UNIVERSITÉ DU LUXEMBOURG

University of Luxembourg

Master in Mathematics

Modular forms and Maeda's conjecture

Author:
Lucija Čalić

Supervisors:
Prof. G. Wiese
P. Tsaknias

Contents

Chapter 1

Introduction

In this work I am going to introduce modular forms and Maeda's conjecture. My colleague Valnea Skansi will present Galois theory which connects field theory and group theory.

I will start with some basic definitions and introduce modular forms as functions on the complex upper half plane. Each nonzero modular form has two associated integers k and N called weight and level respectively. The modular forms form a vector space. Linear operators called the Hecke operators, T_n, act on these vector spaces. An eigenform is a modular form that is a simultaneous eigenvector for all the Hecke operators. The Hecke operators respect the decomposition of $S_k(N)$ into old and new subspaces. We call the normalized eigenforms for $S_k(\Gamma_0(N))_{\text{new}}$, newforms. After presenting the basic theory of newforms, I will present Maeda's conjectures.

The goal of this project is to either disapprove or experimentally verify Maeda's conjecture which says that the Galois group of coefficient field of newforms is the symmetric group of degree of the field dimension.

In the end, Valnea and I will present the code and the computational results which will be obtained using SAGE. With these results we shall test the conjecture, mentioned above, for the squarefree levels.

Chapter 2

Modular forms

This chapter introduces modular forms, the upper half plane and the group $\mathrm{SL}_2(\mathbb{Z})$.

The *modular forms* are functions on the complex upper plane. A matrix group called the modular group acts on the upper half plane, and modular forms are the functions that transform in an invariant way under the action and satisfy a holomorphy condition.

2.1 First defintions

The *modular group* is the group of 2-by-2 matrices with integer entries and its determinant is 1,

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

and it acts on the *complex upper half plane*

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

by *linear transfomation*, as follows. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then for $\forall z \in \mathcal{H}$ we let

$$\gamma(z) = \frac{az + b}{cz + d} \in \mathcal{H} \tag{2.1}$$

Definition 2.1.1. Let R be an open subset of $\mathbb{C}$. A function $f : R \to \mathbb{C}$ is ***holomorphic*** if f is complex differentiable at $\forall z \in R$, i.e.

$$f'(z) = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h}$$

exists, where h may approach 0 along any path. A function $f : R \to \mathbb{C} \cup \{\infty\}$ is ***meromorphic*** if it is holomorphic except at a discrete set S of points in R and at each $\alpha \in S$ there is a positive integer n such that $(z-\alpha)^n f(z)$ is holomorphic at α.

Modular forms are holomorphic functions on $\mathcal{H}$ that transform under a group $\mathrm{SL}_2(\mathbb{Z})$. Before defining general modular forms, we will define modular forms of level 1.

2.2 Modular Forms of Level 1

Definition 2.2.1. Let k be an integer. A meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is a ***weakly modular of weight k*** such that for all $z \in \mathcal{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$f(\gamma(z)) = (cz+d)^k f(z) \tag{2.2}$$

The constant functions are weakly modular of weight 0. The product of two weakly modular functions of weight k_1 and k_2 is a weakly modular function of weight $k_1 + k_2$.

Definition 2.2.2. ***Modular forms*** are weakly modular functions that are also holomorphic on the upper half plane and holomorphic at ∞.

Let $D = \{q \in \mathbb{C} : |q| < 1\}$ be an open complex unit disk and let $D' = D \backslash \{0\}$. From the complex analysis we know that the holomorphic map $z \mapsto e^{2\pi i z} = q(z)$ defines a map $\mathcal{H}$ to D'. There is a function $F : D' \to \mathbb{C}$ such that $F(q(z)) = f(z)$. In particular, $q \to 0$ corresponds to $z \to \infty$. This function F is a complex-valued function on D', but it may or may not be well behaved at 0. Suppose that F is well behaved at 0, then

$$F(q) = \sum_{n=m}^{\infty} a_n q^n \tag{2.3}$$

If this is the case, we say that f is meromorphic at ∞. If, $m \geq 0$, we say that f is holomorphic at ∞. We also call (2.3) the ***q-expansion*** of f about ∞.

Definition 2.2.3. Let k be an integer. A function $f : \mathcal{H} \to \mathbb{C}$ is a ***modular form of weight k*** if

i) f is holomorphic on $\mathcal{H}$

ii) f is weakly modular of weight k

iii) f is holomorphic at ∞

The set of modular forms of weight k is denoted $\mathcal{M}_k(\Gamma)$ where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

Definition 2.2.4. Let $k \in \mathbb{Z}$. A modular form of weight k is called ***a cusp form*** if $f(\infty) = a_0$, i.e. if $f(\infty) = 0$. The set of such functions is denoted $S_k(\Gamma)$. It forms a $\mathbb{C}$-vector space.

Remark 1. f a modular function of odd weight $\Rightarrow f \equiv 0$

$$-I \in \Gamma, f(z) = f(-Iz) = (-1)^k f(z) \overset{k \text{ odd}}{=} -f(z) \Rightarrow f \equiv 0$$

The dimensions of the modular forms and the cusp forms of level 1 are finite and their dimensions can be computed thanks to the Riemann-Roch theorem. For k even, their respective formulae are given as

$$M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor, & \text{if } k \equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor + 1, & \text{if } k \not\equiv 2 \mod 12 \end{cases}$$

$$S_k = \begin{cases} \lfloor \frac{k}{12} \rfloor - 1, & \text{if } k \equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor, & \text{if } k \not\equiv 2 \mod 12 \\ 0, & \text{if } k = 2 \end{cases}$$

Modular forms are geometric, arithmetic, and topological objects that are of interest all over mathematics such as in Diophantine equations, Fermat's last theorem, congruent number problems, cryptography and coding theory, etc.

Chapter 3

Hecke Theory

There are operators $T_n, n \in \mathbb{N}$ acting on the space M_k of modular forms of weight k. The space M_k has a canonical basis of simultaneous eigenvectors of all the T_n; these special modular forms have the property that their Fourier coefficients a_n are algebraic integers and satisfy the multiplicative property $a_{nm} = a_n a_m$ whenever n and m are relatively prime. We will define the operators T_n, present the theory through their action on q-expansion and describe their eigenforms.

3.1 Hecke operators

Definition 3.1.1. Let k be an integer. Define the ***weight k right action*** of $\mathrm{SL}_2(\mathbb{Z})$ on the set of all functions $f : \mathcal{H} \to \mathbb{C}$ as follows. If $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, let

$$(f^{[\gamma]k})(z) = \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma(z)) \tag{3.1}$$

For any $n \geq 0$, let

$$X_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \geq 1, ad = n, \text{ and } 0 \leq b < d \right\}$$

Definition 3.1.2. The nth ***Hecke operator*** $T_{n,k}$ of weight k is the operator on the set of functions on $\mathcal{H}$ defined by

$$T_{n,k}(f) = \sum_{\gamma \in X_n} f^{[\gamma]k},$$

Proposition 1. *If f is a (weakly) modular function of weight k, then so is $T_{n,k}(f)$ a (weakly) modular function.*

Proposition 2. *On weight k modular functions we have*

$$T_{mn} = T_m T_n \text{ if } gcd(m,n) = 1$$
$$T_m T_n = T_n T_m \ \forall m,n \in \mathbb{N}$$
$$T_{p^n} = T_{p^{n-1}} T_p - p^{k-1} T_{p^{n-2}} \text{ if } p \text{ is prime}$$

Proposition 3. *Let $f = \sum_{n \in \mathbb{N}_{\not{0}}} a_n q^n$ be a modular function of weight k. Then*

$$T_n(f) = \sum_{m \geq 0} \left(\sum_{d|m,n} d^{k-1} a_{\frac{mn}{d^2}} \right) q^m. \tag{3.2}$$

and

$$T_n T_m = \sum_{m \geq 0} \left(\sum_{d|m,n} d^{k-1} a_{\frac{mn}{d^2}} \right) q^m. \tag{3.3}$$

If $n = p$ is prime, then

$$T_p(f) = \sum_{m \geq 0} (a_{mp} + p^{k-1} a_{\frac{m}{p}}) q^m,$$

where $a_{\frac{m}{p}} = 0$ if $\frac{m}{p} \notin \mathbb{N}$.

3.2 Eigenforms

Suppose that $f(z) = \sum_{m=0}^{\infty} a_m q^m$ is an ***eigenvector*** of all the T_n, i.e.

$$T_n f = \lambda_n f, \quad \forall n \in \mathbb{N}, \lambda_n \in \mathbb{C} \tag{3.4}$$

From (3.2) and (3.4) we obtain the identity

$$\lambda_n a_m = \sum_{d|n,m} d^{k-1} a_{\frac{nm}{d^2}}$$

by comparing the coefficients of q^m on both sides of (3.4). In particular $\lambda_n a_1 = a_n, \forall n$. It follows that $a_1 \neq 0$ if $f \not\equiv 0$, so we can normalize f by requiring that $a_1 = 1$. We call a modular form satisfying (3.4) and the extra condition $a_1 = 1$, a ***Hecke form (normalized Hecke eigenform)***. From what we have just said, it follows that a Hecke form has the property

$$\lambda_n = a_n, \quad \forall n \geq 0$$

i.e., the Fourrier coefficients of f are equal to its eigenvalues under the Hecke operators. Equations (3.3) and (3.4) imply the property

$$a_n a_m = \sum_{d|m,n} d^{k-1} a_{\frac{mn}{d^2}}$$

for the coefficients of a Hecke form. The sequence of Fourier coefficients $\{a_n\}$ is multiplicative, i.e. $a_1 = 1$ and $a_{nm} = a_n a_m$ whenever n and m are coprime. Especially, $a_{p_1^{r_1} \dots p_l^{r_l}} = a_{p_1^{r_1}} \dots a_{p_l}^{r_l}$ for distinct primes $p_1, \dots, p_l$, so the a_n are determined if we know the values of a_p^r for all primes p.

Theorem 3.2.1. *The Hecke forms in M_k form a basis for every k.*

Definition 3.2.1. The ***Petersson inner product*** is given as mapping $\langle \cdot, \cdot \rangle : M_k \times S_k \longrightarrow \mathbb{C}$

$$\langle f, g \rangle = \int_{\mathcal{H}} f(z)\overline{g(z)}\,\mathrm{Im}(z)\, y^{-2}\, dx\, dy.$$

Theorem 3.2.2. *The Hecke operators are Hermitian with respect to the Petersson inner product.*

Remark 2. It follows that every Hecke operator can be diagnolised.

Remark 3. For a Hecke form we have

$$a_n \langle f, f \rangle = \langle a_n f, f \rangle = \langle \lambda_n f, f \rangle = \langle T_n f, f \rangle = \langle f, T_n f \rangle = \langle f, \lambda_n f \rangle = \langle f, a_n f \rangle = \overline{a_n} \langle f, f \rangle$$

by the self-adjointness of T_n and the linearity of the scalar product. Therefore the Fourier coefficients of f are real. If $g = \sum_{n \geq 0} b_n q^n$ is a second Hecke form in S_k, then the same computation shows that

$$a_n \langle f, g \rangle = \langle T_n f, g \rangle = \overline{b_n} \langle f, g \rangle = b_n \langle f, g \rangle$$

and hence that $\langle f, g \rangle = 0$ if $f \neq g$. Thus the various Hecke forms in S_k are mutually orthogonal and linearly independent.

3.3 Forms of higher level

Definition 3.3.1. A ***congruence subgroup*** of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ is any subgroup of $\mathrm{SL}_2(\mathbb{Z})$ that contains

$$\Gamma(N) = \mathrm{Ker}(\mathrm{SL}_2(\mathbb{Z})) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$$

for some $N \in \mathbb{N}$. The smallest N is the ***level*** of Γ.

The most important congruence subgroup will be

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod N \right\}$$

where $*$ means any element and the group has level N. A central object in the theory of modular forms is the ***sets of cusps***

$$\mathbb{P}^1(\mathbb{Z}) = \mathbb{Z} \cup \{\infty\}.$$

An element $\gamma \in \Gamma$ acts on $\mathbb{P}^1(\mathbb{Z})$ by

$$\gamma(z) = \begin{cases} \frac{az+b}{cz+d}, & \text{if } z \neq \infty \\ \frac{a}{c}, & \text{if } z = \infty \end{cases}$$

The set of ***cusps for a congruence group*** Γ is the set $C(\Gamma)$ of Γ-orbits of $\mathbb{P}^1(\mathbb{Z})$. We will often identify elements of $C(\Gamma)$ with a representative element from the orbit. On the modular forms of level 1 we restricted notion to the group Γ because most of the aspects of the theory are visible here.

However, in the case of the theory of Hecke operators there are some important differences, which we now describe. We will restrict attention to the subgroup Γ_0. Now we will modify the definition of T_n and replace Γ with Γ_0. Let's add an extra condition

$$X_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \geq 1, ad = n, \text{ and } \gcd(a, N) = 1 \right\}$$

meaning that we have fewer representatives than before. For general n, the operation T_n is given by the same formulae (3.2) and (3.3) but with the extra condition that $\gcd(d, N) = 1$. The other main difference with the case of modular forms of level 1 comes from the existance of *old forms*. If $N'|N \implies \Gamma_0(N) \subset \Gamma_0(N')$. $f(Mz)$ is a modular form of weight k on $\Gamma_0(N)$ for each positive divisor M of N/N' since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \Rightarrow \begin{pmatrix} a & bM \\ c/M & d \end{pmatrix} \in \Gamma_0(N') \Rightarrow f\left(M\frac{az+b}{cz+d}\right) = (cz+d)^k f(Mz).$$

The subspace of $M_k(\Gamma_0(N))$ spanned by all forms $f(Mz)$ with $f \in M_k(\Gamma_0(N'))$, $MN'|N$, $N' \neq N$ is called the ***space of old forms***. $M_k(\Gamma_0(N))$ has a canonical splitting as the direct sum of the subspaces $M_k(\Gamma_0(N))_{\mathrm{old}}$ of old forms and a complementary space $M_k(\Gamma_0(N))_{\mathrm{new}}$. For cusp forms, $S_k(\Gamma_0(N))_{\mathrm{new}}$ is the orthogonal complement of $S_k(\Gamma_0(N))_{\mathrm{old}}$ with respect to the Petersson scalar product.), i.e.

$$S_k(\Gamma_0(N)) = S_k(\Gamma_0(N))_{\mathrm{new}} \bigoplus S_k(\Gamma_0(N))_{\mathrm{old}}.$$

If we define a ***Hecke form of level N*** to be a form in $M_k(\Gamma_0(N))_{\text{new}}$ which is an eigenvector of T_n for all n prime to N and with $a_1 = 1$, the Hecke forms are eigenvectors of all the T_n, they form a basis of $M_k(\Gamma_0(N))_{\text{new}}$ and their Fourier coefficients are real algebraic integers just as before. We call the normalized eigenforms of $S_k(N)_{\text{new}}$, ***newforms***. We have a canonical direct sum composition

$$M_k(\Gamma_0(N)) = \bigoplus_{MN'|N} \{f(Mz) : f \in M_k(\Gamma_0(N'))_{\text{new}}\}$$

3.4 Maeda's conjecture

Definition 3.4.1. Let L be an extension field of K. The ***Galois group*** of L/K is $\text{Gal}(L/K) = \left\{\sigma : L \to L \text{ automorphism} : \sigma_{|K} = id\right\}$, with respect to composition.

Definition 3.4.2. Let L/K and $\alpha \in L$. Then $\{\sigma(\alpha) : \sigma \in \text{Gal}(L/K)\}$ is the set of ***Galois conjugates*** of $\alpha \in L$.

Let K_f be the number field obtained by adjoining all Fourier coefficients of eigenform f to $G = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. There is a natural action of G on the basis $B = \{f_1, \ldots, f_s\}$ of newforms, by simply letting $\sigma \in G$ acts on the Fourier coefficients as follows

$$\sigma * \sum_{n\geq 1} a_n q^n = \sum_{n\geq 1} \sigma(a_n) q^n$$

The newform $\sigma(f_i)$ is clearly another newform in B. i.e. σ is the permutation on $\{1, \ldots, s\}$ induced by σ. Let $O_1, \ldots, O_r, r \leq s$ be the orbits of the action of G on B such that $B = O_1 \sqcup \cdots \sqcup O_r$.

Conjecture 3.4.1 (Strong Maeda)**.** *The Galois group of the coefficient field of a newform of level 1 is the symmetric group of degree of the field dimension, i.e.*

$$Gal(K_f^{gal})/\mathbb{Q} \cong S_n \text{ where } n = [K : \mathbb{Q}].$$

The strong conjecture implies the following weaker one:

Conjecture 3.4.2 (Weak Maeda)**.** $S_k(1)$ *consists of a single newform Galois orbit for all k.*

There is a generalisation of the weak conjecture to arbitrary levels by P. Tsaknias:

Conjecture 3.4.3 (General Weak form)**.** *Fix a squarefree integer* $N = p_1 \ldots p_s \geq 1$*, where the* p_i*'s are pairwise distinct prime numbers and* $s \geq 0$*. Then the number of newform Galois orbits* $S_k(N)_{new}$ *is equal to* 2^s*.*

In work currently in preparation, P. Tsaknias and L. Dieulefait suspect the following generalisation of the strong conjecture:

Question 3.4.1. *Let $N \in \mathbb{Z}$ squarefree and $k \in \mathbb{Z}$ and consider the basis $B_{N,k}$ of newforms for the space $S_k(N)_{new}$. Let $f_1, \ldots, f_r$ be a choice of representatives of the orbits of the action $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $B_{N,k}$. Is it true that $Gal(K_{f_i}^{gal})/\mathbb{Q} \cong S_{n_i}$ where n_i is the size of the orbit of f_i?*

In the following examples we will show how to preform computations on newforms in SAGE.

Example 3.4.1. *Let's find a defining polynomial of a number field of a_0 and a_1 of a newform of weight 3 and level 36 in Sage.*

```
sage: N = Newforms(3,36,names='a'); N
[q + a0*q^2 + 129140163*q^3 + (-60912*a0 + 26716334080)*q^4 +
(-1210960*a0 - 703770746130)*q^5 + O(q^6),
q + a1*q^2 - 129140163*q^3 + (a1^2 - 34359738368)*q^4 +
(-1434/23*a1^2 - 404609972/23*a1 + 74249810225802/23)*q^5 + O(q^6)]
sage: N[0].hecke_eigenvalue_field()
Number Field in a0 with defining polynomial x^2 + 60912*x - 61076072448
sage: N[1].hecke_eigenvalue_field()
Number Field in a1 with defining polynomial x^3 +
87330*x^2 - 63970719552*x + 2419568332406784
```

Example 3.4.2. *Let's find a characteristic polynomial of a Hecke operator of a cusp form of level 36 and weight 1.*

```
sage: C = CuspForms(1,36)
sage: C.hecke_operator(36).charpoly('x')
x^3 + 6052204716130590310531484352*x^2 +
9819857737523432243569351753575320838758685918539808768*x +
3495721884239676456038571655466194427407221
```

Chapter 4

Sage computation

To experimentally verify Maeda's conjecture that the Galois group of coefficient field of newforms is the symmetric group of degree of the field dimension, we will use the following theorem to check conditions concerning the cycle length of a transitive subgroup.

Theorem 4.0.1. *Let G be a transitive subgroup of S_n. Suppose G contains a 2-cycle and (n-1)-cycle. Then $G \cong S_n$.*

How does our algorithm work?

First, we generate a newform of level N and weight k. Then we extract a defining polynomial f of a number field of coefficients of a newform. We apply a function *is_symm* on the polynomial f in order to validate the conjecture. We have defined a bool function *is_symm* that takes two arguments, an irreducible monic polynomial f with coefficients in $\mathbb{Z}$ and range (if undefined, we take range = 10000). For the prime numbers p within the range, we do the factorisation of polynomial f modulo p. Since we know that degrees of the factors correspond to cycle lenghts in the Galois group, we are checking the sufficient criterion (Theorem 4.0.1) for the subgroup of symmetric group to be the whole symmetric group. In other words, if $n = deg(f)$, we are trying to find an irreducible factor of degree $n - 1$ and an irreducible factor of degree 2 (but we then also require all the other degrees to be odd). If both conditions are satisfied, our function returns true.

RESULTS

On a 2.4GHz Mac computer from Mid 2010 with a 4GB of RAM, we managed to compute results for levels $N = 2, 3, 5, 6$ and 7 (even though we wanted to check squarefree levels up to 21) and weight $k \leq 32$. Our results experimentally verified the conjecture.

Appendix A

Code

In this appendix you will find the code used to experimentally verify the conjecture.

```
R.<x>=PolynomialRing(ZZ,'x',implementation='NTL')

def is_symm(f,r=10000):
 c1=0
 c2=0
 p=0
 p1=0
 p2=0
  for p in range(r):
    if is_prime(p):
     #print 'prime',p
          L=f.factor_mod(p)
          #print 'factorisation of the polynomial reduced mod p:',L
          i=0
          n=0
          j=0
          while i<len(L):
           n=n+(L[i][0]^L[i][1]).degree()
           i=i+1
          count_odd=0
          while j<len(L):
           if (L[j][0]^L[j][1]).degree()==n-1:
           c1=1
           p1=p
           if (L[j][0]^L[j][1]).degree()==2:
            k=0
```

```
        while k<len(L):
                    if gcd((L[k][0]^L[k][1]).degree(),2)==1:
                     count_odd=count_odd+1
                     k=k+1
                    else:
                     k=k+1
                     continue
             if (count_odd==(len(L)-1)):
                    c2=1
                    p2=p
          j=j+1
    #print 'count odd:',count_odd
    #print 'c1:',c1
    #print 'c2:',c2
    if (c1==1) and (c2==1):
     #print 'first prime',p1
          #print 'second prime',p2
          return true
    else: continue
  else: continue
return false
#
L = [2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 35]
i=0
for i in range(14):
 n = L[i]
 k=2
 while k<40:
   N = Newforms(n,k,names='a')
   j=0
          while(j < len( N )):
                    if(N[j].hecke_eigenvalue_field().degree()>2):
                     f=(N[j].hecke_eigenvalue_field().defining_polynomial())
                     #print f
                     print 'is_symm:',is_symm(f)
               j = j + 1
          k=k+2
```

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