{PSL}_2(\mathcal{O}) \mid \sigma\mathcal{F}_K \cap \mathcal{F}_K \neq \varnothing\}$ is finite.*
5. $\mathrm{PSL}_2(\mathcal{O})$ *is a geometrically finite group.*

# Chapter 2

# Algorithms to compute the quotient space

The aim of this chapter is to describe an algorithm that, given any Bianchi group, computes a fundamental domain for its action on the 3-dimensional hyperbolic space. I will mainly follow the structure of [1].

## 2.1 Swan's concept

In this section, we recall Richard G. Swan's work (see [2]): From the theoretical viewpoint, he described an algorithm to compute the Bianchi fundamental polyhedron. This algorithm has been put into practice in the realization described in Section 2.2.

### 2.1.1 Defining the Bianchi fundamental polyhedron

For this chapter, we will use $K = \mathbb{Q}(\sqrt{-m}\,) \subset \mathbb{C}$ as an imaginary quadratic number field, where $m \in \mathbb{N}$ is a squarefree positive integer, and then $\mathcal{O}_{-m} := \mathcal{O} \subset K$ as its ring of imaginary quadratic integers.

We will consider the familiar group action of the group $\Gamma := \mathrm{SL}_2(\mathcal{O}) \subset \mathrm{GL}_2(\mathbb{C})$ on the 3-dimensional hyperbolic space, for which we will use the upper-half space model $\mathcal{H}$ described in Section 1.2. An explicit formula for the mentioned group action was given in Lemma 1 (see Chapter 1).

Recall that we defined the upper half-space model $\mathcal{H}$ as a set:

$$\mathcal{H} = \{(z, \zeta) \in \mathbb{C} \times \mathbb{R} \mid \zeta > 0\}.$$

**Definition 7.** The coordinate $\zeta$ for a point $(z, \zeta)$ in $\mathcal{H}$ is called the *height.*

The Bianchi/Humbert theory ([4], [5]) gives a fundamental domain for the group action mentioned above. Let us start by presenting a geometric description of this domain, and the arguments why it is in fact a fundamental domain.

The boundary of $\mathcal{H}$ is the Riemann sphere $\partial\mathcal{H} := \mathbb{C} \cup \{\infty\}$ (as a set). The Riemann sphere clearly contains the complex plane $\mathbb{C}$. Recall that the totally geodesic surfaces in $\mathcal{H}$ are the Euclidean vertical planes and the Euclidean hemispheres whose centers lie on the complex plane $\mathbb{C}$. **Note** that we define here *vertical* as to be orthogonal to the complex plane in the Euclidean sense.

**Notation 3.** For a given unimodular pair $(\mu,\ \lambda) \in \mathcal{O}^2$ with $\mu \neq 0$, we denote by $S_{\mu,\lambda} \subset \mathcal{H}$ the hemisphere given by the equation $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$.

Then this hemisphere is defined by its center $\frac{\lambda}{\mu}$ which lies on the complex plane $\mathbb{C}$, and by its radius $\frac{1}{|\mu|}$.

Let us define the set

$$B := \Big\{ (z, \zeta) \in \mathcal{H} \ : \ \text{The inequality } |\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1 \text{ is fulfilled for all unimodular pairs } (\mu,\ \lambda) \in \mathcal{O}^2 \text{ with } \mu \neq 0 \Big\}.$$

Then the set $B$ contains all the points in $\mathcal{H}$ which lie on or above all hemispheres $S_{\mu,\lambda}$, with $(\mu,\ \lambda)$ a unimodular pair.

**Example 10.** Let $\mathcal{O}$ be an arbitrary ring of integers. We choose $(\mu, \lambda) = (2, 3) \in \mathcal{O}^2$ which is a unimodular pair ( as $2 \cdot (-1) + 3 \cdot 1 = 1 \in 2\mathcal{O} + 3\mathcal{O}$, with $2, -1, 3, 1 \in \mathcal{O}$).
Then we denote $S_{2,3} \subset \mathcal{H}$ the hemisphere given by the equation

$$|2z - 3|^2 + |2|^2\zeta^2 = 1 \ ,$$

i.e.

$$|2z - 3|^2 + 4\zeta^2 = 1 \ .$$

This hemisphere has center $\frac{\lambda}{\mu} = \frac{3}{2}$ on the complex plane $\mathbb{C}$, and has radius $\frac{1}{|\mu|} = \frac{1}{2}$.

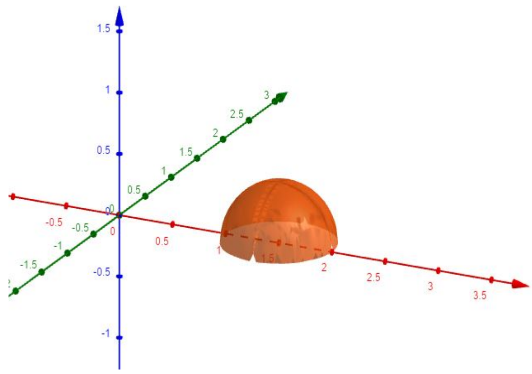


Figure 2.1: The hemisphere $S_{2,3}$ of center $3/2$ and of radius $1/2$

**Lemma 3** (See [2]). *The set $B$ contains representatives for all the orbits of points under the action of* $\mathrm{SL}_2(\mathcal{O})$ *on* $\mathcal{H}$.

*Proof.* Consider the 3-dimensional hyperbolic space as the set of positive definite Hermitian forms $f$ in two complex variables, modulo homotheties (i.e. a geometric transformation corresponding to an extension or a reduction). The action of $\mathrm{GL}_2(\mathbb{C})$ on the variables by linear automorphisms of $\mathbb{C}^2$ induces an action on this set by the formula $\gamma \cdot f(z) := f(\gamma^{-1}z)$ for $\gamma \in \mathrm{GL}_2(\mathbb{C})$, $z \in \mathbb{C}^2$. This action corresponds to the familiar action on $\mathcal{H}$. This latter action was even defined by Swan this way.
Now we have that the set $B$ corresponds to the Hermitian forms which take their proper minimum at the argument $(1,0)$. Then from Humbert [5], it follows that for any binary Hermitian form $f \in B$, there exists an element $\gamma \in \mathrm{SL}_2(\mathcal{O})$ such that $\gamma \cdot f$ takes its proper minimum at $(1,0)$. $\square$

Using another notation, we have that Lemma 3 states $\Gamma \cdot B = \mathcal{H}$.

The group action extends continuously to the boundary $\partial\mathcal{H}$, which is a Riemann sphere.

Now in $\Gamma := \mathrm{SL}_2(\mathcal{O})$, we will denote by $\Gamma_\infty$ the stabilizer subgroup of the point $\infty \in \partial\mathcal{H}$.

**Remark 6.** For $m = 1$ and $m = 3$, $\Gamma_\infty$ contains some rotation matrices like $\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$, which we want to exclude. These two cases have been treated amongst others in [8] and in [9].

So for the following, we assume $m \neq 1$, $m \neq 3$ . Then,

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \middle| \lambda \in \mathcal{O} \right\},$$

which performs translations by the elements of $\mathcal{O}$ with respect to the Euclidean geometry of the upper-half space $\mathcal{H}$.

**Notation 4.** A fundamental domain for $\Gamma_\infty$ is given by

- the rectangle

$$D_0 := \begin{cases} \{x + y\sqrt{-m} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}, & m \equiv 1 \text{ or } 2 \mod 4, \\ \{x + y\sqrt{-m} \in \mathbb{C} \mid -\frac{1}{2} \leqslant x \leqslant \frac{1}{2},\ 0 \leqslant y \leqslant \frac{1}{2}\}, & m \equiv 3 \mod 4. \end{cases}$$

  in the complex plane $\mathbb{C}$ considered as a subset of $\partial\mathcal{H}$,

- $D_\infty := \{(z, \zeta) \in \mathcal{H} \mid z \in D_0\}$ in $\mathcal{H}$; i.e. we can write $\Gamma_\infty \cdot D_\infty = \mathcal{H}$.

Now we are able to give the important definition of the Bianchi fundamental polyhedron, which is one of the main objects in this thesis.

**Definition 8.** We define the *Bianchi fundamental polyhedron* as

$$D := D_\infty \cap B\ .$$

It is a polyhedron in hyperbolic space up to the missing vertex $\infty$, and up to missing vertices at the singular points if $\mathcal{O}$ is not a principal ideal domain (see for more details Subsection 2.1.3).

**Example 11.** In the picture below, there is a cutout of a fundamental domain, for $m = 37$, represented which is computed with BianchiGP and visualized by M. Fuchs:

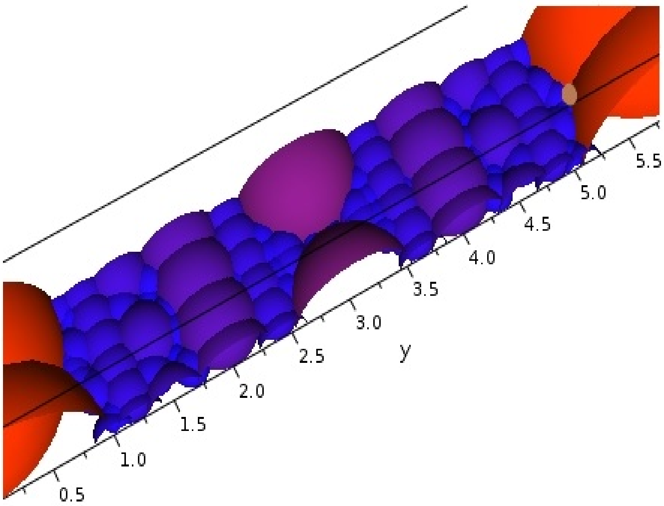



Figure 2.2: (See [16]) Visualization of a fundamental domain

**Remark 7.** Note that, since $\Gamma_\infty \cdot D_\infty = \mathcal{H}$, this implies $\Gamma_\infty \cdot D = \Gamma_\infty \cdot (D_\infty \cap B) = \mathcal{H} \cap B = B$. Now, as Lemma 3 states $\Gamma \cdot B = \mathcal{H}$, and as $\Gamma_\infty \cdot D = B$, we obtain $\Gamma \cdot D = \mathcal{H}$.

Moreover, we observe the following notion of the *strict* fundamental domain: the interior of the Bianchi fundamental polyhedron doesn't contain two points which are identified by $\Gamma$.

The following theorem implies that the boundary of the Bianchi fundamental polyhedron only consists of finitely many cells.

**Theorem 2** (See [2])**.** *There is only a finite number of unimodular pairs* $(\lambda, \mu) \in \mathcal{O}^2$ *such that the intersection of* $S_{\mu,\lambda}$ *with the Bianchi fundamental polyhedron is non-empty.*

Swan also proved a corollary, from which it can be deduced that the action of $\Gamma$ on $\mathcal{H}$ is properly discontinuous. Before we state the mentioned corollary, let us first give the definition of a properly discontinuous group action.

**Definition 9.** Let $G$ be a topological group and $E$ be a topological space. Then the action of $G$ on a $E$ is called *properly discontinuous* if $G$ has the discrete topology and if every point $x \in E$ has a neighborhood $U_x$ such that the intersection $g(U_x) \cap U_x$ with its translate under the group action via some element $g \in G$ is non-empty only for the neutral element $e \in G$. In other words, if $g(U_x) \cap U_x \neq \varnothing$, then this implies that $g = e$.

**Corollary 1** (See [2])**.** *There are only finitely many matrices* $\gamma \in \mathrm{SL}_2(\mathcal{O})$ *such that* $D \cap \gamma \cdot D \neq \varnothing$*.*

### 2.1.2 Determining the Bianchi fundamental polyhedron

By Definition 8, we have that the set $B$, which has been defined using infinitely many hemispheres, determines the Bianchi fundamental polyhedron. But we will know from Theorem 2 that only a finite number of hemispheres are significant for this intention and need to be computed. For this, we will state a criterion for what is an appropriate choice that gives us precisely the set $B$. In addition, we will see later that this criterion will be easy to verify in practice.

For this purpose, suppose that we have a finite number $n \in \mathbb{N}$ of hemispheres. We denote the $i$-th hemisphere by $S(\alpha_i)$, with $i \in \{1, \dots, n\}$, and where $\alpha_i$ is its center given by a fraction $\alpha_i = \frac{\lambda_i}{\mu_i}$ in the number field $K$. We require that the ideal $(\lambda_i, \mu_i)$ is the whole ring of integers $\mathcal{O}$; i.e. $(\lambda_i, \mu_i)$ is a unimodular pair for each $i$. This requirement has also be made for all the hemispheres in the definition of the set $B$.

Now, we are able to give an approximation of Notation 3, using, modulo the translation group $\Gamma_\infty$, a finite number of hemispheres.

**Notation 5.** Let $B(\alpha_1, \dots, \alpha_n) := \Big\{(z, \zeta) \in \mathcal{H}$:

> The inequality $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 \geqslant 1$ is fulfilled for all unimodular pairs $(\mu,\, \lambda) \in \mathcal{O}^2$ with $\frac{\lambda}{\mu} = \alpha_i + \gamma$, for some $i \in \{1, \dots, n\}$ and some $\gamma \in \mathcal{O}$ $\Big\}$.

Then, using a finite selection of $n$ hemispheres, we have that $B(\alpha_1, \dots, \alpha_n)$ is the set of all points in $\mathcal{H}$ lying above or on all hemispheres $S(\alpha_i + \gamma)$, for $i = 1, \dots, n$, for any $\gamma \in \mathcal{O}$.

**Fact:** The intersection $B(\alpha_1, \dots, \alpha_n) \cap D_\infty$ of $B(\alpha_1, \dots, \alpha_n)$ with the fundamental domain $D_\infty$ for the translation group $\Gamma_\infty$ is our candidate to equal the Bianchi fundamental polyhedron defined in Definition 8.

**Convergence of the approximation.** Our goal is to give a method to decide when $B(\alpha_1, \dots, \alpha_n) = B$. This gives us an effective way to find $B$, which determines the Bianchi fundamental polyhedron, by adding more and more elements to the set $\{\alpha_1, \dots, \alpha_n\}$ until we find $B(\alpha_1, \dots, \alpha_n) = B$.

For this purpose, we consider the boundary $\partial B(\alpha_1, \dots, \alpha_n)$ of $B(\alpha_1, \dots, \alpha_n)$ in $\mathcal{H} \cup \mathbb{C}$. The boundary $\partial B(\alpha_1, \dots, \alpha_n)$

- contains the points $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ which satisfy all the non-strict inequalities $|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 \geqslant 1$ that we have used to define $B(\alpha_1, \dots, \alpha_n)$,
- satisfy the additional condition that at least one of these non-strict inequalities is in fact an equality,
- has a natural cell structure (See below Subsection 2.1.5).

This, together with the following definitions, makes it possible to state the important criterion (see Subsection 2.1.4) which tells us when we have found all the significant hemispheres for the Bianchi fundamental polyhedron.

**Definition 10.** Let $(\mu, \lambda) \in \mathcal{O}^2$ and $(\beta, \alpha) \in \mathcal{O}^2$ be two unimodular pairs with $\mu \neq 0$ and $\beta \neq 0$.

1. We say that the hemisphere $S_{\mu,\lambda}$ is *strictly below* the hemisphere $S_{\beta,\alpha}$ at a point $z \in \mathbb{C}$ if the following inequality is satisfied:
$$\left|z - \frac{\alpha}{\beta}\right|^2 - \frac{1}{|\beta|^2} < \left|z - \frac{\lambda}{\mu}\right|^2 - \frac{1}{|\mu|^2} .$$
2. We say that a point $(z, \zeta) \in \mathcal{H} \cup \mathbb{C}$ is *strictly below* a hemisphere $S_{\beta,\alpha}$, if there is a point $(z, \zeta') \in S_{\beta,\alpha}$ with $\zeta' > \zeta$. This case is also illustrated in the following Figure 2.3.

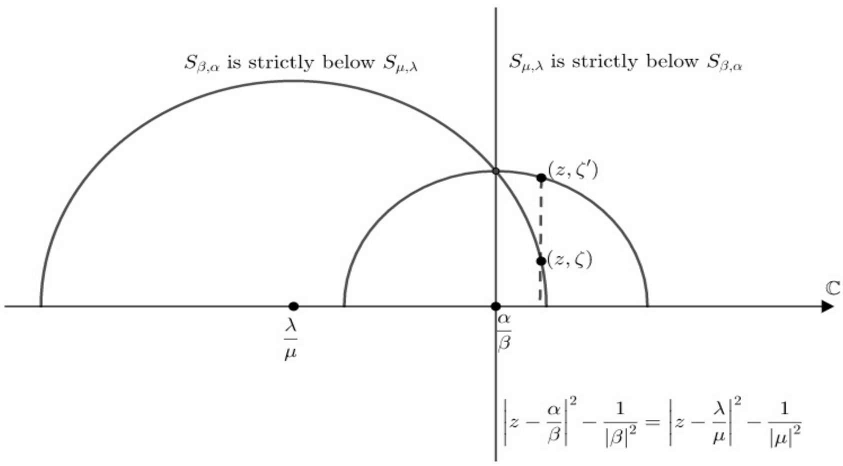



Figure 2.3: Strictly below

**Remark 8.** The first statement of the latter definition is, of course, an abuse of language because there may not be any points on $S_{\beta,\alpha}$ or $S_{\mu,\lambda}$ with coordinate $z \in \mathbb{C}$. But if there is a point $(z, \zeta)$ on $S_{\mu,\lambda}$, another point $(z, \zeta') \in S_{\beta,\alpha}$ and if we assume that $S_{\mu,\lambda}$ is strictly below below $S_{\beta,\alpha}$, then the right hand side of the inequality stated in Definition 10 is just $-\zeta^2$. Thus the left hand side is negative and is of the form $-(\zeta')^2$. Clearly, $(z, \zeta') \in S_{\beta,\alpha}$ and $\zeta' > \zeta$.

Moreover, let us give an additional explanation why the right hand side of the inequality equals to $-\zeta^2$ if there is a point $(z, \zeta)$ on $S_{\mu,\lambda}$:
If $(z, \zeta)$ is a point which lies on $S_{\mu,\lambda}$, then it satisfies the equation

$$|\mu z - \lambda|^2 + |\mu|^2 \zeta^2 = 1 \ ,$$

or equivalently,

$$\left| z - \frac{\lambda}{\mu} \right|^2 + \zeta^2 = \frac{1}{|\mu|^2} \ ,$$

i.e.

$$\left| z - \frac{\lambda}{\mu} \right|^2 - \frac{1}{|\mu|^2} = -\zeta^2 \ .$$

The same reasoning follows for the point $(z, \zeta')$ on $S_{\beta,\alpha}$.

**Observation 1.** We observe that the set of $z \in \mathbb{C}$ over which some hemisphere is strictly below another is either $\mathbb{C}$ or an open half-plane.

### 2.1.3 Singular points

Before we are able to state Swan's termination criterion, we study the notion of cusps and singular points.

**Definition 11.** Let $K$ be a number field. A *cusp* is an element of the number field $K$ considered as a point in the boundary of the 3-dimensional hyperbolic space, via the inclusion $K \subset \mathbb{C} \cup \{\infty\} \cong \partial\mathcal{H}$.

Generally, we represent cusps in the form $\frac{\lambda}{\mu}$, where $\mu, \lambda \in \mathcal{O}$ not both zero. By convention, we write $\infty = \frac{1}{0}$, which is also considered as a cusp. To each representation $\frac{\lambda}{\mu}$, we associate the ideal $(\lambda, \mu) = \lambda\mathcal{O} + \mu\mathcal{O}$ and its ideal class $[(\lambda, \mu)]$.

The set of cusps is closed under the action of $\mathrm{SL}_2(\mathcal{O})$ on $\partial\mathcal{H}$. Moreover, we have the following bijective correspondence between the $\mathrm{SL}_2(\mathcal{O})$-orbits of

cusps and the ideal classes in $\mathcal{O}$:
A cusp $\frac{\lambda}{\mu}$ is in the $\mathrm{SL}_2(\mathcal{O})$-orbit of the cusp $\frac{\lambda'}{\mu'}$, if and only if the ideals $(\lambda', \mu')$ and $(\lambda, \mu)$ are in the same ideal class (See [15], proof of Theorem 1.1).
So we can say that the $\mathrm{SL}_2(\mathcal{O})$-orbit of a cusp $\frac{\lambda}{\mu}$ in $K \cup \{\infty\}$ corresponds to the ideal class $[(\lambda, \mu)]$ of $\mathcal{O}$. From this follows that the orbit of the cusp $\infty = \frac{1}{0}$ corresponds to the principal ideals.

**Lemma 4** (See [3])**.** *Let* $(\lambda, \mu)$, $(\lambda', \mu') \in K \times K$. *Then the following statements are equivalent:*

*(1)* $[(\lambda, \mu)] = [(\lambda', \mu')]$

*(2) there exists* $\gamma \in \mathrm{SL}_2(\mathcal{O})$ *such that* $\gamma \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix}$.

**Theorem 3** (See [11])**.** *Let* $K = \mathbb{Q}(\sqrt{-m})$ *be an imaginary quadratic number field with* $m > 0$. *Then* $\mathcal{O}_K$ *is a principal ideal domain if and only if*

$$m \in \{1,\ 2,\ 3,\ 7,\ 11,\ 19,\ 43,\ 67,\ 163\}\ .$$

**Definition 12.**

- The cusp $\frac{\lambda}{\mu}$, such that the ideal $(\lambda, \mu)$ is not principal, is called *singular*.
- Let us call *singular points* the singular cusps which lie in $\partial B$.

From the characterization of the singular points by Bianchi, it follows that they are precisely the points in $\mathbb{C} \subset \partial\mathcal{H}$ which cannot be strictly below any hemisphere. In the case where $\mathcal{O}$ is a principal ideal domain, $K \cup \{\infty\}$ consists of only one $\mathrm{SL}_2(\mathcal{O})$-orbit. Hence, there are no singular points.

To compute representatives, modulo the translations by $\Gamma_\infty$, of the singular points, we will use the following formula derived by Swan.

**Lemma 5** (See [2])**.** *The singular points of* $K$, $\mod \mathcal{O}$, *are given by* $\frac{p(r+\sqrt{-m})}{s}$, *where* $p, r, s \in \mathbb{Z}$, $s > 0$, $\quad \frac{-s}{2} < r \leqslant \frac{s}{2}$, $\quad s^2 \leqslant r^2 + m$, *and*

- *if* $m \equiv 1$ *or* $2 \mod 4$,
  $s \neq 1$, $s \mid r^2 + m$, *the numbers* $p$ *and* $s$ *are coprime (i.e.* $gcd(p, s) = 1$*), and* $p$ *is taken* $\mod s$*;*

- *if* $m \equiv 3 \mod 4$,
  *s is even,* $s \neq 2$, $2s \mid r^2 + m$, *the numbers* $p$ *and* $\frac{s}{2}$ *are coprime;* $p$ *is taken* $\mod \frac{s}{2}$.

**Example 12.** 1. Let $m = 5 \equiv 1 \mod 4$, then in $\mathcal{O}_{-5} = \mathbb{Z} \oplus \sqrt{-5}\mathbb{Z}$, there exists the ideal $(\sqrt{-5}+1, 2)$ which is not principal.

Indeed, let us start by proving that 2 is an irreducible element of $\mathcal{O}_{-5}$. We will do this by contradiction:
Let us suppose that 2 is reducible; i.e. $2 = a \cdot b$ where $a, b \in \mathcal{O}_{-5}$ and both are no units. Then, by considering the norm $N(x + y\sqrt{-5}) = |x^2 + 5y^2|$ with $x, y \in \mathbb{Z}$, we have $N(2) = 4 = 2 \cdot 2 = N(a) \cdot N(b)$.
But this implies that $N(a) = 2$ as $a$ is not a unit. Thus, since $a \in \mathcal{O}_{-5}$, $a$ can be written as $a = x + y\sqrt{-5}$ with $x, y \in \mathbb{Z}$. So we get $N(a) = |x^2 + 5y^2| = 2$.
But this is impossible for any $x, y \in \mathbb{Z}$. Hence, the assumption is wrong; i.e. 2 is an irreducible element of $\mathcal{O}_{-5}$.

Now let us check that the ideal $(\sqrt{-5}+1, 2)$ is not principal in $\mathcal{O}_{-5}$. We will proceed again by contradiction:
Let us suppose that the ideal $(\sqrt{-5}+1, 2)$ is principal. Thus, the ideal $(\sqrt{-5}+1, 2)$ is of the form $(n)$ where $1 \neq n \in \mathcal{O}_{-5}$. So $2 = (\sqrt{-5}+1) \cdot 0 + 2 \cdot 1 \in (n)$.
But, as $2 \in (n)$, this implies that $2 = n \cdot m$ for some $m \in \mathcal{O}_{-5}$. Since 2 is irreducible in $\mathcal{O}_{-5}$, then we must have that $n = 2$ and $m = 1$. Then we have that $\sqrt{-5}+1 \in (n) = (2)$; i.e. $\sqrt{-5}+1 = 2 \cdot (x + y\sqrt{-5})$, where $x, y \in \mathbb{Z}$. This implies that $x = \frac{1}{2}$.
But this leads us to a contradiction as $x$ was supposed to be in $\mathbb{Z}$. Hence, the assumption is wrong; i.e. the ideal $(\sqrt{-5}+1, 2)$ is not principal in $\mathcal{O}_{-5}$.

This means that the cusp $\frac{\sqrt{-5}+1}{2}$ is singular and corresponds to the singular point $\frac{1+\sqrt{-5}}{2}$ in the fundamental polyhedron.
This can also be proven using Lemma 5: All the conditions are fulfilled for $m \equiv 1 \mod 4$, $s = 2$, and $p = 1 = r$.

2. Let $m = 6 \equiv 2 \mod 4$, then in $\mathcal{O}_{-6} = \mathbb{Z} \oplus \sqrt{-6}\mathbb{Z}$, the ideal $(\sqrt{-6}, 2) = \sqrt{-6}\mathcal{O}_{-6} + 2\mathcal{O}_{-6}$ is not principal.

So let us check that the ideal $(\sqrt{-6}, 2)$ is not principal in $\mathcal{O}_{-6}$. We will do this by contradiction:
Let us suppose that $(\sqrt{-6}, 2)$ is a principal ideal. Thus, the ideal $(\sqrt{-6}, 2)$ is of the form $(n)$, where $n \in \mathcal{O}_{-6}$ and $n \neq 1$ (If $n = 1$, then $(\sqrt{-6}, 2) = \mathcal{O}_{-6}$ which is not possible here). As we can write $2 = \sqrt{-6} \cdot 0 + 2 \cdot 1$ (since $2 \in (\sqrt{-6}, 2)$), we also have that $2 = 2 \cdot 1 \in (n)$.
But this implies that 2 is of the form $2 = n \cdot m$ for some $m \in \mathcal{O}_{-6}$ (i.e. $m = x + y\sqrt{-6}$ with $x, y \in \mathbb{Z}$). Using the same procedure as in Example 12.1, we can prove that 2 is an irreducible element in $\mathcal{O}_{-6}$. Since 2 is an irreducible element in $\mathcal{O}_{-6}$, then we must have that $n = 2$ and $m = 1$. Consequently, $(n) = (2)$. Similarly, if $\sqrt{-6} \in (2)$, then this implies that $\sqrt{-6} = 2 \cdot (x + y\sqrt{-6})$, where $x, y \in \mathbb{Z}$. This implies that $x = 0$ and $y = \frac{1}{2}$. But this leads us to a contradiction as $y$ was supposed to be in $\mathbb{Z}$. Hence, the assumption is wrong; i.e. the ideal $(\sqrt{-6}, 2)$ is not principal in $\mathcal{O}_{-6}$. This means that we have the singular point $\frac{\sqrt{-6}}{2}$.

**Fact:** The singular points need not be considered in Swan's termination criterion, because they cannot be strictly below any hemisphere $S_{\mu,\lambda}$.

### 2.1.4 Swan's termination criterion

Consider Observation 1. In the case, where the set of $z \in \mathbb{C}$ over which some hemisphere is strictly below another is an open half-plane, then the boundary of this half-plane is a line.

**Notation 6.** We denote by $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ the set of $z \in \mathbb{C}$ over which neither $S_{\beta,\alpha}$ is strictly below $S_{\mu,\lambda}$, nor vice versa.

The line $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ is computed by turning the inequality in Definition 10 into an equation. Swan calls it the line over which two hemispheres *agree*. We can see this also on Figure 2.3. Later in this thesis, we will see that the most important edges of the Bianchi fundamental polyhedron lie on the preimages of such lines.

Let us now return to a finite set of hemispheres, modulo the translations in $\Gamma_\infty$.

**Definition 13.** Let $S(\alpha_i+\gamma)$ be a finite set of hemispheres, where $i \in \{1,\ldots,n\}$, and $\gamma \in \mathcal{O}$. We call this set of hemispheres a *collection*, if every non-singular point $z \in \mathbb{C} \subset \partial\mathcal{H}$ (as defined in Section 2.1.3) is strictly below some hemisphere in this set.

**Definition 14.** The intersection point of more than two hemispheres of the collection $S(\alpha_i+\gamma)$ is called *vertex*.

**Example 13.** In Figure 2.2 of Example 11, the brown point illustrates one example of such a vertex.

Now consider a set $B(\alpha_1,\ldots,\alpha_n)$ which is determined by such a collection of hemispheres defined in Definition 13. Finally, we are able to state the following important criterion:

**Theorem 4** (Swan's termination criterion, see [2])**.** *We have $B(\alpha_1,\ldots,\alpha_n) = B$ if and only if no vertex $v$ of $\partial B(\alpha_1,\ldots,\alpha_n)$ can be strictly below any hemisphere $S_{\mu,\lambda}$.*

In other words, no vertex $v$ of the boundary $\partial B(\alpha_1,\ldots,\alpha_n)$ can lie strictly below (or can be covered by) any hemisphere $S_{\mu,\lambda}$; i.e. the value of the height $\zeta$ of the lowest intersection point is greater than the radiuses $\frac{1}{|\mu|}$ of the remaining hemispheres.

This criterion implies that it is enough to compute the cell structure of $\partial B(\alpha_1,\ldots,\alpha_n)$ to see if our choice of hemispheres gives us the Bianchi fundamental polyhedron. This has only to be done modulo the translations of $\Gamma_\infty$, which preserve the height and, thus, the situations of being strictly below. Consequently, the computations only need to be done for a finite number of hemispheres.

### 2.1.5 Computing the cell structure in the complex plane

Let us first recall the meaning of cell structure:

- 2-cells are hyperbolic polygons, lying on associated hemispheres, which are projected on Euclidean polygons in $\mathbb{C}$.
- 1-cells are geodesic hyperbolic line segments, lying on vertical semicircles, which are projected on straight line segments in $\mathbb{C}$.

- 0-cells are nothing else than intersection points.

Now, in a first step, we will compute the image of the cell structure via the homeomorphism

$$\mathrm{pr}: \partial B(\alpha_1, \ldots, \alpha_n) \to \mathbb{C}$$

which is actually the vertical projection. To each 2-cell of this structure, we can associate a hemisphere $S_{\mu,\lambda}$. The interior of this 2-cell consists of the points $z \in \mathbb{C}$ where all other hemispheres in our collection are strictly below $S_{\mu,\lambda}$. Swan shows that the interior of this 2-cell is in fact a convex polygon. Remember that a convex polygon is a polygon for which a line segment between two points in the interior (i.e. one cannot choose one point inside and one point outside the polygon) lies completely within the figure. In Euclidean geometry, all interior angles in a convex polygon are equal to or less than 180°. Simple examples for convex polygons are triangles, rectangles, pentagons; in short, all regular polygons are always convex.

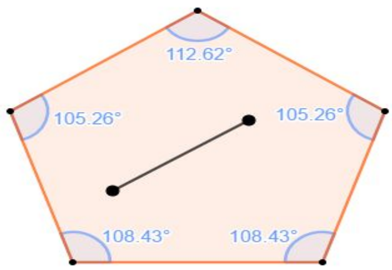



Figure 2.4: Example of a convex polygon

The edges of these polygons lie on real lines in $\mathbb{C}$ which were specified in Notation 6.
Then we actually have that a vertex is an intersection point $z$ of any two of these lines involving the same hemisphere $S_{\mu,\lambda}$, if all other hemispheres in our collection are strictly below, or agree with, $S_{\mu,\lambda}$ at this $z$. See Figure 2.12 to clarify the ideas.

### Lifting the cell structure back to hyperbolic space

Now, using the projection homeomorphism pr onto $\mathbb{C}$, we can lift the cell structure back to $\partial B(\alpha_1, \ldots, \alpha_n)$:

The preimages of the convex polygons of the cell structure on $\mathbb{C}$, are totally geodesic hyperbolic polygons. Each of these hyperbolic polygons lies on one of the hemispheres in our collection. These are the 2-cells of $\partial B(\alpha_1, \dots, \alpha_n)$. The edges of these hyperbolic polygons lie on the intersection arcs between two hemispheres in our collection.

Notice, if two Euclidean 2-spheres (often simply called *sphere*) intersect in a non-trivial way, we obtain a circle. Moreover, the center of this latter circle lies on the straight line connecting the centers of the two Euclidean 2-spheres. This line is called the *connecting line*. The plane determined by this intersection circle is orthogonal to the connecting line. The facts in this paragraph are illustrated in the following figure for the real intersection of two arbitrary Euclidean spheres.

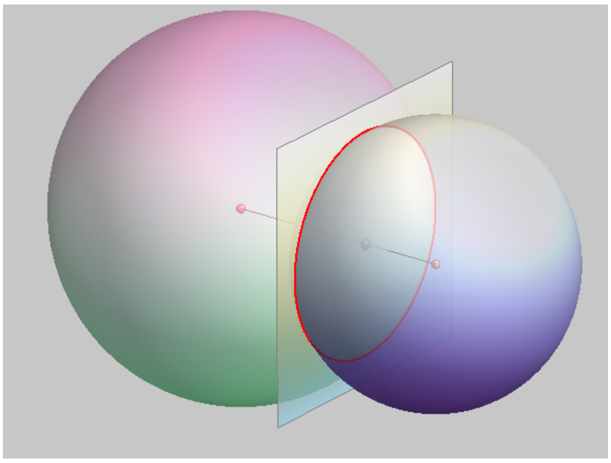


Figure 2.5: Non-trivial intersection of two arbitrary Euclidean spheres (see [17])

In our situation (in the case of a non-trivial intersection of two hemispheres in our collection), we have that such an intersection arc (on which lie the edges of the hyperbolic polygons) lies on a semicircle centered in the complex plane. The plane which contains this semicircle is orthogonal to the connecting line, hence we have a vertical plane in $\mathcal{H}$. Alternatively, we can also conclude the latter facts when we consider the observation that an edge which two totally geodesic polygons have in common must be a geodesic segment.
Now, it becomes clear from the definition of the vertices, what it means to lift them. This allows us to check Swan's termination criterion.

We will now give a sketch of the proof of Swan's criterion stated in Theorem 4. The full detailed proof can be found in [2], Proposition 8.4.

*Proof.* Let $P$ be one of the convex polygons of the cell structure on $\mathbb{C}$. The preimage of $P$ lies on one hemisphere $S(\alpha_i)$ of our collection, for some $i \in \{1, \ldots, n\}$. Now, the points where $S(\alpha_i)$ can be strictly below some other hemisphere in our collection constitute an open half-plane in $\mathbb{C}$ (see Observation 1), and hence cannot lie in the convex hull of the vertices of $P$, which is actually $P$. Thus, the condition stated in Theorem 4, which says that at the vertices of $P$, the hemisphere $S(\alpha_i)$ cannot be strictly below any other hemisphere, is fulfilled. Finally, as $\mathbb{C}$ is tessellated by these convex polygons, the statement of Theorem 4 follows immediately. $\square$

## 2.2 Realization of Swan's algorithm

The aim of this section is to put Swan's concept into practice.

First, we reduce the set of hemispheres on which we realize our computations, using the following definition.

**Definition 15.** Let $(\mu, \lambda) \in \mathcal{O}^2$ and $(\beta, \alpha) \in \mathcal{O}^2$ be two unimodular pairs with $\mu \neq 0$ and $\beta \neq 0$. Consider a hemisphere $S_{\mu,\lambda} \subset \mathcal{H}$ with center $\frac{\lambda}{\mu}$ on $\mathbb{C}$ and of radius $\frac{1}{|\mu|}$ and a hemisphere $S_{\beta,\alpha} \subset \mathcal{H}$ with center $\frac{\alpha}{\beta}$ on $\mathbb{C}$ and of radius $\frac{1}{|\beta|}$. Then $S_{\mu,\lambda}$ is said to be *everywhere below* $S_{\beta,\alpha}$ when:

$$\left|\frac{\lambda}{\mu} - \frac{\alpha}{\beta}\right| \leqslant \frac{1}{|\beta|} - \frac{1}{|\mu|} \ ,$$

i.e. the difference of the centers is smaller than or equal to the difference of the radiuses.

**Note** that this is also the case when $S_{\mu,\lambda} = S_{\beta,\alpha}$.

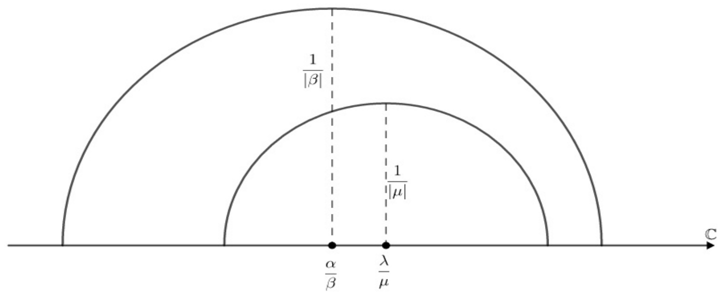



Figure 2.6: everywhere below

**Example 14.** Let us consider again Example 11. Then first of all, we see on Figure 2.2 that there is only a finite number of hemispheres represented, and we can see that the blue hemispheres are not everywhere below the violet ones.

Any hemisphere which is everywhere below another one, does not contribute to the Bianchi fundamental polyhedron, in the following sense:

**Proposition 3** (See [1]). *Consider a finite selection of $n$ hemispheres. Let $S(\alpha_n)$ be a hemisphere everywhere below some other hemisphere $S(\alpha_i)$, where $i \in \{1, \ldots, n-1\}$.*
*Then $B(\alpha_1, \ldots, \alpha_n) = B(\alpha_1, \ldots, \alpha_{n-1})$.*

*Proof.* Write $\alpha_n = \frac{\lambda}{\mu}$ and $\alpha_i = \frac{\theta}{\tau}$ with $\lambda, \mu, \theta, \tau \in \mathcal{O}$. We take any point $(z, \zeta)$ strictly below $S_{\mu,\lambda}$ with center $\frac{\lambda}{\mu}$ on $\mathbb{C}$ and we want to show that it is also strictly below $S_{\tau,\theta}$ with center $\frac{\theta}{\tau}$ on $\mathbb{C}$. In terms of Notation 5, this problem looks as follows:
We assume that the inequality $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 < 1$ is satisfied, and we have to show that this implies the inequality $|\tau z - \theta|^2 + |\tau|^2\zeta^2 < 1$.
The first inequality is equivalent to $\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2 < \frac{1}{|\mu|^2}$. Hence, $\sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2} < \frac{1}{|\mu|}$. We will insert this into the triangle inequality for the Euclidean distance $d$ in $\mathbb{C} \times \mathbb{R}$ applied to the three points $A = (z, \zeta)$, $B = (\frac{\lambda}{\mu}, 0)$ and $C = (\frac{\theta}{\tau}, 0)$, which is

$$d(A, C) \ < \ d(B, C) + d(A, B) \ .$$

Thus,

$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + |\zeta - 0|^2} \ < \ \sqrt{\left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right|^2 + |0 - 0|^2} + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + |\zeta - 0|^2} \ ,$$

i.e.

$$\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} \ < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \sqrt{\left|z - \frac{\lambda}{\mu}\right|^2 + \zeta^2} \ .$$

So, by assumption, we obtain $\sqrt{\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2} < \left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|}$. Using Definition 15, then the expression on the right hand side becomes smaller than or equal to $\frac{1}{|\tau|}$. Indeed,

$$\left|\frac{\lambda}{\mu} - \frac{\theta}{\tau}\right| + \frac{1}{|\mu|} \ \leqslant \ \frac{1}{|\tau|} - \frac{1}{|\mu|} + \frac{1}{|\mu|} \ = \ \frac{1}{|\tau|} \ .$$

Therefore, we take the square and obtain

$$\left|z - \frac{\theta}{\tau}\right|^2 + \zeta^2 < \frac{1}{|\tau|^2} \ ,$$

which is equivalent to the claimed inequality. □

The following definition is another notion that will be useful for our algorithm.

**Definition 16.** Let $z \in \mathbb{C}$ be a point lying within the vertical projection of $S_{\mu,\lambda}$. We define the *lift on the hemisphere* $S_{\mu,\lambda}$ of $z$ as the point $(z, \zeta)$ on $S_{\mu,\lambda}$ the vertical projection of which is $z$.

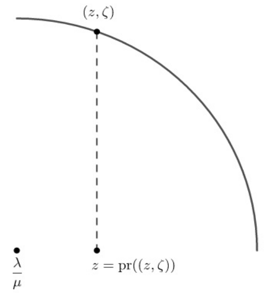



Figure 2.7: Lift on the hemisphere $S_{\lambda,\mu}$ of $z$

**Notation 7.** Let us denote by the *hemisphere list* a list into which we will record a finite number of hemispheres $S(\alpha_1)$,...,$S(\alpha_n)$. Its purpose is to determine a set $B(\alpha_1,\dots,\alpha_n)$ in order to approximate, and finally obtain, the Bianchi fundamental polyhedron.

### 2.2.1 The algorithm computing the Bianchi fundamental polyhedron (See [1])

Now, using Swan's description, we state the algorithm to compute the Bianchi fundamental polyhedron.

**Initial step.** We start with the smallest value which the norm of a non-zero element $\mu \in \mathcal{O}$ can take, namely 1. Then $\mu$ is a unit in $\mathcal{O}$, and for any $\lambda \in \mathcal{O}$, the pair $(\mu,\lambda)$ is unimodular. And we can rewrite the fraction $\frac{\lambda}{\mu} = \lambda$ as $\mu = 1$. We obtain the unit hemispheres of radius $\frac{1}{|\mu|} = 1$, centered at the imaginary quadratic integers $\lambda \in \mathcal{O}$. We record into the hemisphere list the hemispheres which touch the Bianchi fundamental polyhedron, i.e. the ones whose centers lie in the fundamental rectangle $D_0$ (determined in Notation 4) for the action of $\Gamma_\infty$ on the complex plane $\mathbb{C}$.

**Step A.** Now we increase $|\mu|$ to the next higher value which the norm takes on elements of $\mathcal{O}$ and run through all the finitely many $\mu$ which have the same norm. For each of these $\mu$, we have to run through all the finitely many $\lambda \in \mathcal{O}$ with $\frac{\lambda}{\mu}$ in the fundamental rectangle $D_0$. Moreover, we have to check that $(\mu,\lambda)$ are unimodular pairs and, that the hemisphere $S_{\mu,\lambda}$ is not everywhere below another hemisphere $S_{\beta,\alpha}$ in the hemisphere list. If these two checks are passed, we record $S_{\mu,\lambda}$ into the hemisphere list.

We repeat **step A** until $|\mu|$ has reached an expected value. Then we check if we have found all the hemispheres touching the Bianchi fundamental polyhedron, as follows:

**Step B.** We compute the lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ defined in Notation 6, over which two hemispheres agree, for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another. We add the edges of the fundamental rectangle $D_0$ to these lines.
Then, for each hemisphere $S_{\beta,\alpha}$, we compute the intersection points of each two lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ and $L(\frac{\alpha}{\beta}, \frac{\theta}{\tau})$ referring to $\frac{\alpha}{\beta}$ (i.e. we compute the intersection points of those two lines involving the same hemisphere $S_{\beta,\alpha}$).
We drop the intersection points at which $S_{\beta,\alpha}$ is strictly below some hemisphere in the list.
We erase the hemispheres from our list, for which less than three intersection points remain. We can do this because a hemisphere which touches the Bianchi fundamental polyhedron only in two vertices shares only an edge with it but no 2-cell.
Now, the vertices of $B(\alpha_1, \ldots, \alpha_n) \cap D_\infty$ are the lifts of the remaining intersection points. Thus we can check Swan's termination criterion (see Theorem 4) as follows: We pick the lowest value $\zeta > 0$ for which $(z, \zeta) \in \mathcal{H}$ is the lift inside the 3-dimensional Hyperbolic Space of a remaining intersection point $z$.
If $\zeta \geqslant \frac{1}{|\mu|}$, then all (infinitely many) remaining hemispheres have radius equal or smaller than $\zeta$, i.e. $(z, \zeta)$ cannot be strictly below them. So Swan's termination criterion is fulfilled, we have found the Bianchi fundamental polyhedron, and now we can proceed by determining its cell structure.
Else (i.e. if $\zeta < \frac{1}{|\mu|}$), then $\zeta$ becomes the new expected value for the radius $\frac{1}{|\mu|}$.
We repeat **step A** until $|\mu|$ reaches $\frac{1}{\zeta}$ and then proceed again with **step B**.

### 2.2.2 Computation of the Bianchi fundamental polyhedron for $m = 2$

Let us apply the algorithm for $m = 2$. So we consider $K = \mathbb{Q}(\sqrt{-2})$ of discriminant $d_K = -4 \cdot 2 = -8$ (as $-2 \equiv 2 \mod 4$, see Note 1), and its ring of integers $\mathcal{O}_{-2} = \mathbb{Z} \oplus \sqrt{-2}\mathbb{Z}$.

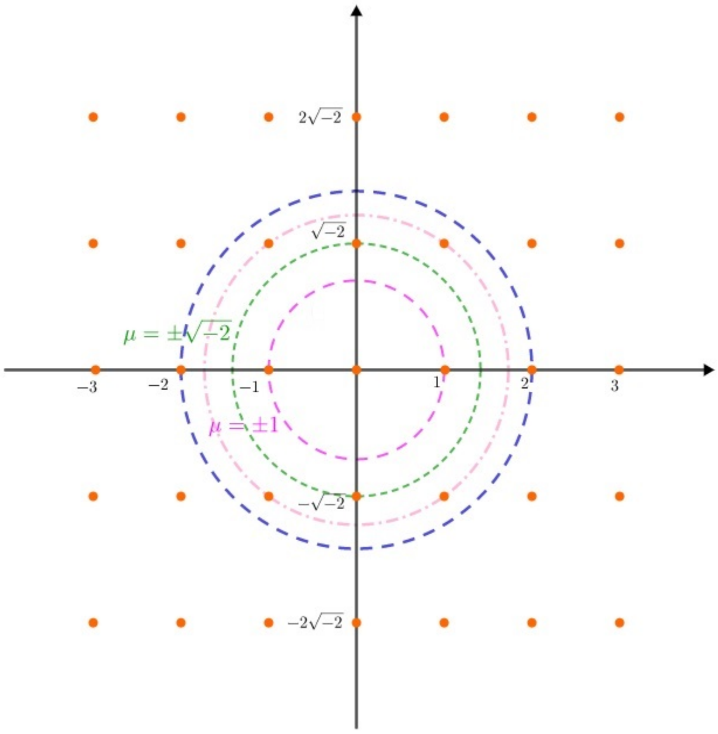



Figure 2.8: $\mathcal{O}_{-2}$

Note that, in terms of Theorem 3, $\mathcal{O}_{-2} \subset \mathbb{Q}(\sqrt{-2})$ is a principal ideal domain. Thus, there are no singular points.

To choose $\mu \in \mathcal{O}$ in a way that the value for $|\mu|$ stays minimal, we have to go outwards in concentric circles (i.e. they have all the same center, but different radiuses). These concentric circles are represented in Figure 2.8.
Then we have to choose $\lambda \in \mathcal{O}$ such that

i) $(\mu, \lambda) \in \mathcal{O}^2$ is a unimodular pair,

ii) $\frac{\lambda}{\mu} \in D_0$, where we denote by $D_0$ the fundamental rectangle.

Here, using Notation 4, the fundamental rectangle $D_0$ for $m = 2 \equiv 2 \mod 4$ is given by

$$D_0 = \{x + y\sqrt{-2} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}\ .$$

In Figure 2.12 below, the fundamental rectangle is represented by the blue rectangle.

Let us start with $\mu = 1 \in \mathcal{O}$, then $|\mu| = 1$. So the only values for $\lambda \in \mathcal{O}$ which satisfy both conditions are $0$, $1$, $\sqrt{-2}$ and $1+\sqrt{-2}$. Indeed, if we choose $(\alpha, \beta) = (0,1) \in \mathcal{O}^2$, then, in terms of Definition 6, we have that the pairs $(1,0)$, $(1,\sqrt{-2})$, $(1,1)$ and $(1, 1+\sqrt{-2})$ are unimodular.
Moreover, as $\mu = 1$, we have that $\frac{\lambda}{\mu} = \lambda$ and $0$, $1$, $\sqrt{-2}$, $1+\sqrt{-2} \in D_0$.
So we obtained the four biggest hemispheres of radius $\frac{1}{|\mu|} = \frac{1}{1} = 1$, centered at the four values of $\lambda \in \mathcal{O}$. Hence, those four hemispheres are recorded in our hemisphere list.
In the following picture, we have a 3-dimensional representation of these four hemispheres.

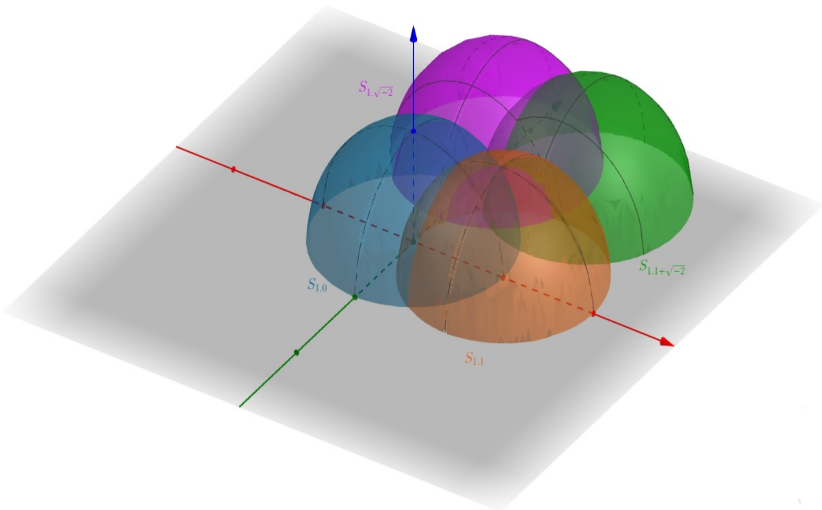



Figure 2.9: $S_{1,0}$, $S_{1,1}$, $S_{1,\sqrt{-2}}$ and $S_{1,1+\sqrt{-2}}$

(Note that this figure is not reduced to the fundamental rectangle $D_0$.)

Now we have to "cut" the hemispheres in semicircles which are vertical to the boundary. Then via the vertical projection to $\mathbb{C}$, they are projected onto a line segment (see Figure 2.12, where those vertical projections are represented by the green line segments).

Now, in terms of Notation 6, let us compute the lines $L(\frac{\alpha}{\beta}, \frac{\lambda}{\mu})$ over which two hemispheres agree, for all pairs $S_{\beta,\alpha}$, $S_{\mu,\lambda}$ in the hemisphere list which touch one another.

1. For $L(\frac{0}{1}, \frac{\sqrt{-2}}{1})$, we have
$$\left|z - \frac{0}{1}\right|^2 - \frac{1}{|\mu|^2} = \left|z - \frac{\sqrt{-2}}{1}\right|^2 - \frac{1}{|\mu|^2} \ ,$$
i.e.
$$|z|^2 = |z - \sqrt{-2}|^2 \ \Leftrightarrow \ |z|^2 = |z - i\sqrt{2}|^2 \ .$$
Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, we get
$$|x + iy|^2 = |x + i(y - \sqrt{2})|^2 \ .$$
Now using the modulus of a complex number $|z| = \sqrt{x^2 + y^2}$, we obtain
$$x^2 + y^2 = x^2 + (y - \sqrt{2})^2 \ \Leftrightarrow \ x^2 + y^2 = x^2 + y^2 - 2y\sqrt{2} + 2 \ ,$$
i.e.
$$0 = -2y\sqrt{2} + 2 \ .$$
Hence, we obtain for $L(\frac{0}{1}, \frac{\sqrt{-2}}{1})$ the equation
$$y = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \ .$$
We obtain the same equation for the line $L(\frac{1}{1}, \frac{1+\sqrt{-2}}{1})$.

2. By a similar procedure, we obtain for the lines $L(\frac{0}{1}, \frac{1}{1})$ and $L(\frac{\sqrt{-2}}{1}, \frac{1+\sqrt{-2}}{1})$ the following equation
$$x = \frac{1}{2} \ .$$

3. For $L(\frac{0}{1}, \frac{1+\sqrt{-2}}{1})$, we have
$$|z - 0|^2 - 1 = \left|z - 1 - \sqrt{-2}\right|^2 - 1 \ , \ \text{i.e. } |z|^2 = |z - 1 - \sqrt{-2}|^2 \ .$$
Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$ and using the modulus, we get
$$|x + iy|^2 = |(x - 1) + i(y - \sqrt{2})|^2 \ \Leftrightarrow \ x^2 + y^2 = x^2 - 2x + 1 + y^2 - 2\sqrt{2}y + 2 \ .$$
Thus, we have
$$2y\sqrt{2} = -2x + 3 \ .$$
Hence, we obtain for $L(\frac{0}{1}, \frac{1+\sqrt{-2}}{1})$ the equation
$$y = -\frac{\sqrt{2}}{2}x + \frac{3\sqrt{2}}{4} \ .$$

4. Following the same instructions as above, we obtain for the line $L(\frac{1}{1}, \frac{\sqrt{-2}}{1})$ the equation
$$y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{4} \ .$$

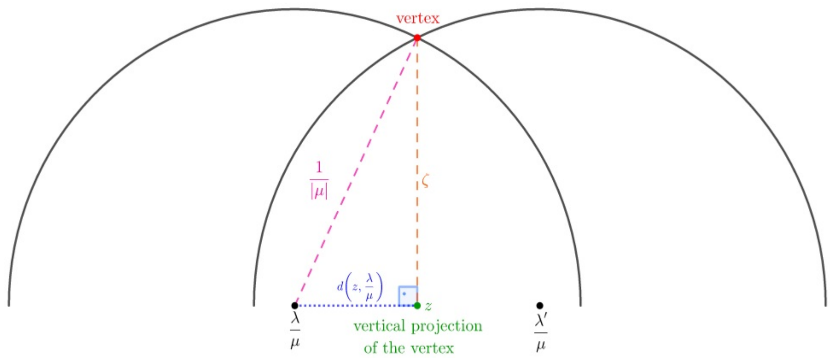
As we can see in Figure 2.11, the four lines only intersect in one point. To compute this intersection point, we need to consider the following sytem of equations

$$\begin{cases} x &= \frac{1}{2} \\ y &= \frac{\sqrt{2}}{2} \\ y &= -\frac{\sqrt{2}}{2}x + \frac{3\sqrt{2}}{4} \\ y &= \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{4} \end{cases}$$

Then the intersection point of these lines has obviously for $z$-coordinate

$$z = \frac{1}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{2} + \frac{1}{2}\sqrt{-2} \in D_0 \ .$$

It remains to check Swan's termination criterion (see Theorem 4). For this, we determine the height $\zeta$ of the vertex $(\frac{1}{2} + \frac{1}{2}\sqrt{-2}, \zeta)$ via the Pythagorean theorem. So consider the following picture (which represents the situation in a vertical cut) to clarify the ideas:




Consider $z = \frac{1}{2} + \frac{\sqrt{-2}}{2}$ as the point $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$ in the Euclidean plane $\mathbb{R} \times \mathbb{R}$, and the center $\frac{\lambda}{\mu} = 0$ of the hemisphere $S_{1,0}$ as the point $(0,0)$ (Note that we can also use the center of one of the three remaining hemispheres). Then the distance (represented as the blue dotted line in the picture above) between these two points is

$$d\left((0,0); \left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)\right) = \sqrt{\left(0 - \frac{1}{2}\right)^2 + \left(0 - \frac{\sqrt{2}}{2}\right)^2} = \frac{\sqrt{3}}{2} \ .$$

Using the Pythagorean theorem, we get:

$$\left(\frac{\sqrt{3}}{2}\right)^2 + \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 \Leftrightarrow \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 ,$$

i.e. for $\mu = 1$

$$\zeta^2 = 1 - \frac{3}{4} = \frac{1}{4} .$$

As $\zeta$ is the height of a point $(z, \zeta) \in \mathcal{H}$, this implies that $\zeta > 0$. Hence, we finally have that $\zeta = \frac{1}{2}$.

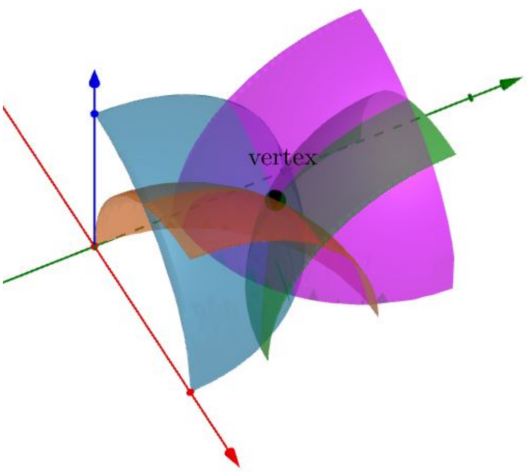



Figure 2.10: Fundamental domain for $m = 2$

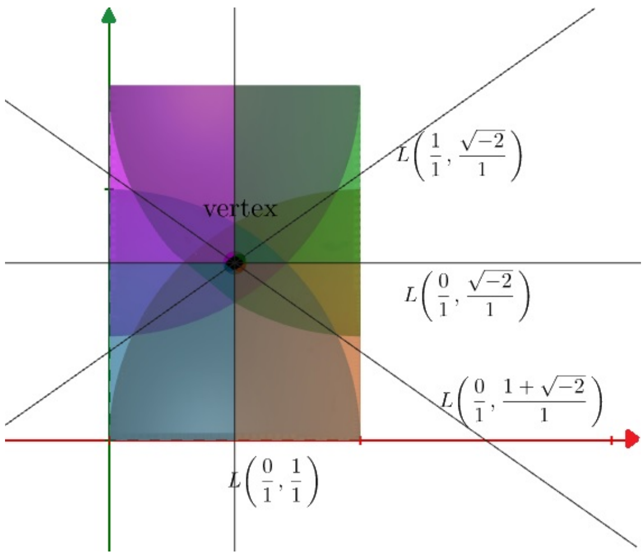



Figure 2.11: View from above

The next value for $\mu$ would be $\pm\sqrt{-2}$, thus the radius $\frac{1}{|\mu|}$ would be equal to $\frac{1}{2}$. But since $\zeta \geqslant \frac{1}{2}$, we have that $(z, \zeta) = \left(\frac{1}{2} + \frac{\sqrt{-2}}{2}, \frac{1}{2}\right) \in \mathcal{H}$ cannot be strictly below the remaining hemispheres which have radius equal or smaller than $\zeta$. In other words, the highest point of any remaining hemisphere cannot lie higher than $(z, \zeta)$, and hence these hemispheres cannot contribute to the structure of $B$. Thus, Theorem 4 is fulfilled, and we have computed the Bianchi fundamental polyhedron.

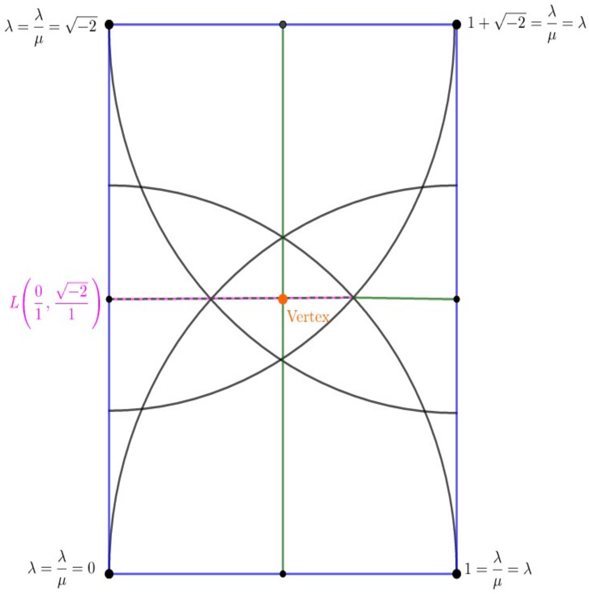



Figure 2.12: Bianchi fundamental polyhedron for $m = 2$

In the following picture, the cell structure of the Bianchi fundamental polyhedron is illustrated. This cell structure consists of four 2-cells of the boundary which are in fact rectangles here:

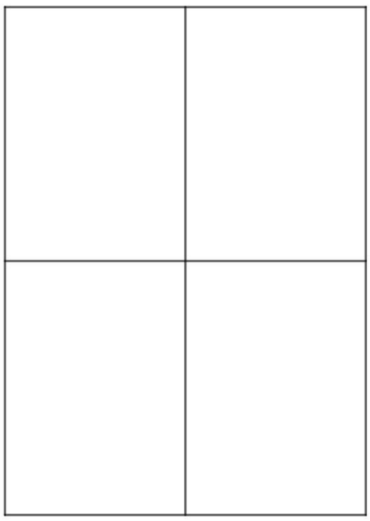


Figure 2.13: Cell structure for the Bianchi fundamental polyhedron for $m = 2$

**Remark 9.** Note that we also have to add the edges of the fundamental rectangle $D_0$ to the agreeing lines. For the intersection point between $L(\frac{0}{1}, \frac{1}{1})$ and $\{y = 0\}$, we obtain as $z$-coordinate

$$z = \frac{1}{2} + 0i = \frac{1}{2} \in D_0 .$$

It remains to determine the height $\zeta$ of $(\frac{1}{2}, \zeta)$. So the distance between the points $(\frac{1}{2}, 0)$ and $(0,0)$ is

$$d\left((0,0), \left(\frac{1}{2}, 0\right)\right) = \frac{1}{2} .$$

So, via the Pythagorean theorem, we obtain

$$\zeta^2 = 1^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} .$$

This implies that $\zeta = \frac{\sqrt{3}}{2}$. But, as $\frac{\sqrt{3}}{2} > \frac{1}{2}$, we have that the height of the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is not minimal. Hence, it is not considered as a relevant vertex on which to check Swan's termination criterion.
We draw the same conclusion for the intersection points of $L(\frac{\sqrt{-2}}{1}, \frac{1+\sqrt{-2}}{1})$ and $\{y = \sqrt{2}\}$; of $L(\frac{0}{1}, \frac{\sqrt{-2}}{1})$ and $\{x = 0\}$; of $L(\frac{1}{1}, \frac{1+\sqrt{-2}}{1})$ and $\{x = 1\}$; of $L(\frac{0}{1}, \frac{1\sqrt{-2}}{1})$, $\{x = 0\}$ and $\{x = 1\}$; and of $L(\frac{1}{1}, \frac{\sqrt{-2}}{1})$, $\{x = 0\}$ and $\{x = 1\}$.

### 2.2.3 Computation of the Bianchi fundamental polyhedron for $m = 5$

Let us apply the algorithm for $m = 5$. Thus, we consider $K = \mathbb{Q}(\sqrt{-5})$ of discriminant $d_K = -4 \cdot 5 = -20$ (as $-5 \equiv 3 \mod 4$, see Note 1), and its ring of integers $\mathcal{O} = \mathbb{Z} \oplus \sqrt{-5}\mathbb{Z}$.

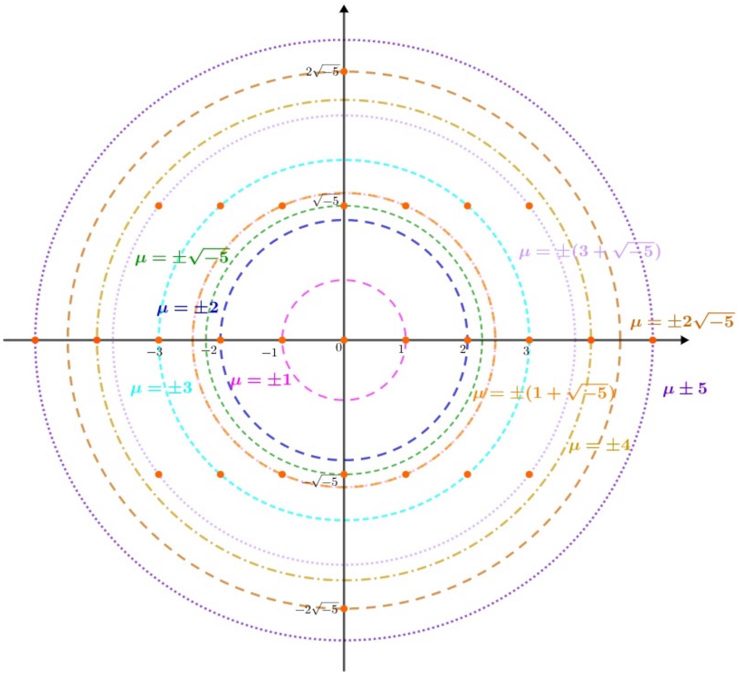



Figure 2.14: $\mathcal{O}_{-5}$

Using Notation 4, the fundamental rectangle $D_0$ for $m = 5 \equiv 1 \mod 4$ is given by

$$D_0 = \{x + y\sqrt{-5} \in \mathbb{C} \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1\}\ .$$

In Figure 2.25 below, the fundamental rectangle is represented by the blue rectangle.

Let us start with $\mu = 1 \in \mathcal{O}$, then $|\mu| = 1$. The only values for $\lambda \in \mathcal{O}$, such that

- $(\mu, \lambda) \in \mathcal{O}^2$ is a unimodular pair, and
- the center $\frac{\lambda}{\mu}$ lies in the fundamental rectangle $D_0$ for the action of $\Gamma_\infty$ on the complex plane $\mathbb{C}$,

are $0,\ 1,\ \sqrt{-5},\ 1 + \sqrt{-5}$.
Indeed, by choosing $(\alpha, \beta) = (0, 1) \in \mathcal{O}^2$, then, in terms of Definition 6, we have that the pairs $(1, 0)$, $(1, \sqrt{-5})$, $(1, 1)$ and $(1, 1 + \sqrt{-5})$ are unimodular. Furthermore, as $\mu = 1$, we have that $\frac{\lambda}{\mu} = \lambda$ and $0,\ 1,\ \sqrt{-5},\ 1 + \sqrt{-5}\ \in D_0$.
So we obtained the four biggest hemispheres of radius $\frac{1}{|\mu|} = \frac{1}{1} = 1$, centered

at the values of $\lambda \in \mathcal{O}$. Hence, those four hemispheres are recorded in our hemisphere list.
In the following picture, we see a 3-dimensional representation of these four hemispheres.

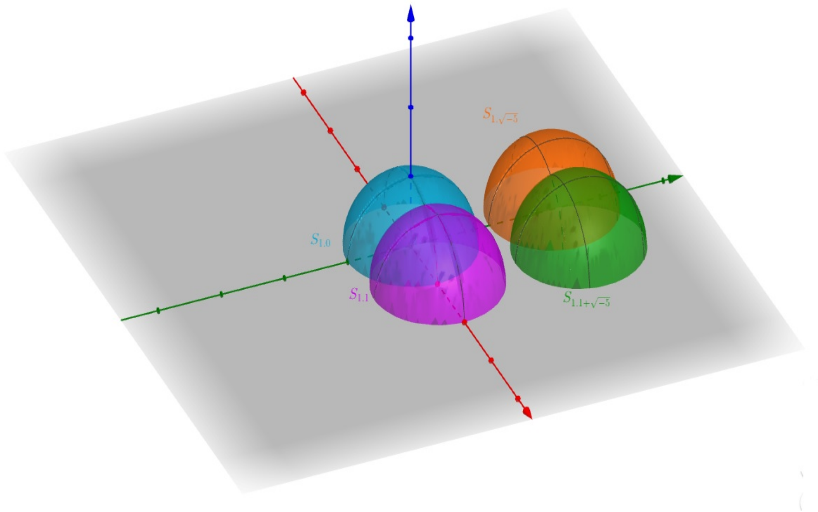



Figure 2.15: $S_{1,0}$, $S_{1,1}$, $S_{1,\sqrt{-5}}$ and $S_{1,1+\sqrt{-5}}$

(Note that this figure is not reduced to the fundamental rectangle $D_0$.)

As the radius of these four hemispheres is equal to 1, we have that $S_{1,0}$ cannot touch $S_{1,\sqrt{-5}}$ and $S_{1,1+\sqrt{-5}}$; it only touches the hemisphere $S_{1,1}$. The same, $S_{1,\sqrt{-5}}$ only touches the hemisphere $S_{1,1+\sqrt{-5}}$. So we don't need to compute the lines $L(\frac{0}{1}, \frac{\sqrt{-5}}{1})$, $L(\frac{0}{1}, \frac{1+\sqrt{-5}}{1})$, $L(\frac{1}{1}, \frac{\sqrt{-5}}{1})$ and $L(\frac{1}{1}, \frac{1+\sqrt{-5}}{1})$.
We can also see this in the picture above.

Thus, let us compute the line $L(\frac{0}{1}, \frac{1}{1})$ in terms of Notation 6.

$$\left|z - \frac{0}{1}\right|^2 - 1 = \left|z - \frac{1}{1}\right|^2 - 1 ,$$

i.e.

$$|z|^2 = |z-1|^2 .$$

Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, and using the modulus of a complex number $|z| = \sqrt{x^2 + y^2}$, we get

$$|x + iy|^2 = |(x-1) + iy|^2 \iff x^2 + y^2 = x^2 - 2x + 1 + y^2 .$$

Hence, we obtain for $L(\frac{0}{1}, \frac{1}{1})$ the equation

$$x = \frac{1}{2} \ .$$

We obtain the same equation for $L(\frac{\sqrt{-5}}{1}, \frac{1+\sqrt{-5}}{1})$.
Consequently, there does not exist a relevant intersection point.

It remains to add the edges of the fundamental rectangle $D_0$ to the agreeing lines. For the intersection point between $L(\frac{0}{1}, \frac{1}{1})$ and $\{y = 0\}$, we obviously obtain as $z$-coordinate

$$z = \frac{1}{2} + 0i = \frac{1}{2} \ \in D_0 \ .$$

It remains to determine the height $\zeta$ of the lift $(\frac{1}{2}, \zeta)$ of this intersection point via the Pythagorean theorem. For this, we consider $z = \frac{1}{2}$ as the point $\left(\frac{1}{2}, 0\right)$ in the Euclidean plane $\mathbb{R} \times \mathbb{R}$, and the center $\frac{\lambda}{\mu} = 0$ of the hemisphere $S_{1,0}$ as the point $(0,0)$ (Note that we can also use the center 1 of the hemisphere $S_{1,1}$). Then the distance between the points $(\frac{1}{2}, 0)$ and $(0,0)$ is

$$d\left((0,0), \left(\frac{1}{2}, 0\right)\right) = \sqrt{\left(0 - \frac{1}{2}\right)^2 + (0-0)^2} = \frac{1}{2} \ .$$

So, via the Pythagorean theorem, we obtain

$$\left(\frac{1}{2}\right)^2 + \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 \ \Leftrightarrow \ \zeta^2 = 1^2 - \left(\frac{1}{2}\right)^2 \ ,$$

i.e. $\zeta^2 = \frac{3}{4}$. As $\zeta$ is the height of the point $(z, \zeta) \in \mathcal{H}$, this implies that $\zeta > 0$. Hence, we have that $\zeta = \frac{\sqrt{3}}{2}$. Later in this computation, we will find out that the height of the point $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is not minimal. Hence, it is not considered as a relevant vertex to check Swan's termination criterion on it.
We draw the same conclusion for the intersection point between $L(\frac{\sqrt{-5}}{1}, \frac{1+\sqrt{-5}}{1})$ and $\{y = \sqrt{5}\}$.

By considering Figure 2.14, we increase $\mu$ to the next value $\pm 2 \in \mathcal{O}$; i.e. $|\mu| = \sqrt{0^2 + (\pm 2)^2} = 2$. Let us take $\mu = 2$. Then the only values for $\lambda \in \mathcal{O}$ which satisfy all the conditions, described in **Step A** in the algorithm computing the Bianchi fundamental polyhedron (see Subsection 2.2.1), are $\lambda = \sqrt{-5}$ and $\lambda = 2 + \sqrt{-5}$. Indeed,

◇ 1. for $\lambda = \sqrt{-5}$, we have $\frac{\lambda}{\mu} = \frac{1}{2}\sqrt{-5} \in D_0$ ; and

2. for $\lambda = 2 + \sqrt{-5}$, we have $\frac{\lambda}{\mu} = 1 + \frac{1}{2}\sqrt{-5} \in D_0$.

Moreover, note that, in view of Lemma 5, that both centers $\frac{1}{2}\sqrt{-5}$ and $1 + \frac{1}{2}\sqrt{-5}$ are no singular points:

1. For $\frac{1}{2}\sqrt{-5} = \frac{\sqrt{-5}}{2}$, we have that the condition $s \,|\, r^2 + m$ does not hold for $s = 2$, $r = 0$ and $m = 5$.
2. For $1 + \frac{1}{2}\sqrt{-5} = \frac{2+\sqrt{-5}}{2}$, we have that the condition $\frac{-s}{2} < r \leqslant \frac{s}{2}$ does not hold for $s = 2 = r$.

◇ In terms of Definition 6, we have

1. by choosing $(\alpha, \beta) = (\sqrt{-5}, 3) \in \mathcal{O}^2$, that the pair $(\mu, \lambda) = (2, \sqrt{-5}) \in \mathcal{O}^2$ is unimodular as $\sqrt{-5} \cdot \sqrt{-5} + 2 \cdot 3 = 1 \in 2\mathcal{O} + \sqrt{-5}\mathcal{O}$; and
2. by choosing $(\alpha, \beta) = (2 - \sqrt{-5}, -4) \in \mathcal{O}^2$, that the pair $(\mu, \lambda) = (2, 2 + \sqrt{-5}) \in \mathcal{O}^2$ is unimodular as $(2 + \sqrt{-5}) \cdot (2 - \sqrt{-5}) + 2 \cdot (-4) = 1 \in 2\mathcal{O} + (2 + \sqrt{-5})\mathcal{O}$.

◇ In terms of Definition 15, the hemispheres $S_{2,\sqrt{-5}}$ and $S_{2,2+\sqrt{-5}}$ are not everywhere below a hemisphere of radius 1 in the hemisphere list. For example:

– $S_{2,\sqrt{-5}}$ is not everywhere below $S_{1,0}$:

$$\begin{aligned} \left|\frac{1}{2}\sqrt{-5} - \frac{0}{1}\right| = \left|\frac{i}{2}\sqrt{5}\right| &= \sqrt{\left(\frac{1}{2}\sqrt{5}\right)^2} \\ &= \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2} \cong 1,12 \\ &\nleq \frac{1}{1} - \frac{1}{2} = \frac{1}{2} = 0,5 \ . \end{aligned}$$

– $S_{2,2+\sqrt{-5}}$ is not everywhere below $S_{1,1}$:

$$\begin{aligned} \left|1 + \frac{1}{2}\sqrt{-5} - \frac{1}{1}\right| = \left|\frac{i}{2}\sqrt{5}\right| &= \frac{\sqrt{5}}{2} \cong 1,12 \\ &\nleq 1 - \frac{1}{2} = 0,5 \ . \end{aligned}$$

Doing the same computation for the remaining hemispheres in the list, we can draw the same conclusion.

Hence, the hemispheres $S_{2,\sqrt{-5}}$ and $S_{2,2+\sqrt{-5}}$ are recorded into the hemisphere list.
In the following picture, we see a 3-dimensional representation of the six hemispheres in list.

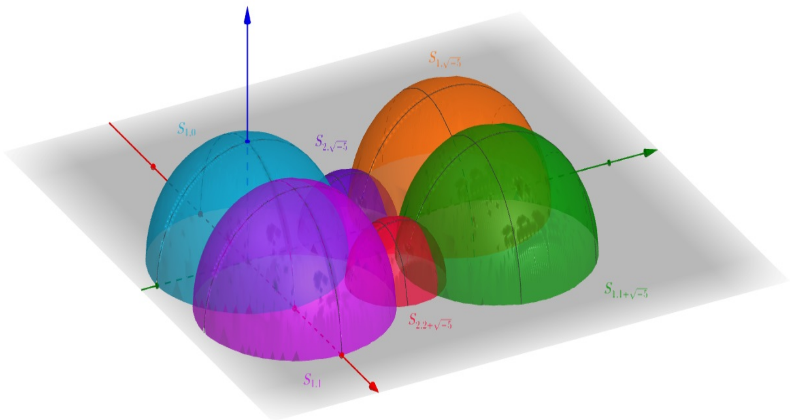


Figure 2.16: $S_{1,0}$, $S_{1,1}$, $S_{1,\sqrt{-5}}$, $S_{1,1+\sqrt{-5}}$, $S_{2,\sqrt{-5}}$ and $S_{2,2+\sqrt{-5}}$

(Note that this figure is not reduced to the fundamental rectangle $D_0$.)

Now, using Notation 6, let us compute the agreeing lines over which two hemispheres in the list touch one another.

1. For $L(\frac{\sqrt{-5}}{2}, \frac{2+\sqrt{-5}}{2})$, we obtain
$$\left|z - \frac{\sqrt{-5}}{2}\right|^2 - \frac{1}{4} = \left|z - \frac{2+\sqrt{-5}}{2}\right|^2 - \frac{1}{4} ,$$
i.e.
$$\left|z - \frac{\sqrt{-5}}{2}\right|^2 = \left|z - \frac{2+\sqrt{-5}}{2}\right|^2 .$$
Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, and using the modulus $|z| = \sqrt{x^2 + y^2}$, we get
$$\left|x + i\left(y - \frac{\sqrt{5}}{2}\right)\right|^2 = \left|(x-1) + i\left(y - \frac{\sqrt{5}}{2}\right)\right|^2 \Leftrightarrow x^2 + \left(y - \frac{\sqrt{5}}{2}\right)^2 = (x-1)^2 + \left(y - \frac{\sqrt{5}}{2}\right)^2 .$$
Thus, we get
$$x^2 = x^2 - 2x + 1 \Leftrightarrow 0 = -2x + 1 .$$

Hence, we obtain the equation

$$x = \frac{1}{2} \ .$$

But actually the hemispheres $S_{2,\sqrt{-5}}$ and $S_{2,2+\sqrt{-5}}$ only touch one another in one point of $z$-coordinate

$$z = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \ \in D_0 \ .$$

But it is in fact a singular point in the fundamental polyhedron (see Example 12.1). So we don't need to consider it in Swan's termination criterion as it cannot be strictly below any hemisphere recorded in the list.

2. For $L(\frac{0}{1}, \frac{\sqrt{-5}}{2})$, we get

$$\left|z - \frac{0}{1}\right|^2 - 1 = \left|z - \frac{\sqrt{-5}}{2}\right|^2 - \frac{1}{4} \ , \text{ i.e. } |z|^2 - 1 = \left|z - \frac{\sqrt{-5}}{2}\right|^2 - \frac{1}{4} \ .$$

Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, we get

$$|x + iy|^2 - 1 = \left|x + i\left(y - \frac{\sqrt{5}}{2}\right)\right|^2 - \frac{1}{4} \ .$$

Using the modulus $|z| = \sqrt{x^2 + y^2}$, we obtain

$$x^2 + y^2 - 1 = x^2 + y^2 - y\sqrt{5} + \frac{5}{4} - \frac{1}{4} \ .$$

Hence, we obtain the equation

$$y = \frac{2\sqrt{5}}{5} \ .$$

We obtain the same equation for the line $L(\frac{1}{1}, \frac{2+\sqrt{-5}}{2})$.

3. By a similar procedure, we obtain for the lines $L(\frac{\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$ and $L(\frac{1+\sqrt{-5}}{1}, \frac{2+\sqrt{-5}}{2})$ the following equation

$$y = \frac{3\sqrt{5}}{5} \ .$$

Note that the hemisphere $S_{2,\sqrt{-5}}$ doesn't touch the hemispheres $S_{1,1}$ and $S_{1,1+\sqrt{-5}}$. So we don't need to compute the lines $L(\frac{1}{1}, \frac{\sqrt{-5}}{2})$ and $L(\frac{1+\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$. Similarly, the hemisphere $S_{2,2+\sqrt{-5}}$ doesn't touch the hemisphere $S_{1,0}$ and $S_{1,\sqrt{-5}}$. So we don't need to compute the lines $L(\frac{0}{1}, \frac{2+\sqrt{-5}}{2})$ and $L(\frac{\sqrt{-5}}{1}, \frac{2+\sqrt{-5}}{2})$.

Then, to compute the intersection point of $L(\frac{0}{1}, \frac{1}{1})$ and $L(\frac{0}{1}, \frac{\sqrt{-5}}{2})$, we need to consider the following system of equations

$$\begin{cases} x &= \frac{1}{2} \\ y &= \frac{2\sqrt{5}}{5} \end{cases} .$$

Thus, the intersection point of these two lines has for $z$-coordinate

$$z = \frac{1}{2} + i\frac{2}{5}\sqrt{5} = \frac{1}{2} + \frac{2}{5}\sqrt{-5} \in D_0 .$$

But there is no point $(z, \zeta)$ on the hemispheres $S_{1,0}$, $S_{1,1}$, $S_{2,\sqrt{-5}}$ and $S_{2,2+\sqrt{-5}}$ which has this $z$-coordinate. So we can assume that the height $\zeta \leqslant 0$.
We can verify that by using the equation $|\mu z - \lambda|^2 + |\mu|^2\zeta^2 = 1$ of a hemisphere $S_{\mu,\lambda}$ for a given unimodular pair $(\mu, \lambda) \in \mathcal{O}^2$ (see Notation 3). We will do it only for the hemisphere $S_{1,0}$:

$$\left|1 \cdot \left(\frac{1}{2} + i\frac{2}{5}\sqrt{5}\right) - 0\right|^2 + 1^2 \cdot \zeta^2 = 1 \iff \frac{21}{20} + \zeta^2 = 1 .$$

So we obtain that $\zeta^2 = 1 - \frac{21}{20} = -\frac{1}{20}$, which is impossible as $\zeta$ denotes the height for a point in $\mathcal{H}$ (i.e. $0 < \zeta \in \mathbb{R}$).

We can draw the same conclusion (that the height $\zeta \leqslant 0$) for the intersection point between $L(\frac{\sqrt{-5}}{1}, \frac{1+\sqrt{-5}}{1})$ and $L(\frac{\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$ which has

$$z = \frac{1}{2} + \frac{3}{5}\sqrt{-5} \in D_0$$

as $z$-coordinate.

It remains to add the edges of the fundamental rectangle $D_0$ to the agreeing lines. For the intersection point between $L(\frac{0}{1}, \frac{\sqrt{-5}}{2})$ and $\{x = 0\}$, we obtain as $z$-coordinate

$$z = 0 + i\frac{2\sqrt{5}}{5} = \frac{2}{5}\sqrt{-5} \in D_0 .$$

Let us determine, via the Pythagorean theorem, the height $\zeta$ of the lift $(\frac{2}{5}\sqrt{-5},\zeta)$ of this intersection point. For this, we consider this $z$-coordinate as the point $\left(0, \frac{2\sqrt{5}}{5}\right)$ in $\mathbb{R}\times\mathbb{R}$, and the center of the hemisphere $S_{1,0}$ as the point $(0,0)$. Then the distance between these two points is

$$d\Bigg((0,0), \left(0, \frac{2\sqrt{5}}{5}\right)\Bigg) = \frac{2\sqrt{5}}{5} \; .$$

So, via the Pythagorean theorem, we obtain

$$\left(\frac{2\sqrt{5}}{5}\right)^2 + \zeta^2 = 1^2 \;\Leftrightarrow\; \zeta^2 = 1^2 - \left(\frac{2\sqrt{5}}{5}\right)^2 ,$$

i.e. $\zeta^2 = \frac{1}{5}$. As $\zeta$ denotes the height of the point $(z,\zeta) \in \mathcal{H}$, this implies that $\zeta > 0$. Hence, we have that $\zeta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$. Later in this computation, we will find another value for the height of a point in $\mathcal{H}$, which is minimal. Hence, the point $(\frac{2}{5}\sqrt{-5}, \frac{\sqrt{5}}{5})$ is not considered as a relevant vertex to check Swan's termination criterion.
We can draw the same conclusion for the intersection points between $L(\frac{\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$ and $\{x = 0\}$; between $L(\frac{1}{1}, \frac{2+\sqrt{-5}}{2})$ and $\{x = 1\}$; and between $L(\frac{1+\sqrt{-5}}{1}, \frac{2+\sqrt{-5}}{2})$ and $\{x = 1\}$.

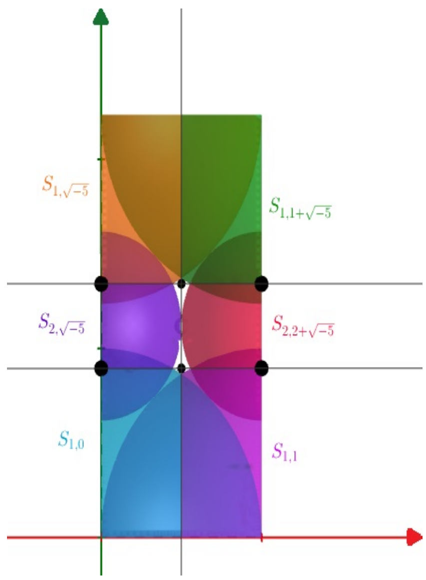



Figure 2.17: View from above

As we have points $(z,\zeta)$ with height $\zeta = 0$, Swan's termination criterion is not fulfilled. So we have to add more hemispheres to our hemisphere list.

Moreover, in the picture above, we can see that the fundamental domain $D_0$ for $\begin{pmatrix} 1 & \mathcal{O}_K \\ 0 & 1 \end{pmatrix}$ in $\mathbb{C}$ is not completely covered by hemispheres yet.

Let us increase $\mu$ to the next higher value; i.e. $\mu = \pm\sqrt{-5} \in \mathcal{O}$. Let us take $\mu = \sqrt{-5}$ $(= i\sqrt{5} \in \mathbb{C})$ and, hence, $|\mu| = \sqrt{(\sqrt{5})^2} = \sqrt{5}$.
Then the finitely many values for $\lambda \in \mathcal{O}$, such that $\frac{\lambda}{\mu} \in D_0$, are

$0,\ -1,\ -2,\ -3,\ -4,\ -5, \sqrt{-5},\ -1+\sqrt{-5},\ -2+\sqrt{-5},\ -3+\sqrt{-5},\ -4+\sqrt{-5},\ -5+\sqrt{-5}$ .

Indeed, for $\lambda = -2$ for example, we have that

$$\frac{\lambda}{\mu} = \frac{-2}{\sqrt{-5}} \cdot \frac{\sqrt{-5}}{\sqrt{-5}} = \frac{-2\sqrt{-5}}{-5} = \frac{2}{5}\sqrt{-5} \ \in D_0 \ .$$

Or, for $\lambda = -5 + \sqrt{-5}$, we have

$$\frac{\lambda}{\mu} = \frac{-5 + \sqrt{-5}}{\sqrt{-5}} = \frac{-5\sqrt{-5} - 5}{-5} = 1 + \sqrt{-5} \ \in D_0 \ .$$

Note that for $\lambda = a + b\sqrt{-5} \in \mathcal{O}$, with $a \in \mathbb{N}$ and $b \in \mathbb{Z}$, or with $a \in \mathbb{Z}_-$ and $b \geqslant 2$, then $\frac{\lambda}{\mu} \notin D_0$.

But, in terms of height, every hemisphere $S_{\sqrt{-5},\lambda}$, with the corresponding center $\frac{\lambda}{\sqrt{-5}}$, and of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{5}} \cong 0,45$ is covered by one or two of the six hemispheres, which are already recorded in the list.
For example, if $\lambda = -5 + \sqrt{-5}$, then $S_{\sqrt{-5},-5+\sqrt{-5}}$ is everywhere below $S_{1,1+\sqrt{-5}}$. Indeed, using Definition 15, we have

$$0 = \left|1 + \sqrt{-5} - 1 - \sqrt{-5}\right| \overset{!}{\leqslant} 1 - \frac{1}{\sqrt{5}} \cong 0,55 \ .$$

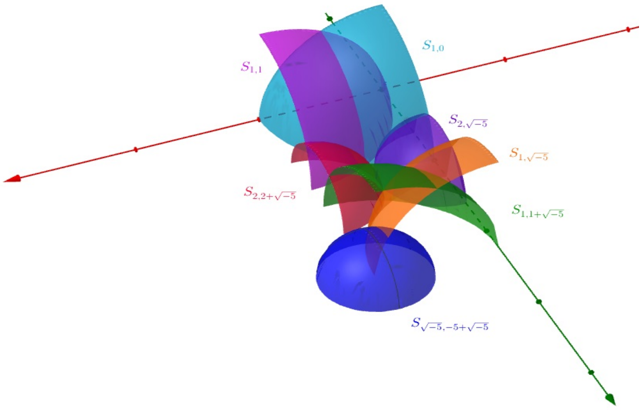



Figure 2.18: $S_{\sqrt{-5},-5+\sqrt{-5}}$ everywhere below $S_{1,1+\sqrt{-5}}$

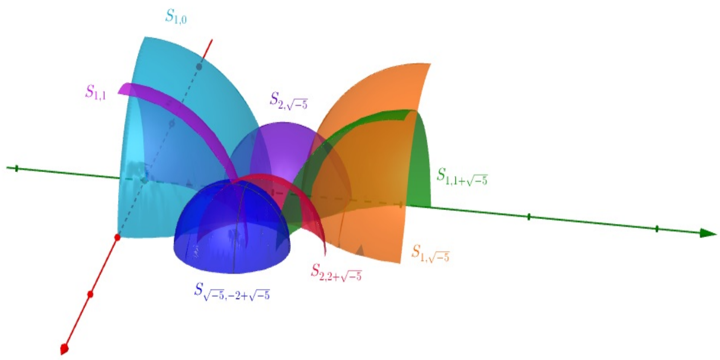



Figure 2.19: $S_{\sqrt{-5},-2+\sqrt{-5}}$ covered by $S_{1,1}$ and $S_{2,2+\sqrt{-5}}$

We can make the same observation for $\mu = -\sqrt{-5}$.
Hence, there is no hemisphere of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{5}}$ recorded in the hemisphere list.

By considering Figure 2.14, we increase $\mu$ to the next value $\pm 1 \pm \sqrt{-5} \in \mathcal{O}$; i.e. $\mu = \pm 1 \pm i\sqrt{5} \in \mathbb{C}$. Moreover, we have $|\mu| = \sqrt{(\pm 1)^2 + (\pm\sqrt{5})^2} = \sqrt{6}$. Let us take $\mu = 1 + \sqrt{-5}$.
Then the finitely many values for $\lambda \in \mathcal{O}$, such that $\frac{\lambda}{\mu} \in D_0$, are

$0,\ \sqrt{-5},\ -1+\sqrt{-5},\ -2+\sqrt{-5},\ -3+\sqrt{-5},\ -5+\sqrt{-5},\ -5+\sqrt{-5},\ 1+\sqrt{-5},\ -4+2\sqrt{-5}\,.$

For instance, if $\lambda = \sqrt{-5}$, then we have

$$\frac{\lambda}{\mu} = \frac{\sqrt{-5}}{1+\sqrt{-5}} \cdot \frac{1-\sqrt{-5}}{1-\sqrt{-5}} = \frac{\sqrt{-5}-(-5)}{1-(-5)} = \frac{\sqrt{-5}+5}{6} = \frac{5}{6} + \frac{1}{6}\sqrt{-5} \in D_0 .$$

But, in terms of height, every hemisphere $S_{1+\sqrt{-5},\lambda}$, with the corresponding center $\frac{\lambda}{1+\sqrt{-5}}$ (except for the center with $\lambda = -2+\sqrt{-5}$), and of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{6}} \cong 0,41$ is covered by at least one of the six hemispheres in the list. For example, if $\lambda = 0$, $\lambda = 1+\sqrt{-5}$, $\lambda = -5+\sqrt{-5}$ or if $\lambda = -4+2\sqrt{-5}$, then the centers of the associated hemispheres are 0, 1, $\sqrt{-5}$ and $1+\sqrt{-5}$ respectively. But these are also the centers of the four biggest hemispheres of radius 1, which are recorded in our list. Hence, the four hemispheres of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{6}}$ are obviously everywhere below the corresponding hemispheres of radius 1.

If $\lambda = -2+\sqrt{-5}$, then $S_{1+\sqrt{-5},-2+\sqrt{-5}}$ would be the only hemisphere of radius $\frac{1}{\sqrt{6}}$ which would cover $D_0$ (see Figure 2.20). But its center equals to $\frac{\lambda}{\mu} = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \in D_0$, which is the singular point we already mentioned. Consequently, $S_{1+\sqrt{-5},-2+\sqrt{-5}}$ is not recorded in the hemisphere list as $(\mu,\lambda) = (1+\sqrt{-5},-2+\sqrt{-5})$ is not a unimodular pair.

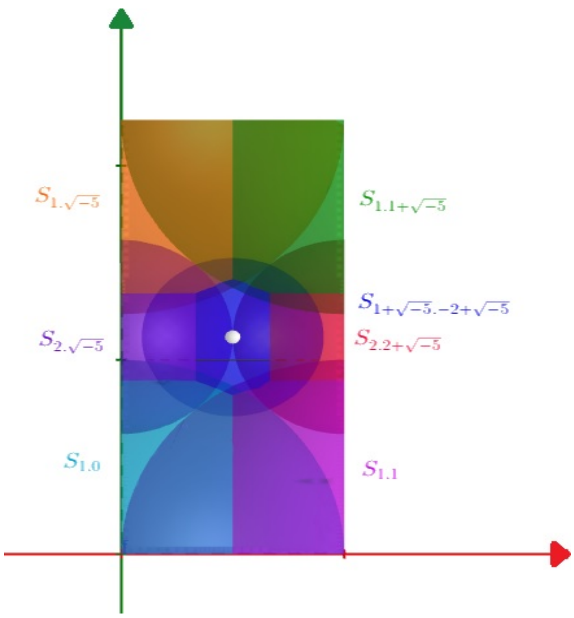



Figure 2.20: $S_{1+\sqrt{-5},-2+\sqrt{-5}}$

We can make the same observation for $\mu = 1 - \sqrt{-5}$, $\mu = -1 + \sqrt{-5}$ and for $\mu = -1 - \sqrt{-5}$.
Hence, there is no hemisphere of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{6}}$ recorded in the hemisphere list.

**Remark 10.** In general, note that a singular point cannot be strictly below any hemisphere $S_{\mu,\lambda}$, with $(\mu, \lambda)$ a unimodular pair.

Let us increase $\mu$ to the next higher value; i.e. $\mu = \pm 3 \in \mathcal{O}$ or $\mu = \pm 2 \pm \sqrt{-5} \in \mathcal{O}$, such that $|\mu| = \sqrt{(\pm 3)^2} = \sqrt{(\pm 2)^2 + (\pm\sqrt{5})^2} = 3$. Let us take $\mu = 3$.
Then the finitely many values for $\lambda \in \mathcal{O}$, such that $\frac{\lambda}{\mu} \in D_0$, are

$$\begin{gathered}0,\ 1,\ 2,\ 3,\ \sqrt{-5},\ 1+\sqrt{-5},\ 2+\sqrt{-5},\ 3+\sqrt{-5},\ 2\sqrt{-5},\ 1+2\sqrt{-5},\\ 2+2\sqrt{-5},\ 3+2\sqrt{-5},\ 3\sqrt{-5},\ 1+3\sqrt{-5},\ 2+3\sqrt{-5},\ 3+3\sqrt{-5}\ .\end{gathered}$$

Note that for $\lambda = a + b\sqrt{-5} \in \mathcal{O}$, with $a \in \mathbb{Z}_-$ and $b \in \mathbb{Z}$, or with $a \in \mathbb{N}_{\geqslant 4}$ and $b \in \{0, 1, 2, 3\}$, or with $a \in \{0, 1, 2, 3\}$ and $b \in \mathbb{N}_{\geqslant 4}$, then $\frac{\lambda}{\mu} \notin D_0$.
For example, if $\lambda = -3 + 4\sqrt{-5}$, then

$$\frac{\lambda}{\mu} = \frac{-3+4\sqrt{-5}}{3} = -1 + \frac{4}{3}\sqrt{-5} \notin D_0\ .$$

But, in terms of height, every hemisphere $S_{3,\lambda}$, with the corresponding center $\frac{\lambda}{3}$, and of radius $\frac{1}{|\mu|} = \frac{1}{3} \cong 0,33$ is covered by the hemispheres, which are already recorded in the list.
For example, if $\lambda = 2 + 3\sqrt{-5}$, then $S_{3,2+3\sqrt{-5}}$ of center $\frac{\lambda}{\mu} = \frac{2}{3} + \sqrt{-5} \in D_0$ is everywhere below $S_{1,1+\sqrt{-5}}$. Indeed, using Definition 15, we have

$$\left|\frac{2}{3} + \sqrt{-5} - 1 - \sqrt{-5}\right| = \left|-\frac{1}{3}\right| = \sqrt{\left(-\frac{1}{3}\right)^2} = \frac{1}{3} \overset{!}{\leqslant} 1 - \frac{1}{3} = \frac{2}{3}\ .$$

We can make the same observation for $\mu = -3$, $\mu = \pm 2 \pm \sqrt{-5}$.
Hence, there is no hemisphere of radius $\frac{1}{|\mu|} = \frac{1}{3}$ recorded in the hemisphere list.

By considering Figure 2.14, we increase $\mu$ to the next value $\pm 3 \pm \sqrt{-5} \in \mathcal{O}$; i.e. $\mu = \pm 3 \pm i\sqrt{5} \in \mathbb{C}$. Moreover, we have $|\mu| = \sqrt{(\pm 3)^2 + (\pm\sqrt{5})^2} = \sqrt{14}$. Let us

take $\mu = 3 + \sqrt{-5}$.
Then the finitely many values for $\lambda \in \mathcal{O}$, such that $\frac{\lambda}{\mu} \in D_0$, are

$$0,\ \sqrt{-5},\ -1+\sqrt{-5},\ 3+\sqrt{-5},\ 2+\sqrt{-5},\ 1+\sqrt{-5},\ 2\sqrt{-5},\ 1+2\sqrt{-5},$$
$$-1+2\sqrt{-5},\ -2+2\sqrt{-5},\ -3+2\sqrt{-5},\ -1+3\sqrt{-5},\ -2+3\sqrt{-5},\ -3+3\sqrt{-5},$$
$$-4+3\sqrt{-5},\ -5+3\sqrt{-5},\ -2+4\sqrt{-5}\ .$$

For instance, if $\lambda = \sqrt{-5}$, then we have

$$\frac{\lambda}{\mu} = \frac{\sqrt{-5}}{3+\sqrt{-5}} \cdot \frac{3-\sqrt{-5}}{3-\sqrt{-5}} = \frac{3\sqrt{-5}-(-5)}{9-(-5)} = \frac{3\sqrt{-5}+5}{14} = \frac{5}{14} + \frac{3}{14}\sqrt{-5}\ \in D_0\ .$$

But, in terms of height, every hemisphere $S_{3+\sqrt{-5},\lambda}$, with the corresponding center $\frac{\lambda}{3+\sqrt{-5}}$ (except for the center with $\lambda = -1+2\sqrt{-5}$), and of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{14}} \cong 0,27$ is covered by at least one of the six hemispheres in the list.

Note that, if $\lambda = -1+2\sqrt{-5} \in \mathcal{O}$, then $\frac{\lambda}{\mu} = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \in D_0$, which is again the singular point we already mentioned. Consequently, $S_{3+\sqrt{-5},-1+2\sqrt{-5}}$ is not recorded in the hemisphere list as $(\mu,\lambda) = (3+\sqrt{-5}, -1+2\sqrt{-5})$ is not a unimodular pair.

We can make the same observation for $\mu = 3-\sqrt{-5}$, $\mu = -3+\sqrt{-5}$ and for $\mu = -3-\sqrt{-5}$.
Hence, there is no hemisphere of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{14}}$ recorded in the hemisphere list.

Let us increase $\mu$ to the next higher value; i.e. $\mu = \pm 4 \in \mathcal{O}$ and $|\mu| = \sqrt{(\pm 4)^2} = 4$. Let us take $\mu = 4$.
Then the finitely many values for $\lambda \in \mathcal{O}$, such that $\frac{\lambda}{\mu} \in D_0$, are

$$0,\ 1,\ 2,\ 3,\ 4,\ \sqrt{-5},\ 1+\sqrt{-5},\ 2+\sqrt{-5},\ 3+\sqrt{-5},\ 4+\sqrt{-5},\ 2\sqrt{-5},\ 1+2\sqrt{-5},$$
$$3+2\sqrt{-5},\ 4+2\sqrt{-5},\ 3\sqrt{-5},\ 1+3\sqrt{-5},\ 2+3\sqrt{-5},\ 3+3\sqrt{-5},\ 4+3\sqrt{-5},$$
$$4\sqrt{-5},\ 1+4\sqrt{-5},\ 2+4\sqrt{-5},\ 3+4\sqrt{-5},\ 4+4\sqrt{-5}\ .$$

But, in terms of height, every hemisphere $S_{4,\lambda}$, with the corresponding center $\frac{\lambda}{4}$, and of radius $\frac{1}{|\mu|} = \frac{1}{4} = 0,25$ is covered by hemispheres, which are already recorded in the list.

For example, if $\lambda = 1 + 2\sqrt{-5}$, then $S_{4,1+2\sqrt{-5}}$ of center $\frac{\lambda}{\mu} = \frac{1}{4} + \frac{1}{2}\sqrt{-5} \in D_0$ is everywhere below $S_{2,\sqrt{-5}}$. Indeed, using Definition 15, we have

$$\left|\frac{1}{4} + \frac{1}{2}\sqrt{-5} - \frac{1}{2}\sqrt{-5}\right| = \left|\frac{1}{4}\right| = \sqrt{\left(\frac{1}{4}\right)^2} = \frac{1}{4} \overset{!}{\leqslant} \frac{1}{2} - \frac{1}{4} = \frac{1}{4}\ .$$

Note that for $\lambda = 2 + 2\sqrt{-5} \in \mathcal{O}$, we have that $\frac{\lambda}{\mu} = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \in D_0$, which is again the singular point we already mentioned. Consequently, the hemisphere $S_{4,2+2\sqrt{-5}}$ of radius $\frac{1}{4}$ is not recorded in the hemisphere list as $(\mu, \lambda) = (4, 2 + 2\sqrt{-5})$ is not a unimodular pair.

We can make the same observation for $\mu = -4$.
Hence, there is no hemisphere of radius $\frac{1}{|\mu|} = \frac{1}{4}$ recorded in the hemisphere list.

The next value for $\mu$ is $\pm 2\sqrt{-5} \in \mathcal{O}$; i.e. $\mu = \pm 2i\sqrt{5} \in \mathbb{C}$. Let us take $\mu = 2\sqrt{-5} = 2i\sqrt{5}$. Thus, we have that $|\mu| = \sqrt{0^2 + (2\sqrt{5})^2} = \sqrt{4 \cdot 5} = \sqrt{20}$. Then the only values for $\lambda \in \mathcal{O}$, which satisfy all the conditions described in **Step A** in the algorithm computing the Bianchi fundamental polyhedron (see Subsection 2.2.1), are $\lambda = -4 + \sqrt{-5}$ and $\lambda = -6 + \sqrt{-5}$. Indeed,

◇ 1. for $\lambda = -4+\sqrt{-5}$, we have $\frac{\lambda}{\mu} = \frac{-4+\sqrt{-5}}{2\sqrt{-5}} \cdot \frac{\sqrt{-5}}{\sqrt{-5}} = \frac{-4\sqrt{-5}-5}{-10} = \frac{1}{2} + \frac{2}{5}\sqrt{-5} \in D_0$; and

2. for $\lambda = -6 + \sqrt{-5}$, we have $\frac{\lambda}{\mu} = \frac{-6+\sqrt{-5}}{2\sqrt{-5}} = \frac{-6\sqrt{-5}-5}{-10} = \frac{1}{2} + \frac{3}{5}\sqrt{-5} \in D_0$.

Moreover, note that, in view of Lemma 5, that both centers $\frac{1}{2} + \frac{2}{5}\sqrt{-5}$ and $\frac{1}{2} + \frac{3}{5}\sqrt{-5}$ are no singular points:

1. For $\frac{1}{2} + \frac{2}{5}\sqrt{-5} = \frac{5+4\sqrt{-5}}{10} = \frac{4(\frac{5}{4}+\sqrt{-5})}{10}$, we have that the condition $r \in \mathbb{Z}$ does not hold as $r = \frac{5}{4}$.
2. For $\frac{1}{2} + \frac{3}{5}\sqrt{-5} = \frac{5+6\sqrt{-5}}{10} = \frac{6(\frac{5}{6}+\sqrt{-5})}{10}$, we have that the condition $r \in \mathbb{Z}$ does not hold as $r = \frac{5}{6}$.

◇ In terms of Definition 6, we have,

1. by choosing $(\alpha, \beta) = (-4 - \sqrt{-5}, 2\sqrt{-5}) \in \mathcal{O}^2$, that the pair $(\mu, \lambda) = (2\sqrt{-5}, -4 + \sqrt{-5}) \in \mathcal{O}^2$ is unimodular as $(-4+\sqrt{-5}) \cdot (-4-\sqrt{-5}) + (2\sqrt{-5}) \cdot (2\sqrt{-5}) = 21 - 20 = 1 \in (2\sqrt{-5})\mathcal{O} + (-4 + \sqrt{-5})\mathcal{O}$; and

2. by choosing $(\alpha,\beta) = (-6-\sqrt{-5}, 4\sqrt{-5}) \in \mathcal{O}^2$, that the pair $(\mu,\lambda) = (2\sqrt{-5}, -6+\sqrt{-5}) \in \mathcal{O}^2$ is unimodular as
$(-6+\sqrt{-5})\cdot(-6-\sqrt{-5}) + (2\sqrt{-5})\cdot(4\sqrt{-5}) = 41-40 = 1 \in (2\sqrt{-5})\mathcal{O} + (-6+\sqrt{-5})\mathcal{O}$.

◇ In view of Definition 15, the hemispheres $S_{2\sqrt{-5},-4+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$ are not everywhere below a hemisphere, which is already recorded in the hemisphere list. For example:

– $S_{2\sqrt{-5},-4+\sqrt{-5}}$ is not everywhere below $S_{2,\sqrt{-5}}$:

$$\left|\frac{1}{2}+\frac{2}{5}\sqrt{-5}-\frac{1}{2}\sqrt{-5}\right| = \left|\frac{1}{2}-i\frac{1}{10}\sqrt{5}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{-\sqrt{5}}{10}\right)^2} = \sqrt{\frac{3}{10}} \cong 0,548$$
$$\nleq \frac{1}{2}-\frac{1}{\sqrt{20}} \cong 0,276\ .$$

– $S_{2\sqrt{-5},-6+\sqrt{-5}}$ is not everywhere below $S_{1,\sqrt{-5}}$:

$$\left|\frac{1}{2}+\frac{3}{5}\sqrt{-5}-\sqrt{-5}\right| = \left|\frac{1}{2}-\frac{2i}{5}\sqrt{5}\right| = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{-2\sqrt{5}}{5}\right)^2} = \sqrt{\frac{21}{20}} \cong 1,025$$
$$\nleq 1-\frac{1}{\sqrt{20}} \cong 0,776\ .$$

Doing the same computation for the remaining hemispheres in the list, we can draw the same conclusion.

Finally, we have that the hemispheres $S_{2\sqrt{-5},-4+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$ of radius $\frac{1}{|\mu|} = \frac{1}{\sqrt{20}}$ are recorded into the hemisphere list.
In the following picture, we have a 3-dimensional representation of the eight hemispheres in the list.

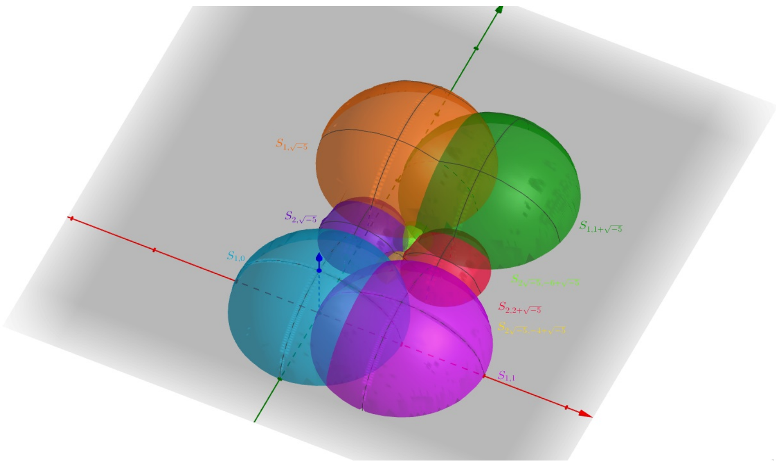



Figure 2.21: $S_{1,0}$, $S_{1,1}$, $S_{1,\sqrt{-5}}$, $S_{1,1+\sqrt{-5}}$, $S_{2,\sqrt{-5}}$, $S_{2,2+\sqrt{-5}}$, $S_{2\sqrt{-5},-4+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$

(Note that this figure is not reduced to the fundamental rectangle $D_0$.)

Now, using Notation 6, let us compute the agreeing lines over which two hemispheres in the list touch one another.

1. For $L(\frac{0}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, we get
$$\left|z - \frac{0}{1}\right|^2 - 1 = \left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} ,$$
i.e.
$$|z|^2 - 1 = \left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} .$$
Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, we get
$$|x + iy|^2 - 1 = \left|\left(x - \frac{1}{2}\right) + i\left(y - \frac{2}{5}\sqrt{5}\right)\right|^2 - \frac{1}{20} .$$
Using the modulus of a complex number $|z| = \sqrt{x^2 + y^2}$, we obtain
$$x^2+y^2-1 = \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{2}{5}\sqrt{5}\right)^2 - \frac{1}{20} \Leftrightarrow x^2+y^2-1 = x^2-x+\frac{1}{4}+y^2-y\frac{4}{5}\sqrt{5}+\frac{4}{5}-\frac{1}{20} .$$
Thus, we have
$$-1 = -x - y\frac{4}{5}\sqrt{5} + 1 ,$$

i.e.

$$y\frac{4}{5}\sqrt{5} = -x + 2 \ .$$

Hence, we obtain for $L(\frac{0}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ the equation

$$y = -\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \ .$$

2. For $L(\frac{1}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, we have

$$\left|z - \frac{1}{1}\right|^2 - 1 = \left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} \ .$$

   Then, by following the same instructions as above, we obtain the equation of the line $L(\frac{1}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$:

$$y = \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{4} \ .$$

3. For $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, we get

$$\left|z - \frac{\sqrt{-5}}{2}\right|^2 - \frac{1}{4} = \left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} \ .$$

   Then, by following the same instructions as previously, we obtain the line equation

$$y = \sqrt{5}x \ .$$

4. Using the same procedure, we obtain for the line $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ the following equation

$$y = -\sqrt{5}x + \sqrt{5} \ .$$

5. For the line $L(\frac{-4+\sqrt{-5}}{2\sqrt{-5}}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we get

$$\left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} = \left|z - \frac{1}{2} - \frac{3}{5}\sqrt{-5}\right|^2 - \frac{1}{20} \ .$$

   Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, we get

$$\left|\left(x - \frac{1}{2}\right) + i\left(y - \frac{2}{5}\sqrt{5}\right)\right|^2 = \left|\left(x - \frac{1}{2}\right) + i\left(y - \frac{3}{5}\sqrt{5}\right)\right|^2 \ .$$

By using the modulus $|z| = \sqrt{x^2+y^2}$, we obtain

$$x^2 - x + \frac{1}{4} + y^2 - y\frac{4}{5}\sqrt{5} + \frac{4}{5} = x^2 - x + \frac{1}{4} + y^2 - y\frac{6}{5}\sqrt{5} + \frac{9}{5} \ ,$$

i.e.

$$-y\frac{4}{5}\sqrt{5} + \frac{4}{5} = -y\frac{6}{5}\sqrt{5} + \frac{9}{5} \ .$$

Hence, for $L(\frac{-4+\sqrt{-5}}{2\sqrt{-5}}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we obtain the equation

$$y = \frac{\sqrt{5}}{2} \ .$$

But actually the hemispheres $S_{2\sqrt{-5},-4+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$ only touch one another in one point, which has as $z$-coordinate

$$z = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \ \in D_0 \ .$$

But this point is in fact the singular point, which we already mentioned. So we don't need to consider it in Swan's termination criterion as it cannot be strictly below any hemisphere recorded in the list.

6. For $L(\frac{\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we get

$$\left| z - \frac{\sqrt{-5}}{1} \right|^2 - 1 = \left| z - \frac{1}{2} - \frac{3}{5}\sqrt{-5} \right|^2 - \frac{1}{20} \ .$$

Then, by setting $z = x + iy$, with $x, y \in \mathbb{R}$, we get

$$|x + i(y - \sqrt{5})|^2 - 1 = \left| \left( x - \frac{1}{2} \right) + i \left( y - \frac{3}{5}\sqrt{5} \right) \right|^2 - \frac{1}{20} \ .$$

By using the modulus $|z| = \sqrt{x^2+y^2}$, we obtain

$$x^2 + y^2 - y2\sqrt{5} + 5 - 1 = x^2 - x + \frac{1}{4} + y^2 - y\frac{6}{5}\sqrt{5} + \frac{9}{5} - \frac{1}{20} \ .$$

Thus, we have

$$-y2\sqrt{5} + 4 = -x - y\frac{6}{5}\sqrt{5} + 2 \ ,$$

i.e.

$$-y\frac{4}{5}\sqrt{5} = -x - 2 \ .$$

We finally obtain for $L(\frac{\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ the equation

$$y = \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \ .$$

7. For the line $L(\frac{1+\sqrt{5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we get
$$|z-1-\sqrt{-5}|^2 - 1 = \left|z - \frac{1}{2} - \frac{3}{5}\sqrt{-5}\right|^2 - \frac{1}{20} \ .$$
Then, by following the same instructions as above, we obtain the equation of the line
$$y = -\frac{\sqrt{5}}{4}x + \frac{3\sqrt{5}}{4} \ .$$

8. Using the same procedure, we obtain for the line $L(\frac{\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ the following equation
$$y = -\sqrt{5}x + \sqrt{5} \ .$$
This is actually the same equation as for the line $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$.

9. For $L(\frac{2+\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we get
$$\left|z - 1 - \frac{\sqrt{-5}}{2}\right|^2 - \frac{1}{4} = \left|z - \frac{1}{2} - \frac{3}{5}\sqrt{-5}\right|^2 - \frac{1}{20} \ .$$
Then, using a similar procedure as previously, we obtain the equation of the line $L(\frac{2+\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$
$$y = \sqrt{5}x \ .$$
This is actually the same equation as for the line $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$.

Note that the hemisphere $S_{2\sqrt{-5},-4+\sqrt{-5}}$ doesn't touch the hemispheres $S_{1,\sqrt{-5}}$ and $S_{1,1+\sqrt{-5}}$. So we don't need to compute the lines $L(\frac{\sqrt{-5}}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{1+\sqrt{-5}}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$.
Similarly, the hemisphere $S_{2\sqrt{-5},-6+\sqrt{-5}}$ doesn't touch the hemispheres $S_{1,0}$ and $S_{1,1}$. Consequently, we don't need to compute the lines $L(\frac{0}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{1}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$.

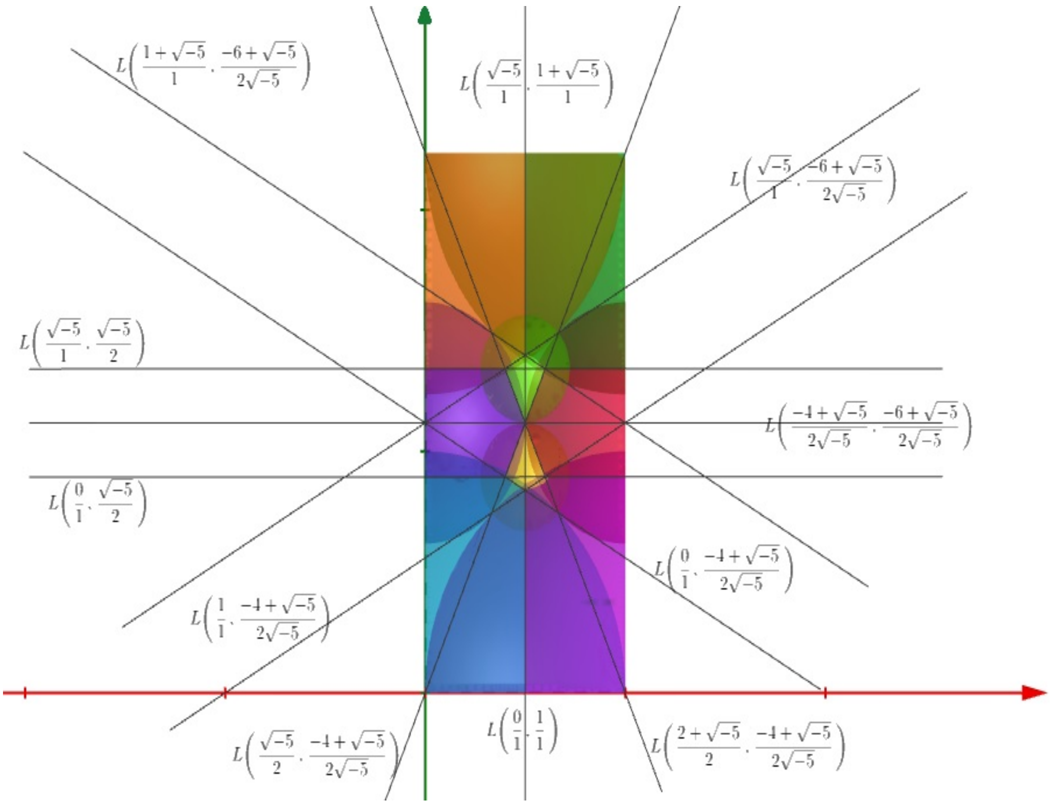



Figure 2.22: View from above to $S_{1,0}$, $S_{1,1}$, $S_{1,\sqrt{-5}}$, $S_{1,1+\sqrt{-5}}$, $S_{2,\sqrt{-5}}$, $S_{2,2+\sqrt{-5}}$, $S_{2\sqrt{-5},-4+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$, and all the agreeing lines

Let us now compute all the possible intersection points of the agreeing lines referring to the same center:

1. Recall that we previously computed the $z$-coordinate
$$z = \frac{1}{2} + \frac{2}{5}\sqrt{-5} \; \in D_0$$
of the intersection point of $L(\frac{0}{1}, \frac{1}{1})$ and $L(\frac{0}{1}, \frac{\sqrt{-5}}{2})$. We concluded there that $\zeta \leqslant 0$. Now, we have that the point $(z, \zeta)$ with this latter $z$-coordinate lies on the hemisphere $S_{2\sqrt{-5},-4+\sqrt{-5}}$. Actually, this $z$-coordinate is the center of this hemisphere. So we can conclude that the height corresponds to the radius of this hemisphere, i.e. $\zeta = \frac{1}{\sqrt{20}} \cong 0,22$.
We can verify that by using the equation of the hemisphere $S_{2\sqrt{-5},-4+\sqrt{-5}}$ (see Notation 3):
$$\left|2\sqrt{-5}\left(\frac{1}{2} + \frac{2}{5}\sqrt{-5}\right) + 4 - \sqrt{-5}\right|^2 + |2\sqrt{-5}|^2 \cdot \frac{1}{\sqrt{20}^2} = |\sqrt{-5} - 4 + 4 - \sqrt{-5}|^2 + 20 \cdot \frac{1}{20} \stackrel{!}{=} 1\,.$$

But, as the value for $\zeta$ is not minimal (we will see this at the end of this computation), $(\frac{1}{2}+\frac{2}{5}\sqrt{-5}, \frac{1}{\sqrt{20}})$ is not considered as a relevant vertex on which to check Theorem 4.

Similarly, for the intersection point of $L(\frac{\sqrt{-5}}{1}, \frac{1+\sqrt{-5}}{1})$ and $L(\frac{\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$, we determined the $z$-coordinate

$$z = \frac{1}{2} + \frac{3}{5}\sqrt{-5} \ \in D_0 \ .$$

Now, we have that this $z$-coordinate corresponds to the center (on the complex plane $\mathbb{C}$) of the hemisphere $S_{2\sqrt{-5},-6+\sqrt{-5}}$. Thus, we can conclude that the height of the lift $(\frac{1}{2}+\frac{3}{5}\sqrt{-5}, \zeta)$, which lies on $S_{2\sqrt{-5},-6+\sqrt{-5}}$, equals to the radius of this hemisphere, i.e. $\zeta = \frac{1}{\sqrt{20}} \cong 0,22$.
Again, as $\zeta$ is not minimal, it is not considered as a relevant vertex.

2. We need to add the edges of the fundamental rectangle $D_0$ to the agreeing lines. Among all the edges of $D_0$, it remains to compute the intersection points of $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $\{y=0\}$; of $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $\{y=0\}$; of $L(\frac{\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $\{y=\sqrt{5}\}$; and of $L(\frac{2+\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $\{y=\sqrt{5}\}$. We can easily see in Figure 2.22, that these four intersection points are the lifts of the centers of the hemispheres of radius 1. Hence, the heights of these four vertices are equal to 1. But, as the heights of all those points (which are lifts on the edges of the rectangle $D_0$) are not minimal, they are not considered as relevant vertices on which to check Swan's termination criterion.

3. The intersection point of the lines $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{-4+\sqrt{-5}}{2\sqrt{-5}}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{\sqrt{-5}}{2}, \frac{2+\sqrt{-5}}{2})$ has for $z$-coordinate

$$z = \frac{1}{2} + \frac{1}{2}\sqrt{-5} \ \in D_0 \ ,$$

which is again the singular point.

4. Let us compute the intersection point of $L(\frac{0}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$; i.e. we have to solve the following system of equations:

$$\begin{cases} y &= -\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= -\sqrt{5}x + \sqrt{5} \end{cases} .$$

This implies that

$$-\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} = -\sqrt{5}x + \sqrt{5}\ .$$

Thus, we get $x = \frac{2}{3}$.
Hence, if we insert $x = \frac{2}{3}$ in one of the equations of the system above, we get $y = \frac{\sqrt{5}}{3}$.
So the intersection point of these two lines has for $z$-coordinate

$$z = \frac{2}{3} + i\frac{\sqrt{5}}{3} = \frac{2}{3} + \frac{1}{3}\sqrt{-5} \in D_0\ .$$

But we have that the hemispheres $S_{1,0}$, $S_{2,2+\sqrt{-5}}$ and $S_{2\sqrt{-5},-4+\sqrt{-5}}$ are strictly below the hemisphere $S_{1,1}$ at this point $z \in \mathbb{C}$. Indeed, in terms of Definition 10, we obtain for the left hand side

$$\left|z - \frac{1}{1}\right|^2 - 1 = \left|\frac{2}{3} + i\frac{\sqrt{5}}{3} - 1\right|^2 - 1 = \left|-\frac{1}{3} + i\frac{\sqrt{5}}{3}\right|^2 - 1 = -\frac{1}{3}\ ,$$

and for the right hand side

$$\left|z - \frac{1}{2} - \frac{2}{5}\sqrt{-5}\right|^2 - \frac{1}{20} = \left|\frac{2}{3} + i\frac{\sqrt{5}}{3} - \frac{1}{2} - i\frac{2\sqrt{5}}{5}\right|^2 - \frac{1}{20} = \left|\frac{1}{6} - i\frac{\sqrt{5}}{15}\right|^2 - \frac{1}{20} = 0\ .$$

Thus, as we clearly have that $-\frac{1}{3} < 0$, we can conclude that $S_{2\sqrt{-5},-4+\sqrt{-5}}$ is strictly below $S_{1,1}$ at this $z \in \mathbb{C}$.
We can draw the same conclusion if we do the same computations for $S_{2,2+\sqrt{-5}}$ and $S_{1,0}$.
Hence, we drop this intersection point.

5. Let us compute the intersection point of $L(\frac{1}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$; i.e. we have to solve the following system

$$\begin{cases} y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{4} \\ y &= \sqrt{5}x + \sqrt{5} \end{cases}\ .$$

Then, by proceeding in the same way as in the previous computation, we get

$$x = \frac{1}{3} \text{ and } y = \frac{\sqrt{5}}{3}\ .$$

So the intersection point of these two lines has the following $z$-coordinate

$$z = \frac{1}{3} + i\frac{\sqrt{5}}{3} = \frac{1}{3} + \frac{1}{3}\sqrt{-5} \in D_0\ .$$

But we have that the hemispheres $S_{1,1}$, $S_{2,\sqrt{-5}}$ and $S_{2\sqrt{-5},-4+\sqrt{-5}}$ are strictly below the hemisphere $S_{1,0}$ at this point $z \in \mathbb{C}$. Indeed, in terms of Definition 10, we obtain for the left hand side

$$|z-0|^2 - 1 = \left|\frac{1}{3} + i\frac{\sqrt{5}}{3}\right|^2 - 1 = -\frac{1}{3} \ ,$$

and for the right hand side

$$|z-1|^2 - 1 = \left|\frac{1}{3} + i\frac{\sqrt{5}}{3} - 1\right|^2 - 1 = \left|-\frac{2}{3} + i\frac{\sqrt{5}}{3}\right|^2 - 1 = 0 \ .$$

Thus, as we clearly have that $-\frac{1}{3} < 0$, we can conclude that $S_{1,1}$ is strictly below $S_{1,0}$ at this $z \in \mathbb{C}$.
We can draw the same conclusion if we do the same computations for $S_{2,\sqrt{-5}}$ and $S_{2\sqrt{-5},-4+\sqrt{-5}}$.
Hence, we drop this intersection point.

6. Now we compute the intersection point of $L(\frac{\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{2+\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$; i.e. we have to solve the following system:

$$\begin{cases} y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= \sqrt{5}x \end{cases} .$$

Then, by proceeding in the same way as previously, we get

$$x = \frac{2}{3} \text{ and } y = \frac{2\sqrt{5}}{3} \ .$$

So the intersection point of these two lines has for $z$-coordinate

$$z = \frac{2}{3} + i\frac{2\sqrt{5}}{3} = \frac{2}{3} + \frac{2}{3}\sqrt{-5} \in D_0 \ .$$

But we have that the hemispheres $S_{1,\sqrt{-5}}$, $S_{2,2+\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$ are strictly below the hemisphere $S_{1,1+\sqrt{-5}}$ at this $z \in \mathbb{C}$. Indeed, in terms of Definition 10, we obtain for the left hand side

$$|z-1-\sqrt{-5}|^2 - 1 = \left|\frac{2}{3} + i\frac{2\sqrt{5}}{3} - 1 - \sqrt{-5}\right|^2 - 1 = \left|-\frac{1}{3} - i\frac{\sqrt{5}}{3}\right|^2 - 1 = -\frac{1}{3} \ ,$$

and for the right hand side

$$\left|z - \frac{1}{2} - \frac{3}{5}\sqrt{-5}\right|^2 - \frac{1}{20} = \left|\frac{2}{3} + i\frac{2\sqrt{5}}{3} - \frac{1}{2} - i\frac{3\sqrt{5}}{5}\right|^2 - \frac{1}{20} = \left|\frac{1}{6} + i\frac{\sqrt{5}}{15}\right|^2 - \frac{1}{20} = 0 \ .$$

Thus, as we clearly have that $-\frac{1}{3} < 0$, we can conclude that $S_{2\sqrt{-5},-6+\sqrt{-5}}$ is strictly below $S_{1,1+\sqrt{-5}}$ at this $z \in \mathbb{C}$.
We can draw the same conclusion if we do the same computations for $S_{2,2+\sqrt{-5}}$ and $S_{1,\sqrt{-5}}$.
Hence, we drop this intersection point.

7. For the computation of the intersection point of $L(\frac{1+\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, we have to solve the following system of equations:

$$\begin{cases} y &= -\frac{\sqrt{5}}{4}x + \frac{3\sqrt{5}}{4} \\ y &= -\sqrt{5}x + \sqrt{5} \end{cases} .$$

Then we obtain

$$x = \frac{1}{3} \text{ and } y = \frac{2\sqrt{5}}{3} .$$

So the intersection point of these two lines has for $z$-coordinate

$$z = \frac{1}{3} + i\frac{2\sqrt{5}}{3} = \frac{1}{3} + \frac{2}{3}\sqrt{-5} \in D_0 .$$

But we have again that the hemispheres $S_{1,1+\sqrt{-5}}$, $S_{2,\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$ are strictly below the hemisphere $S_{1,\sqrt{-5}}$ at this $z \in \mathbb{C}$. Indeed, in terms of Definition 10, we obtain for the left hand side

$$|z-\sqrt{-5}|^2 - 1 = \left|\frac{1}{3} + i\frac{2\sqrt{5}}{3} - i\sqrt{5}\right|^2 - 1 = \left|\frac{1}{3} - i\frac{\sqrt{5}}{3}\right|^2 - 1 = -\frac{1}{3} ,$$

and for the right hand side

$$|z-1-\sqrt{-5}|^2 - 1 = \left|\frac{1}{3} + i\frac{2\sqrt{5}}{3} - 1 - i\sqrt{5}\right|^2 - 1 = \left|-\frac{2}{3} - i\frac{\sqrt{5}}{3}\right|^2 - 1 = 0 .$$

Finally, as we have that $-\frac{1}{3} < 0$, we can conclude that $S_{1,1+\sqrt{-5}}$ is strictly below $S_{1,\sqrt{-5}}$ at this $z \in \mathbb{C}$.
We can draw the same conclusion if we do the same computations for $S_{2,\sqrt{-5}}$ and $S_{2\sqrt{-5},-6+\sqrt{-5}}$.
Hence, we drop this intersection point.

**Remark 11.** Note that, we do not need to check Swan's termination criterion on the corresponding vertices of the last four intersection points, which have height $\zeta = 0$, because they lie strictly below one of the unit hemispheres (each of the four unit hemispheres occurs once here).

8. Let us compute the intersection point of $L(\frac{0}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{1}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{0}{1}, \frac{1}{1})$. Thus, we have to solve the system

$$\begin{cases} y &= -\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{4} \\ x &= \frac{1}{2} \end{cases}$$

Let us insert $x = \frac{1}{2}$ in the first equation of the system. Then we obtain

$$y = -\frac{\sqrt{5}}{4} \cdot \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{8} \ .$$

We can use the second equation of the system as a verification:

$$\frac{3\sqrt{5}}{8} \stackrel{!}{=} \frac{\sqrt{5}}{4} \cdot \frac{1}{2} + \frac{\sqrt{5}}{4} \ .$$

Hence, the intersection point of these three lines has as $z$-coordinate

$$z = \frac{1}{2} + i\frac{3\sqrt{5}}{8} = \frac{1}{2} + \frac{3}{8}\sqrt{-5} \ \in D_0 \ .$$

It remains to check Swan's termination criterion (see Theorem 4). For this, we need to determine the height $\zeta$ of the vertex $(\frac{1}{2}+\frac{3}{8}\sqrt{-5}, \zeta)$, which is the lift of the intersection point above. So we consider $z = \frac{1}{2} + \frac{3}{8}\sqrt{-5}$ as the point $\left(\frac{1}{2}, \frac{3\sqrt{5}}{8}\right)$ in the Euclidean plane $\mathbb{R} \times \mathbb{R}$, and the center $\frac{\lambda}{\mu} = 0$ of the hemisphere $S_{1,0}$ (of radius $\frac{1}{|1|} = 1$) as the point $(0,0)$. Then the distance between these two points is

$$d\left((0,0); \left(\frac{1}{2}, \frac{3\sqrt{5}}{8}\right)\right) = \sqrt{\left(0 - \frac{1}{2}\right)^2 + \left(0 - \frac{3\sqrt{5}}{8}\right)^2} = \frac{\sqrt{61}}{8} \ .$$

Using the Pythagorean theorem, we get

$$\left(\frac{\sqrt{61}}{8}\right)^2 + \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 \Leftrightarrow \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 - \left(\frac{\sqrt{61}}{8}\right)^2 ,$$

i.e. we have for $\mu = 1$

$$\zeta^2 = 1^2 - \left(\frac{\sqrt{61}}{8}\right)^2 = 1 - \frac{61}{64} = \frac{3}{64} \ .$$

As $\zeta$ denotes the height of a point $(z,\zeta)$ in $\mathcal{H}$, this implies that $\zeta > 0$. Hence,

$$\zeta = \sqrt{\frac{3}{64}} = \frac{\sqrt{3}}{8} \cong 0,217 \ .$$

The next value for $\mu$ would be $\pm 5$, thus the radius $\frac{1}{|\mu|}$ would be equal to $\frac{1}{5}$. As $\zeta \geqslant \frac{1}{|\mu|}$ for $\mu = 5$, we have that $(z,\zeta) = (\frac{1}{2} + \frac{3}{8}\sqrt{-5}, \frac{\sqrt{3}}{8}) \in \mathcal{H}$ cannot be strictly below the remaining hemispheres, as they have radius smaller than $\zeta$. In other words, the highest point of any remaining hemisphere cannot lie higher than $(z,\zeta)$, and hence these hemispheres cannot contribute to the structure of $B$. So Theorem 4 is fulfilled.

9. Let us compute the intersection point of $L(\frac{\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{1+\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{\sqrt{-5}}{1}, \frac{1+\sqrt{-5}}{1})$. Thus, we have to solve the system of equations

$$\begin{cases} y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= -\frac{\sqrt{5}}{4}x + \frac{3\sqrt{5}}{4} \\ x &= \frac{1}{2} \end{cases} .$$

If we insert $x = \frac{1}{2}$ in the first equation of the system, then we obtain

$$y = \frac{\sqrt{5}}{4} \cdot \frac{1}{2} + \frac{\sqrt{5}}{2} = \frac{5\sqrt{5}}{8} \ .$$

We can use the second equation of the system as a verification:

$$\frac{5\sqrt{5}}{8} \overset{!}{=} -\frac{\sqrt{5}}{4} \cdot \frac{1}{2} + \frac{3\sqrt{5}}{4} \ .$$

Hence, the intersection point of these three lines has as $z$-coordinate

$$z = \frac{1}{2} + i\frac{5\sqrt{5}}{8} = \frac{1}{2} + \frac{5}{8}\sqrt{-5} \ \in D_0 \ .$$

It remains to check Swan's termination criterion. For this, we determine the height of the lift $(\frac{1}{2} + \frac{5}{8}\sqrt{-5}, \zeta)$ of the intersection point above. So we consider $z = \frac{1}{2} + \frac{5}{8}\sqrt{-5}$ as the point $\left(\frac{1}{2}, \frac{5\sqrt{5}}{8}\right)$ in the Euclidean plane $\mathbb{R} \times \mathbb{R}$ and the center $\frac{\lambda}{\mu} = \sqrt{-5}$ of the hemisphere $S_{1,\sqrt{-5}}$ as the point $(0, \sqrt{5})$. Then the distance between these two points is

$$d\left((0,\sqrt{5}); \left(\frac{1}{2}, \frac{5\sqrt{5}}{8}\right)\right) = \sqrt{\left(0 - \frac{1}{2}\right)^2 + \left(\sqrt{5} - \frac{5\sqrt{5}}{8}\right)^2} = \frac{\sqrt{61}}{8} \ .$$

Thus, using the Pythagorean theorem, we find the same value for $\zeta$ as in the previous bullet point, namely $\zeta = \frac{\sqrt{3}}{8} \cong 0,217$.
Again, as $\zeta = \frac{\sqrt{3}}{8} \geqslant \frac{1}{5}$, we have that Swan's termination criterion is fulfilled.

10. Let us compute the intersection point of $L(\frac{1}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{2+\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{1}{1}, \frac{2+\sqrt{-5}}{2})$. Thus, we have to solve the following system

$$\begin{cases} y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{4} \\ y &= -\sqrt{5}x + \sqrt{5} \\ y &= \frac{2\sqrt{5}}{5} \end{cases} .$$

If we insert $y = \frac{2\sqrt{5}}{5}$ in one of the equations of the system, then we obtain $x = \frac{3}{5}$.

Hence, the intersection point of these three lines has for $z$-coordinate

$$z = \frac{3}{5} + i\frac{2\sqrt{5}}{5} = \frac{3}{5} + \frac{2}{5}\sqrt{-5} \in D_0 .$$

It remains to check Swan's termination criterion (see Theorem 4). For this, we need to determine the height $\zeta$ of the point $(\frac{3}{5} + \frac{2}{5}\sqrt{-5}, \zeta)$, which is the lift of this intersection point. So let us consider $z = \frac{3}{5} + \frac{2}{5}\sqrt{-5}$ as the point $\left(\frac{3}{5}, \frac{2\sqrt{5}}{5}\right)$ in the Euclidean plane $\mathbb{R} \times \mathbb{R}$ and the center $\frac{\lambda}{\mu} = 1$ of the hemisphere $S_{1,1}$ as the point $(1,0)$. Then the distance between these two points is

$$d\left((1,0); \left(\frac{3}{5}, \frac{2\sqrt{5}}{5}\right)\right) = \sqrt{\left(1 - \frac{3}{5}\right)^2 + \left(0 - \frac{2\sqrt{5}}{5}\right)^2} = \frac{2\sqrt{6}}{5} .$$

Using the Pythagorean theorem, we get

$$\left(\frac{2\sqrt{6}}{5}\right)^2 + \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 \Leftrightarrow \zeta^2 = \left(\frac{1}{|\mu|}\right)^2 - \left(\frac{2\sqrt{6}}{5}\right)^2$$

i.e. for $\mu = 1$ we obtain

$$\zeta^2 = 1^2 - \left(\frac{2\sqrt{6}}{5}\right)^2 = 1 - \frac{24}{25} = \frac{1}{25} .$$

Hence, as $\zeta$ is the height of a point $(z,\zeta)$ in $\mathcal{H}$, i.e. $\zeta > 0$, this implies that

$$\zeta = \sqrt{\frac{1}{25}} = \frac{1}{5} = 0,2 \ .$$

The next value for $\mu$ would be $\pm 5$, i.e. the radius would be equal to $\frac{1}{5}$. Since $\zeta \geqslant \frac{1}{|\mu|}$ for $\mu = 5$, we have that all remaining hemispheres have radius equal or smaller than $\frac{1}{5}$, so $(z,\zeta) = (\frac{3}{5} + \frac{2}{5}\sqrt{-5}, \frac{1}{5}) \in \mathcal{H}$ cannot be strictly below them. So Theorem 4 is fulfilled.

11. For the computation for the intersection point of $L(\frac{\sqrt{-5}}{2}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{0}{1}, \frac{-4+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{0}{1}, \frac{\sqrt{-5}}{2})$, we have to solve the system of equations

$$\begin{cases} y &= \sqrt{5}x \\ y &= -\frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= \frac{2\sqrt{5}}{5} \end{cases} .$$

Let us insert $y = \frac{2\sqrt{5}}{5}$ in the first equation of the system above. Then we obtain

$$\frac{2\sqrt{5}}{5} = \sqrt{5}x \iff x = \frac{2}{5} \ .$$

Hence, the intersection point of these three lines has for $z$-coordinate

$$z = \frac{2}{5} + i\frac{2\sqrt{5}}{5} = \frac{2}{5} + \frac{2}{5}\sqrt{-5} \ \in D_0 \ .$$

It remains to determine the height $\zeta$ of the lift $(\frac{2}{5} + \frac{2}{5}\sqrt{-5}, \zeta)$ of this intersection point. For this, we consider $z = \frac{2}{5}+\frac{2}{5}\sqrt{-5}$ as the point $\left(\frac{2}{5}, \frac{2\sqrt{5}}{5}\right)$ in the Euclidean plane $\mathbb{R}\times\mathbb{R}$, and the center $\frac{\lambda}{\mu} = 0$ of the hemisphere $S_{1,0}$ as the point $(0,0)$. Then the distance between these two points is

$$d\left((0,0); \left(\frac{2}{5}, \frac{2\sqrt{5}}{5}\right)\right) = \sqrt{\left(0-\frac{2}{5}\right)^2 + \left(0-\frac{2\sqrt{5}}{5}\right)^2} = \frac{2\sqrt{6}}{5} \ .$$

Thus, using the Pythagorean theorem, we get the same value for $\zeta$ as in the previous bullet point, namely $\zeta = \frac{1}{5}$.
Again, as $\zeta = \frac{1}{5} \geqslant \frac{1}{|\mu|} = \frac{1}{5}$ for $\mu = 5$, we have that Theorem 4 is fulfilled.

12. For the computation for the intersection point of $L(\frac{\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{\sqrt{-5}}{1}, \frac{\sqrt{-5}}{2})$, we have to solve the following system

$$\begin{cases} y &= -\sqrt{5}x + \sqrt{5} \\ y &= \frac{\sqrt{5}}{4}x + \frac{\sqrt{5}}{2} \\ y &= \frac{3\sqrt{5}}{5} \end{cases} .$$

Following the same procedure as previously, then we obtain $x = \frac{2}{5}$. Hence, the intersection point of these three lines has for $z$-coordinate

$$z = \frac{2}{5} + i\frac{3\sqrt{5}}{5} = \frac{2}{5} + \frac{3}{5}\sqrt{-5} \in D_0 .$$

It remains to determine the height $\zeta$ of $(\frac{2}{5} + \frac{3}{5}\sqrt{-5}, \zeta)$. For this, we consider $z = \frac{2}{5} + \frac{3}{5}\sqrt{-5}$ as the point $\left(\frac{2}{5}, \frac{3\sqrt{5}}{5}\right)$ in $\mathbb{R} \times \mathbb{R}$, and the center $\frac{\lambda}{\mu} = \sqrt{-5}$ of the hemisphere $S_{1,\sqrt{-5}}$ as the point $(0, \sqrt{5})$. Then the distance between these two points is

$$d\left((0, \sqrt{5}); \left(\frac{2}{5}, \frac{3\sqrt{5}}{5}\right)\right) = \sqrt{\left(0 - \frac{2}{5}\right)^2 + \left(\sqrt{5} - \frac{3\sqrt{5}}{5}\right)^2} = \frac{2\sqrt{6}}{5} .$$

Thus, we get the same value for $\zeta$ as previously, namely $\zeta = \frac{1}{5}$.
Again, as $\zeta = \frac{1}{5} \geqslant \frac{1}{|\mu|} = \frac{1}{5}$ for $\mu = 5$, we have that Theorem 4 is fulfilled.

13. We compute the intersection point of $L(\frac{2+\sqrt{-5}}{2}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$, $L(\frac{1+\sqrt{-5}}{1}, \frac{-6+\sqrt{-5}}{2\sqrt{-5}})$ and $L(\frac{1+\sqrt{-5}}{1}, \frac{2+\sqrt{-5}}{2})$, i.e we have to solve the following system

$$\begin{cases} y &= \sqrt{5}x \\ y &= -\frac{\sqrt{5}}{4}x + \frac{3\sqrt{5}}{4} \\ y &= \frac{3\sqrt{5}}{5} \end{cases} \Leftrightarrow x = \frac{3}{5} .$$

Hence, the intersection point of these three lines has for $z$-coordinate

$$z = \frac{3}{5} + i\frac{3\sqrt{5}}{5} = \frac{3}{5} + \frac{3}{5}\sqrt{-5} \in D_0 .$$

It remains to check Swan's termination criterion. For this, we determine the height $\zeta$ of the point $(\frac{3}{5} + \frac{3}{5}\sqrt{-5}, \zeta)$. So let us consider $z = \frac{3}{5} + \frac{3}{5}\sqrt{-5}$ as the point $\left(\frac{3}{5}, \frac{3\sqrt{5}}{5}\right)$ in $\mathbb{R} \times \mathbb{R}$, and the center $\frac{\lambda}{\mu} = 1 + \sqrt{-5}$ of the hemisphere

$S_{1,1+\sqrt{-5}}$ as the point $(1, \sqrt{5})$. Then the distance between these two points is

$$d\left((1, \sqrt{5}); \left(\frac{3}{5}, \frac{3\sqrt{5}}{5}\right)\right) = \sqrt{\left(1 - \frac{3}{5}\right)^2 + \left(\sqrt{5} - \frac{3\sqrt{5}}{5}\right)^2} = \frac{2\sqrt{6}}{5}.$$

So we obtain again the same value for $\zeta$ as previously, namely $\zeta = \frac{1}{5}$. Finally, as $\zeta = \frac{1}{5} \geqslant \frac{1}{|\mu|} = \frac{1}{5}$ for $\mu = 5$, we have that Theorem 4 is fulfilled.

**Remark 12.** In terms of Lemma 5, we can easily check that none of these intersection points are singular points.

Comparing the values of the heights for the last six points in $\mathcal{H}$, we notice that $\frac{\sqrt{3}}{8} > \frac{1}{5}$. But, as we have to pick the lowest value for $\zeta > 0$, we can conclude that the points $(\frac{1}{2} + \frac{3}{8}\sqrt{-5}, \frac{\sqrt{3}}{8})$ and $(\frac{1}{2} + \frac{5}{8}\sqrt{-5}, \frac{\sqrt{3}}{8})$ are not considered as relevant vertices. Hence, there are four points left which can be considered as relevant vertices for Swan's termination criterion:

$$(z, \zeta) = (\frac{3}{5} + \frac{2}{5}\sqrt{-5}, \frac{1}{5})$$
$$(z, \zeta) = (\frac{2}{5} + \frac{2}{5}\sqrt{-5}, \frac{1}{5})$$
$$(z, \zeta) = (\frac{2}{5} + \frac{3}{5}\sqrt{-5}, \frac{1}{5})$$
$$(z, \zeta) = (\frac{3}{5} + \frac{3}{5}\sqrt{-5}, \frac{1}{5})$$

Moreover, in the next picture, we can see that the fundamental domain is now completely covered by hemispheres.

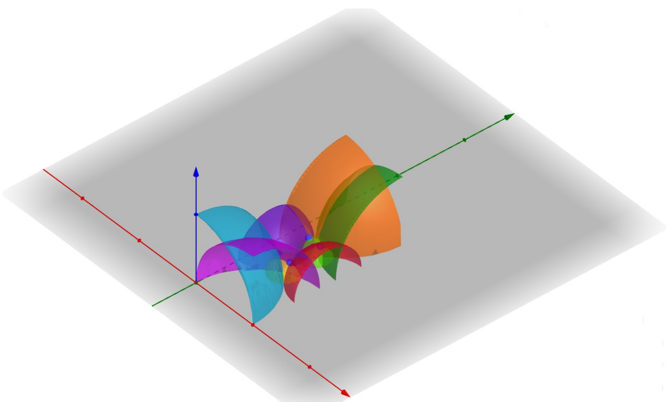


Figure 2.23: Fundamental domain for $m = 5$

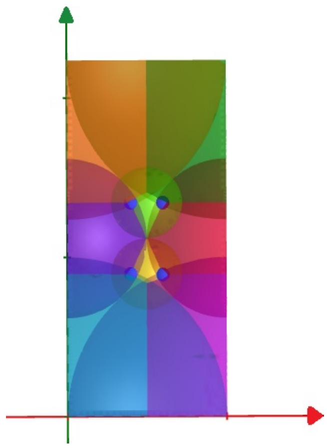


Figure 2.24: View from above

**Remark 13.** Note that in the figures 2.23 and 2.24, there have been marked precisely those vertices which are relevant for checking Swan's criterion (i.e. minimal height amongst non-singular vertices).

Thus, Theorem 4 is fulfilled, and we have computed the Bianchi fundamental polyhedron.

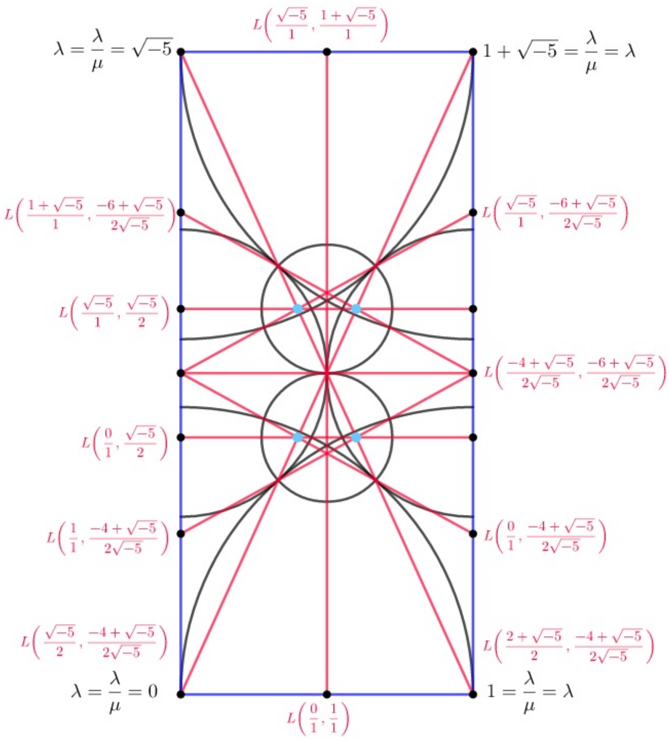



Figure 2.25: Bianchi fundamental polyhedron for $m = 5$

In the following picture, the cell structure of the Bianchi fundamental polyhedron is illustrated.

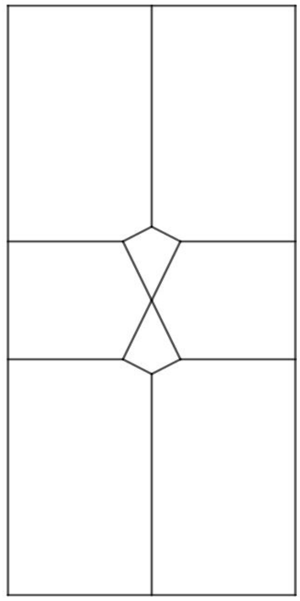


Figure 2.26: Cell structure for the Bianchi fundamental polyhedron for $m = 5$

**Remark 14.**

- We obtain the cell structure while using only those line segments which are projections of arcs on the surface.
  If we compare Figure 2.22 and Figure 2.25, then it becomes more clear how we obtain Figure 2.26.

- I established an animated "GIF"-file of the fundamental domain, which you can find either on the website "Experimental Mathematics Lab" under the category "Image gallery": `http://math.uni.lu/eml/`,
  or on my Dropbox via the following QR-Code:

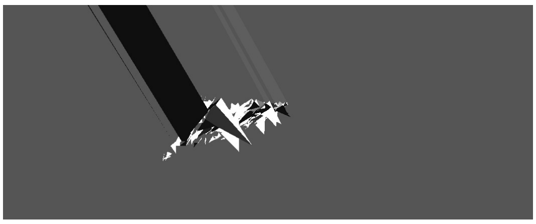
# Appendix

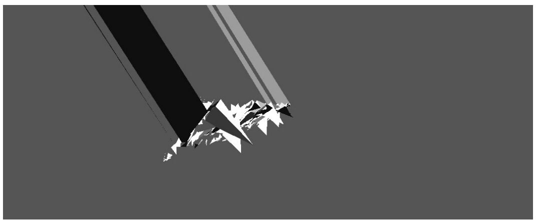
Another part of my Master thesis was to collect screenshots of the fundamental polyhedron for the Bianchi group of discriminant $-427$, computed with Bianchi.gp and visualized with the program "Geomview". Then using these screenshots, I established an animated "GIF"-file, which you can find on the website "Experimental Mathematics Lab" under the category "Image gallery":

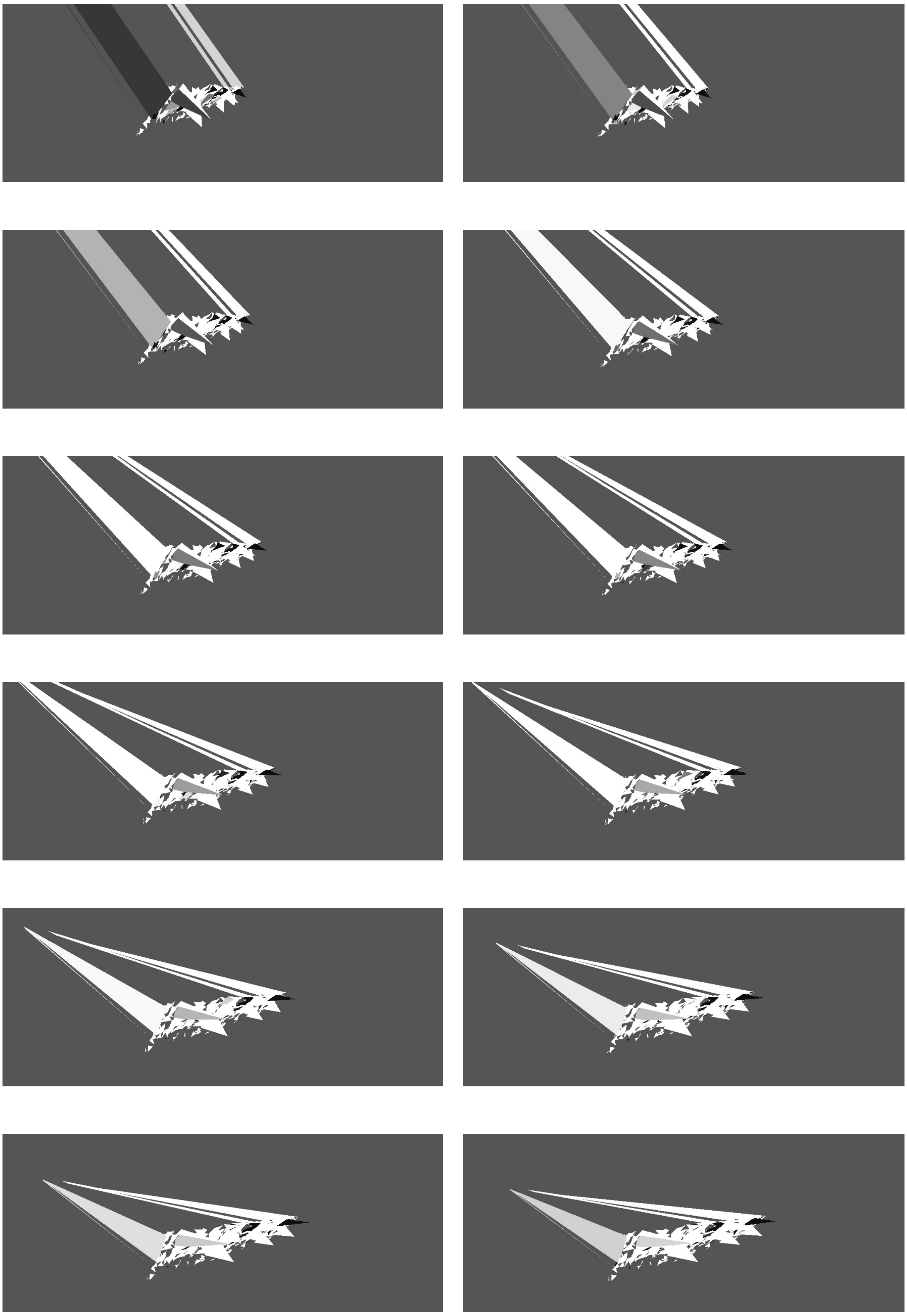
`http://math.uni.lu/eml/`

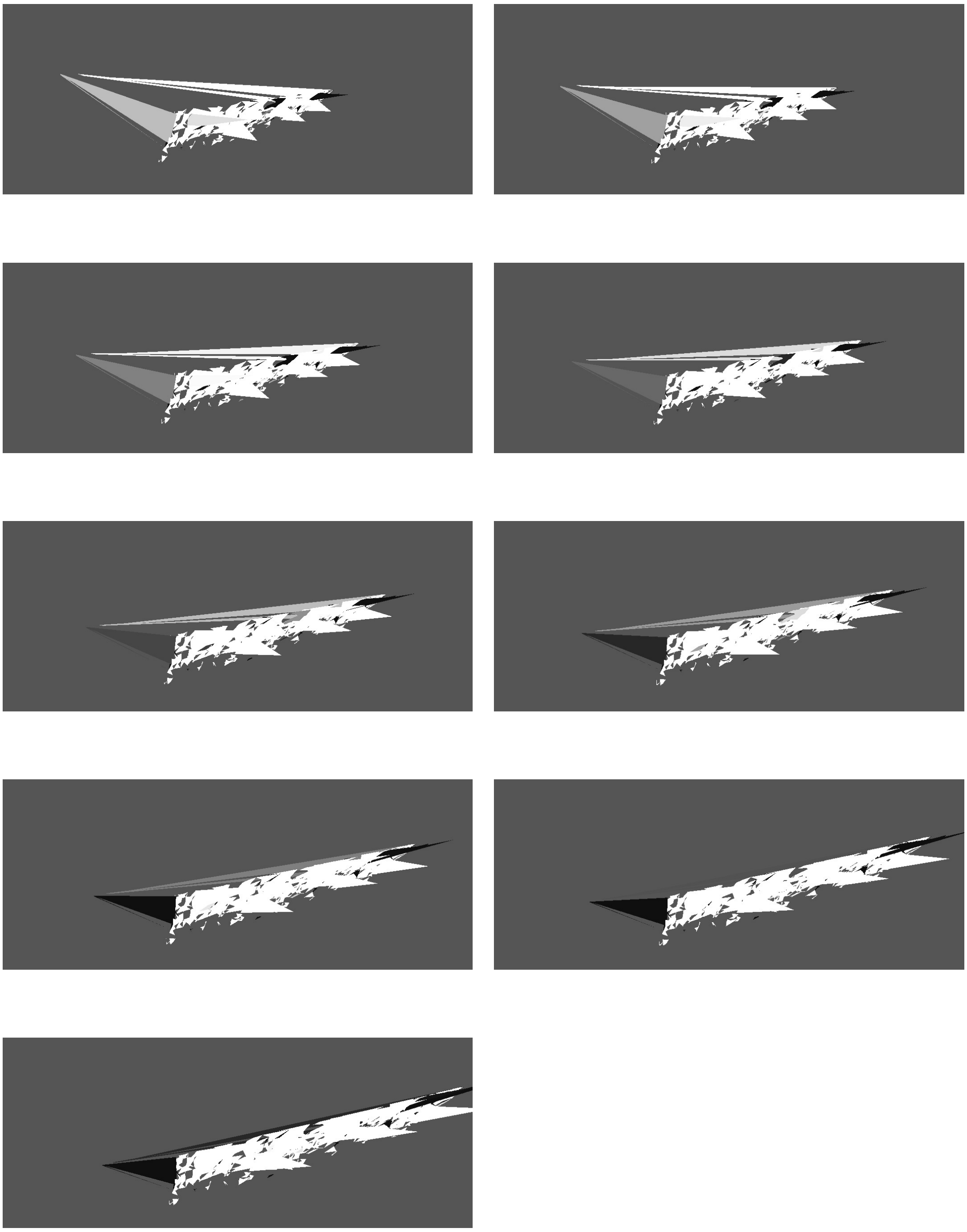
This video is also accessible on my Dropbox via the following QR-Code:

Here are the collected screenshots:









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