Exercise 1 (Iwasawa decomposition and Siegel set in dimension 2) *1. Prove that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}$$

yields an action of $\mathrm{SL}_2(\mathbb{R})$ *on Poincaré upper half-plane*

$$\mathbb{H} = \{z \in \mathbb{C} \,:\, \mathrm{Im}(z) > 0\}.$$

2. Deduce from 1. the Iwasawa decomposition

$$\mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \,:\, x \in \mathbb{R} \right\} \cdot \left\{ \begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix} \,:\, \lambda > 0 \right\} \cdot \mathrm{SO}_2.$$

3. Using the transformations $S = \left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right)$ *et* $T = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ *prove that*

$$\{z = x + iy \in \mathbb{H} \,:\, |x| \leqslant 1/2,\ |z| \geqslant 1\}$$

is a fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ *on* $\mathcal{H}$.

4. Deduce from 3. that

$$\mathrm{SL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{Z}) \cdot \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \,:\, |x| \leqslant 1/2 \right\} \cdot \left\{ \begin{pmatrix} \lambda & \\ & 1/\lambda \end{pmatrix} \,:\, \lambda^2 \geqslant \sqrt{3}/2 \right\} \cdot \mathrm{SO}_2.$$

5. Prove Hermite-Mahler criterion for lattices in $\mathbb{R}^2$.

Exercise 2 (Iwasawa decomposition and Siegel sets in general) *Consider the groups* $G = \mathrm{GL}_m(\mathbf{R})$ *and*

$$K = \mathrm{O}(m) = \{g \in \mathrm{GL}_m(\mathbf{R}) \mid {}^t g g = I_m\}.$$

Let $A \subset G$ *be the subgroup of diagonal matrices with positive diagonal coefficients. Let* $N \subset G$ *be the subgroup of upper triangular matrices whose diagonal coefficients are all equal to* 1.

1. Let $(f_1, \dots, f_m)$ *be a basis of* $\mathbf{R}^m$*. Prove that there exists one and only one orthonormal basis* $(e_1, \dots, e_m)$ *of* $\mathbf{R}^m$ *such that for all* $i \in \{1, \dots, m\}$*, we have*

$$f_i = \alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \cdots + \alpha_{i,i} e_i$$

with $\alpha_{i,1}, \alpha_{i,2}, \dots \alpha_{i,i}$ *real numbers such that* $\alpha_{i,i} > 0$*. Prove moreover that the vectors* e_i *and the coefficients* $\alpha_{i,j}$ *depend continuously of* $(f_1, \dots, f_m)$.

2. Prove that the map

$$K \times A \times N \to G; \quad (k, a, n) \mapsto kan$$

is a homeomorphism. This is the Iwasawa decomposition.

3. Let $\Phi \colon G \to \mathbf{R}$, $\Phi(g) = \|g(\varepsilon_1)\|$*, where* $\|\cdot\|$ *is the norm associated to the canonical scalar product* $\langle \cdot, \cdot \rangle$ *of* $\mathbf{R}^m$*. Compute* $\Phi(g)$ *in termes of a unique matrix coefficient of one of the factors of the Iwasawa decomposition of* g.

4. Let $\Gamma = \mathrm{GL}_m(\mathbf{Z})$*. Prove by induction on* $m \geqslant 1$ *that*

$$\begin{array}{l} \textit{for all } g \in G, \textit{ the restriction } \Phi|_{g\Gamma} \textit{ of } \Phi \textit{ to } g\Gamma \\ \textit{reaches his minimum at a point in } \mathcal{S}^{(m)}_{\frac{2}{\sqrt{3}},\frac{1}{2}}. \end{array} \qquad (**)$$

Where

$$\mathcal{S}^{(m)}_{t,u} = K A_t N_u$$

is a Siegel domain *with*

$$A_t = \{a \in A \mid \frac{a_{i,i}}{a_{i+1,i+1}} \leqslant t \textit{ for all } 1 \leqslant i \leqslant m-1\},$$

and

$$N_u = \{n \in N \mid |n_{i,j}| \leqslant u \textit{ for all } 1 \leqslant i < j \leqslant m\}.$$

5. *Prove that, for all $t \geqslant \frac{2}{\sqrt{3}}$ and for $u \geqslant \frac{1}{2}$, we have*

$$G = \mathcal{S}^{(m)}_{t,u}\,\Gamma. \tag{*}$$

6. *Prove Hermite's inequality: for all $g \in G$, we have*

$$\min_{x \in \mathbf{Z}^m \setminus \{0\}} ||g(x)|| \leqslant \left(\frac{2}{\sqrt{3}}\right)^{\frac{m-1}{2}} |\det(g)|^{\frac{1}{m}}.$$

7. *Compare (*) with Question 4 of Exercise 1.*

Exercise 3 *1. Show that the set of solutions of the 'Pell–Fermat equation'*

$$x^2 - ay^2 = 1,\ (x,y) \in \mathbb{Z}^2$$

is in bijection with the set

$$\mathrm{SL}_2(\mathbb{Z}) \cap \mathrm{O}(q_a),$$

where $q_a = x^2 - ay^2$.

2. *Deduce from the construction of arithmetic groups that, if $a \in \mathbb{N}$ is not a square, Pell–Fermat's equation has an infinite number of solutions.*

Exercise 4 (Minkowski's lemma) *Let F be a finite subgroup of $\mathrm{GL}_N(\mathbb{Z})$.*

1. *Prove that there exists a F-invariant norm on $\mathbb{R}^N$ such that*

- *any vector of $\mathbb{Z}^N$ is of norm $\geqslant 1$, and*
- *span$\{x \in \mathbb{Z}^N$ of norm $1\} = \mathbb{R}^N$.*

2. *Deduce form 1. that for every $\ell \geqslant 3$, the 'mod ℓ restriction map'*

$$\mathrm{GL}_N(\mathbb{Z}) \to \mathrm{GL}_N(\mathbb{Z}/\ell\mathbb{Z})$$

is injective on F.

3. *Give an algebraic direct proof of that.*

Exercise 5 (Weil restriction of scalars) *Let $G \subset \mathrm{GL}(N,\mathbb{C})$ be an algebraic group defined over an algebraic number field $K \subset \mathbb{C}$. Let $\mathcal{O}$ be the ring of integers of K. Let*

$$(\tau_1,\ldots,\tau_d) = (\sigma_1,\ldots,\sigma_{r_1},\sigma_{r_1+1},\overline{\sigma}_{r_1+1},\ldots,\sigma_{r_1+r_2},\overline{\sigma}_{r_1+r_2})$$

be the set of the r_1 real and r_2 complex embeddings of K, so that $r_1 + 2r_2 = d = [K:\mathbb{Q}]$.

1. Prove that the group

$$^{\tau_1}G \times \cdots \times {}^{\tau_d}G \subset \mathrm{GL}(N, \mathbb{C})^d$$

is isomorphic to an algebraic group $R_{K/\mathbb{Q}}(G)$ *in* $\mathrm{GL}(dN, \mathbb{C})$ *defined over* $\mathbb{Q}$ *and such that*

$$R_{K/\mathbb{Q}}(G)(\mathbb{Q}) = \{(\tau_1(g), \dots, \tau_d(g)) \ : \ g \in G(K)\}.$$

2. Identify the group of real points of $R_{K/\mathbb{Q}}(G)$*.*

3. Prove that the projection $p : R_{K/\mathbb{Q}}(G) \to {}^{\tau_1}G = G$ *is defined over* K *and induces a bijection between* $R_{K/\mathbb{Q}}(G)(\mathbb{Q})$ *and* $G(K)$*.*

Exercise 6 *1. Using the last two exercises, prove Selberg's lemma for arithmetic lattices.*

2. What about the general case ? (Hard)

Exercise 7 *Prove that there exist non-arithmetic closed hyperbolic manifolds*

1. in dimension 2, (Easy)

2. in dimension 3, (Not easy)

3. in arbitrary dimension. (Hard)

Exercise 8 *Construct a closed hyperbolic manifold fibering over the circle. (Hard)*

Exercise 9 (Falconer slicing Theorem) *Let* U *be a bounded open set in* $\mathbb{R}^3$ *of volume* 1 *and let* f *be its characteristic function. To any fixed direction* $\theta \in \mathbb{S}^2$ *we associate the function*

$$F_\theta : r \mapsto \int_{\Pi(r,\theta)} f$$

where $\Pi(r, \theta)$ *denotes the plane orthogonal to* θ *and at distance* r *from the origin.*

1. Prove that the Fourier transform of F_θ *is given by*

$$\widehat{F_\theta}(\xi) = \widehat{f}(\theta, \xi)$$

where $\widehat{f}$ *is the Fourier transform of* f*.*

2. Using the Plancherel formula, prove that $||\widehat{f}||_2 = 1$*.*

3. Deduce from 2. that there exists a direction $\theta_0 \in \mathbb{S}^2$ *such that*

$$\int_{\mathbf{R}} |\widehat{f}(\theta_0, r)|^2 r^2 dr \leqslant \frac{1}{2\pi}.$$

4. Deduce from 1. and 3. that

$$\int_{\mathbf{R}} |\widehat{F}_{\theta_0}(\xi)|^2 (1 + |\xi|^2)^2 d\xi < 5.$$

5. Then prove that

$$\int_{\mathbf{R}} |\widehat{F}_{\theta_0}(\xi)| d\xi \leqslant \left(\int_{\mathbf{R}} |\widehat{F}_{\theta_0}(\xi)|^2 (1 + |\xi|^2)^2 d\xi \right)^{1/2} \left(\int_{\mathbf{R}} (1 + |\xi|^2)^{-2} d\xi \right)^{1/2} < \sqrt{10}$$

6. Conclude that there exists a family of parallel planes such that each of these planes intersect U *in a region of area at most* $\sqrt{10}$*.*

7. Is the corresponding statement true with lines in $\mathbb{R}^2$ *?*