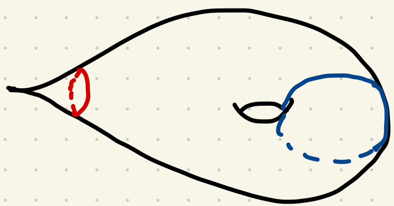
Reading Seminar: The probabilistic nature of McShane's identity: planar tree coding of simple loops. F. Labourie, SP Tan

Day 1 (10/11/2020)

- **Introduction:**

1) Recap on McShane's Identity



$$\boxed{\sum_{[\gamma]\in\mathcal{C}} \frac{1}{1+e^{l(\gamma_g)}} = \frac{1}{2}.}$$

S = Once-punctured torus

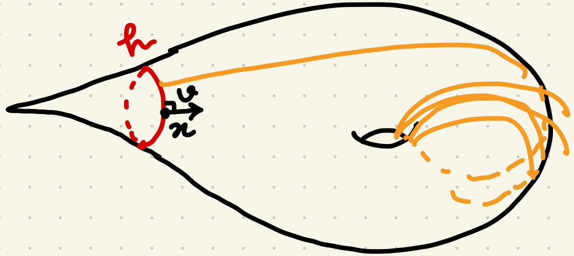
$\mathcal{C}$ = isotopy classes of simple closed geodesics γ_g

$\gamma_g \to$ unique geodesic in $[\gamma]$.

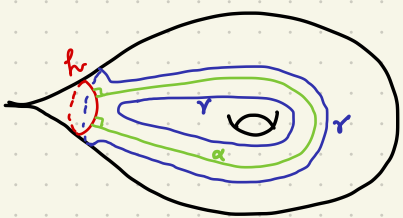
$l(\gamma_g) \to$ length of γ_g using the hyp metric of S.

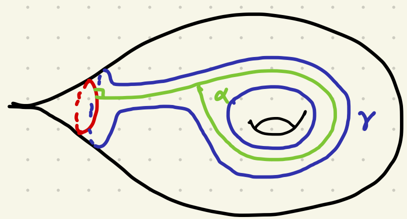
Proof: Two different approaches (Motivation for L-T)

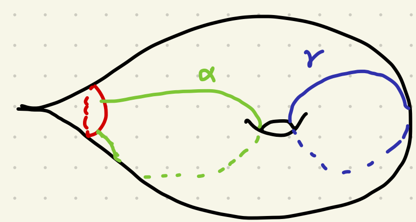
McShane (Geometry, dynamics, measure).



- Take a horoball h of length 1.
- Define $X := \{(x,v) \in T^1S \mid x \in h,\ v \perp h,\ v \text{ points inwards}\}$
- $X \simeq h \Rightarrow \ell(h) = \mathrm{Vol}(X) = 1$.
- Define $Z := \{v \in X \mid$ every geod α s.t. $\alpha(0) \in h$, $\alpha'(0) = v$ has infinite length and 0 self int$\}$
- Elements of $X - Z$: (h, α, γ) pants decomposition







surface

Thm (Birman-Series) On a hyperbolic manifold M, let G_k be the family of geodesics which are either closed and smooth or open infinite length, and have at most $k \geq 0$ transversal self-intersections.

Then $\bigcup_{\gamma \in G_k} \gamma$ is nowhere dense in M.

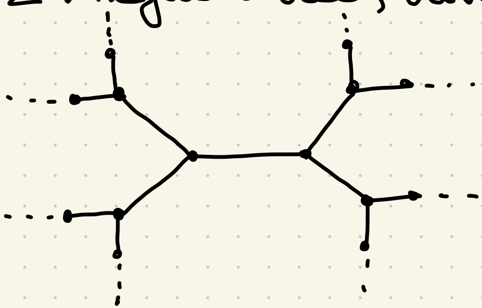
$\Rightarrow$ Z has measure 0.

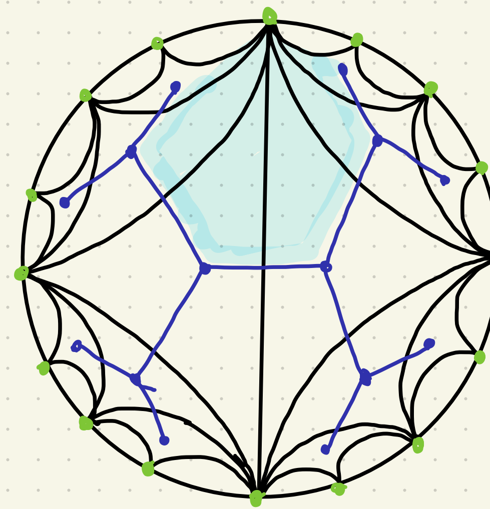
Finally, partition the set $X - Z$ into X_P's where P is an embedded pair of pants.

$$1 = \mathrm{Vol}(X - Z) = \sum_{P \in \mathcal{P}} \mathrm{Vol}(X_P) = \sum \frac{1}{1 + e^{\ell(\gamma)}} .$$

Bowditch (Representations, Markoff maps, regular trivalent trees)

- A metric on S : $[\rho]$ where $\rho : \underset{\langle a,b \rangle}{\underset{\shortparallel}{\pi_1 S}} \longrightarrow PSL_2\mathbb{R}$ holonomy representation with $\operatorname{tr}[a,b] = -2$.
- $\ell(\gamma) = 2\operatorname{arccosh}\left(\frac{\operatorname{tr}(\rho(\gamma))}{2}\right)$
- $\mathcal{C} = \{\text{iso cls s.c.c}\} \cong \mathbb{Q} \cup \{\infty\}$
- Σ : Regular tree, trivalent $\hookrightarrow \mathbb{H}^2$



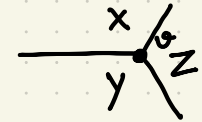


$$\Omega = \left\{ \begin{array}{l} \text{connected components} \\ \text{of } \mathbb{H}^2 \setminus \Sigma \end{array} \right\}$$

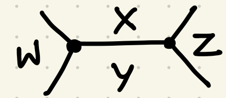
$$= \{\text{complementary regions}\}$$

$$\cong \mathcal{C}$$

- Markoff maps: $\phi : \Omega \longrightarrow \mathbb{C}$ satisfying
  $\phi(X) := x$

1) 

Around each v,
$$x^2 + y^2 + z^2 = xyz$$

2) 

$$xy = z + w.$$

- Given $[\rho] \in \mathrm{Teich}(S)$, define $\phi : \underset{\mathcal{C}}{\underset{\wr\!\wr}{\Omega}} \longrightarrow \mathbb{C}$, $X \longmapsto \mathrm{tr}(\rho(X))$

  - depends only on the conjugacy class.
  - $x > 2$ since $\rho(X)$ is hyp.

Then Φ is Markoff (using trace relations)

- Reformulation of the Identity: $\displaystyle\sum_{X \in \Omega} h(\Phi(X)) = \frac{1}{2}$ where

$$h : (2, \infty) \to \mathbb{R}$$
$$x \longmapsto \frac{1}{2}\left(1 - \sqrt{1 - \frac{4}{x^2}}\right)$$

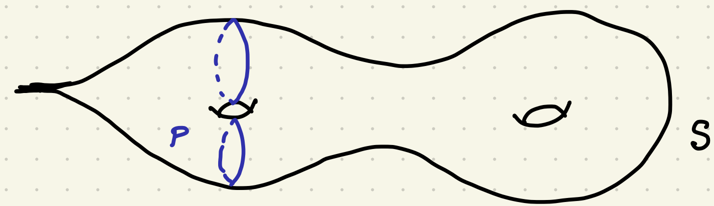
2) The probabilistic interpretation of McShane's Identity (Labourie, Tan)

Idea: To **measure** the $\underbrace{\text{appropriate sets}}_{\text{Approach 1}}$ related to a $\underbrace{\text{rooted planar tree } T}_{\text{Approach 2}}$

McShane's Identity for an orientable surface with genus $g \geq 1$ and exactly one cusp:

$$\underbrace{\sum_{P \in \mathcal{P}} \frac{1}{e^{\ell(\partial P)/2} + 1}}_{\text{interpreted as probability measure}} = 1$$

where $\mathcal{P}$ = embedded 1-cusp pair of pants P with ∂P an $\underbrace{\text{oriented}}_{\text{accounts for 1}}$ simple closed geod.

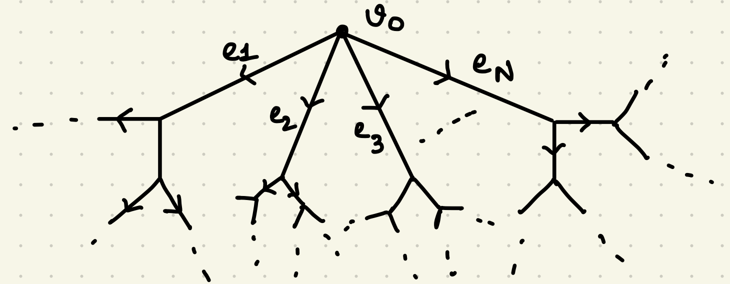


The Planar Tree

vertex set

$T,\ V(T),\ E(T)$ edgeset

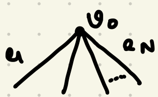
root $v_0 \in V(T)$



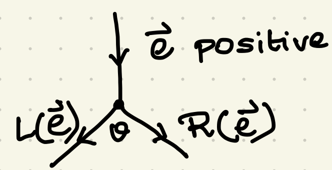
- Trivalent except at v_0
- $e_1, \dots, e_N$ edges at v_0
- Oriented edgset $\vec{E}(T)$: +ve if pointing away from the root

  Orient all the edges positively.

  $\vec{e} \to$ +ve or
  $-\vec{e} \to$ −ve or
- $d(v_0, \vec{e}) := d(v_0, \text{head of } \vec{e})$, $\vec{e} \in \vec{E}(T)$
- Spheres : $S(1)$



- Notion of left and right edges :

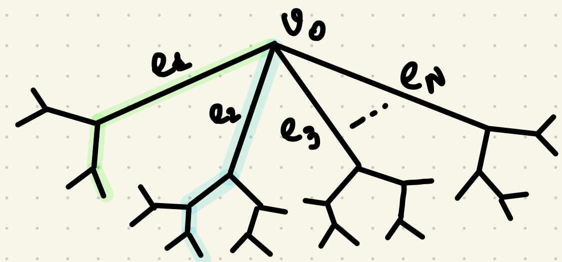


$(-\vec{e}, L, R)$ +vely oriented

Space of embedded paths

$\mathcal{P}$ = space of infinite embedded paths in T.

= { infinite sequences of L, R with base edge e_i }



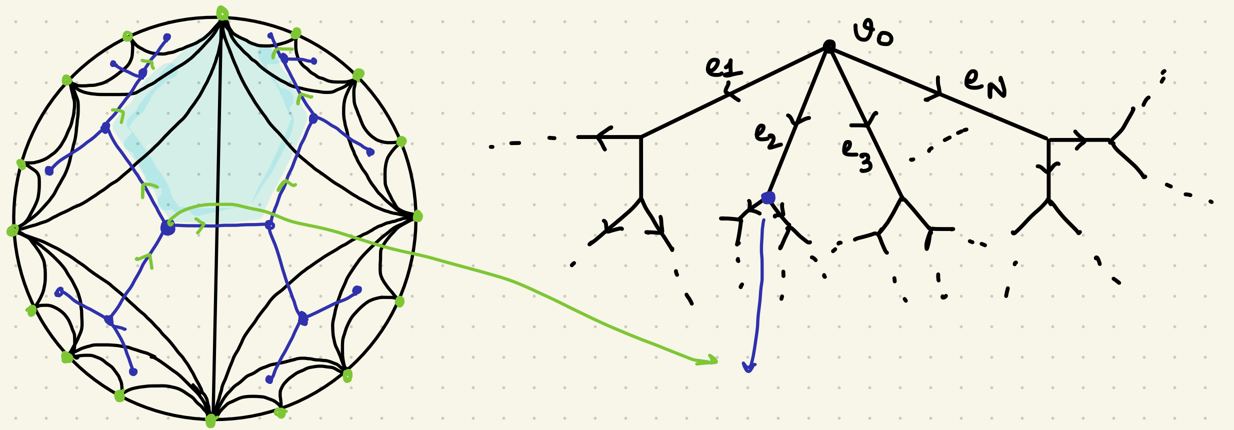
Rational paths := { eventually constant seqs }

Irrational paths := $\mathcal{P} \setminus$ Rational paths

Idea of L-T : { Measure the rational paths of a particular tree
Show that the measure of irrational paths is 0 (Birman Series) }

Complementary regions : Embed the tree on the plane

Ω

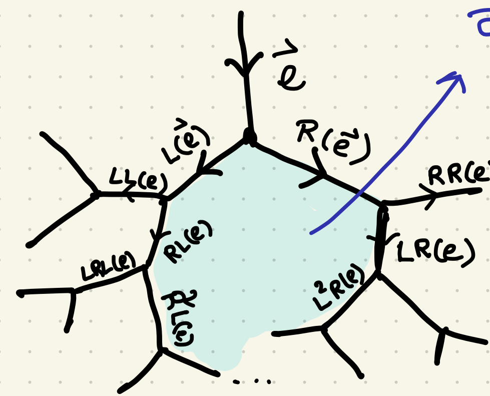


Every positive edge $\vec{e}$

$\rightsquigarrow$

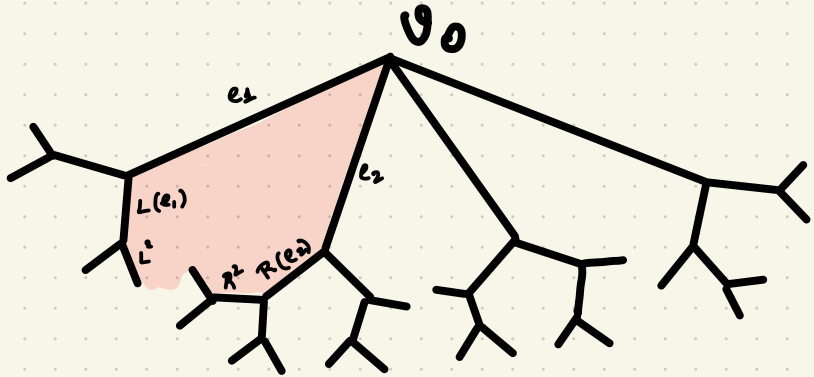
complementary region
(also denoted by 'e')

$$\partial^R e = L^n Re \, ; \quad \partial^L e = R^n L(e)$$



a given by
2 rational
paths of the
form
$L^nR(e)$ and
$R^nL(e$

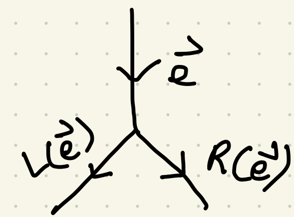
The remaining comp. regions ...



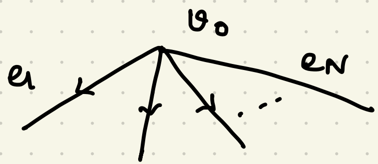
- Denoted by $f = (e_i, e_{i+1})$
- $\partial f = \underbrace{L^n e_i}_{\partial^L f} \cup \underbrace{R^n e_{i+1}}_{\partial^R f}$

Harmonic 1-form

$\Phi : \vec{E}(T) \longrightarrow \mathbb{R}$ satisfying $\begin{cases} \text{a) } \Phi(-\vec{e}) = -\Phi(\vec{e}) \quad \forall \vec{e} \in \vec{E}(T) \\ \text{b) } \Phi(\vec{e}) = \Phi(L(\vec{e})) + \Phi(R(\vec{e})) \end{cases}$



$S(1)$:



Green's formula: $\boxed{\sum_{\vec{e} \in S(n)} \Phi(\vec{e}) = \partial \Phi := \sum_{\vec{e} \in S(1)} \Phi(\vec{e})}$

Harmonic measure

Let Φ be a harmonic 1-form s.t $\Phi(\vec{e}) > 0$ whenever $\vec{e}$ positive.

Define
$$\mu_{\Phi} : \mathcal{P} \longrightarrow \mathbb{R}$$
$$p \longmapsto \lim_{n\to\infty} \Phi(\pi_n(p))$$

$\pi_n(p) \longrightarrow$ the +ve edge after the n-th step $\in S(n)$

$$\mathrm{Gap}_{\Phi}(\vec{e}) := \frac{1}{2}\left(\mu_{\Phi}(\partial^L e) + \mu_{\Phi}(\partial^R e)\right)$$

Thm 1

$$\sum_{\vec{e}\in\Omega} \mathrm{Gap}_{\Phi}(\vec{e}) \leq \partial \Phi$$

$\rightarrow$ measure of the rat. paths

$$\mathrm{Error}(\Phi) := \partial\Phi - \sum_{e\in\Omega} \mathrm{Gap}_{\Phi}(\vec{e})$$

$\rightarrow$ measure of the irrat paths.