CAT(0) cubical complexes

Exercice 1 (M)**.** Prove that a CAT(0) cubical complex whose links are all complete bi-partite graphs is a product of two trees.

Exercice 2 (M)**.** Show that a product of two trees does not embed in $\mathbf{R}^3$, even locally.

Exercice 3 (E)**.** Square a surface group.

Exercice 4 (E)**.** Show that the triple space of a group is empty if and only if the group is elementary.

Exercice 5 (M)**.** Show that if a group G acts cocompactly on a CAT(0) cube complex X, then every hyperplane is acted on cocompactly by its stabilizer.

Exercice 6 (E/O)**.** Draw the CAT(0) cube complex associated to all partitions of n points, where $n = 1, 2, 3$ and 4. Can you give a description for the general case?

Exercice 7 (M)**.** Let L be a discrete collection of lines which has finitely many parallelism classes (for example, think of the plane triangluated by unit isosceles triangles). Consider the set $S = \mathbf{R}^2 - L$. What is the cube complex associated to this space with walls (S, L)?

Exercice 8 (M/H)**.** Let L be a transverse collection of n lines in $\mathbf{H}^2$, with $n > 1$. Then, there exists a number $R = R(L) > 0$ such any line intersecting all the lines in L, intersects the ball of radius R about the origin.

Exercice 9 (M)**.** The Coxeter group $\mathsf{PGL}_2(\mathbf{Z})$ acts by isometries on the upper half-plane preserving a hyperbolic tessellation by isometric triangles, each with one ideal vertex as illustrated on figure 1. This gives the plane a wall space structure, where the walls are the mirrors for the reflections. Describe the associated CAT(0) cube complex.

Exercice 10 (M/H)**.** Show that a codimension one subgroup H in a discrete group G gives G a structure of space with walls.

Exercice 11 (M/H)**.** Let X be a CAT(0) cube complex, $g \in \mathrm{Aut}(X)$ and h be a half-space.

(1) If g skewers h, then g is hyperbolic and any axis for g crosses $\hat{h}$.
(2) If g is hyperbolic and the axis of g crosses h, then for some $n \in \mathbf{Z}$, we have that g^n skewers h.

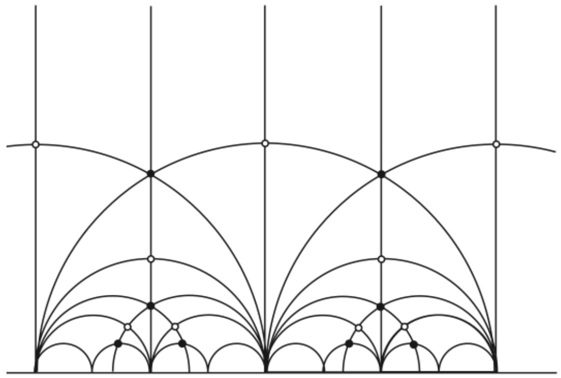

FIGURE 1. The Coxeter complex for the group $\mathsf{PGL}_2(\mathbb{Z})$ embedded in the upper half plane model of the hyperbolic space

Exercice 12 (H)**.** Let X be a CAT(0) cube complex, $g \in \mathrm{Aut}(X)$, let g be a hyperbolic isometry of X and let h be a halfspace. Then one of the following holds.

(1) Some power of g skewers h.
(2) Some power of g flips h.
(3) Some power of g stabilizes $\hat{h}$.

The first of the above possiblities is when the axis of g meets $\hat{h}$ and the last is when the axis for g lies in a bounded neighborhood of $\hat{h}$.

Exercice 13 (O)**.** Let G act properly and cocompactly on a product of two trees, does G contain $F_2 \times \mathbf{Z}$? What about $\mathbf{Z}^2 * \mathbf{Z}$?

E=easy, M=medium, most of those are taken from Sageev's course and require to apply the theory, H=hard, O=open,