Quantum integrable systems and Langlands program

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Algebraic-geometric methods in integrable systems:

	Classical	**Quantum**
Spectral curve	Characteristic polynomial of the Lax operator	Quantum characteristic polynomial
SoV	Divisor of the eigenbundle for the Lax operator	G-oper or Baxter equation
Solution	Abel transform	Hecke symmetries?

Plan

1. **Classical spectral curve method**
   - Hitchin construction
   - Singular curves
   - Gaudin system
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2. **Quantization problem**
   - Deformation quantization paradigm
   - Quantum spectral curve
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   - Quantum separated variables
3. **Langlands correspondence**
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   - Beilinson-Drinfeld scheme
   - Correspondence
4. **Hecke symmetries of Bethe solutions**
   - Monodromic formulation
   - Schlesinger transformation

Hitchin construction

Stable bundles and integrable systems. Duke Math. Journal 1987 V 54 N1 91-114.

Let Σ_0 be an algebraic curve, consider the moduli space $\mathcal{M} = \mathcal{M}_{r,d}(\Sigma)$ of stable holomorphic bundles of fixed rank r and degree d over Σ_0. There is a canonical simplectic structure on $T^*\mathcal{M}$. The deformation technics gives us quite explicit description of this space: a cotangent vector at a point E of the moduli space $\mathcal{M}$ is an element $\Phi \in H^0(End(E) \otimes \mathcal{K})$. There is a well defined function $h_i : T^*\mathcal{M} \to H^0(\mathcal{K}^{\otimes i})$ such that $h_i(E, \Phi) = \frac{1}{i} tr\Phi^i$. The map

$$h : T^*\mathcal{M} \longrightarrow \oplus_{i=1}^r H^0(\mathcal{K}^{\otimes i})$$

is called Hitchin map and realizes the algebraic integrability, which means that the fibers are abelian lagrangian varieties of half dimension. The crucial role is played by the spectral curve that is defined as follows. One can define the bundle morphism (non-linear) $char(\Phi) : \mathcal{K} \to \mathcal{K}^{\otimes r}$ by

$$char(\Phi)(\mu) = det(\Phi - \mu * Id)$$

where μ is a local section of $\mathcal{K}$ and Id is the identical global section of $End(E)$. The spectral curve is the preimage of the zero section in $\mathcal{K}^{\otimes r}$. It is an algebraic curve Σ in the projectivization of the total space of $\mathcal{K}$.

Line bundle

The solution of the Hitchin system (means the action-angle variables) are constructed by the following consideration: let π be the projection corresponding to the canonical bundle $\mathcal{K}$

$$\pi : \mathcal{K} \to \Sigma_0$$

let $\mathcal{L}$ be the line bundle on Σ defined as

$$0 \to \mathcal{L} \longrightarrow \pi^* E \stackrel{\Phi - \mu * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K})$$

The support of this sheaf coincides with the spectral curve and defines a line bundle. The Abel transform A maps the class of divisors to the Jacobian $Jac(\Sigma)$.

Theorem

The linear coordinates on $Jac(\Sigma)$ of the Abel transform image $A(\mathcal{L})$ provide the angle variables of the Hitchin system.

Singular curves

We enlarge the class of systems by considering the singular points and market points.

- Marked points: one can consider the moduli space of holomorphic bundles with the additional data - the trivialization at the market points: $\mathcal{M}_{r,d}(z_1, \dots, z_k)$. This means for example that the tangent vector to $\mathcal{M}_{r,d}(z_1, \dots, z_k)$ at the point E is of the form

$$T_E\mathcal{M}_{r,d}(z_1, \dots, z_k) \simeq H^1(End(E) \otimes \mathcal{O}(-\sum_{i=1}^{k} z_i))$$

  The cotangent vector is now meromorphic

$$\Phi \in H^0(End(E) \otimes \mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^{k} z_i))$$

- Singular points: in the singular situation (means that the curve Σ_0 has singularities for example of the node or cusp type) there is also a consistent formalism of the Hitchin system. One should realize the singular analog of the dualizing sheaf $\mathcal{K}$ and the holomorphic bundle - the algebraic one.

Example: CM system

Consider the rational curve with one node point $z_1 \leftrightarrow z_2$ (the ring of rational function is the subring of rational function f on $\mathbb{C}P^1$ such that $f(z_1) = f(z_2)$) and one marked point z_3. The dualizing sheaf (the sheaf realizing the Serre duality $H^1(\mathcal{F})^* \simeq H^0(\mathcal{F}^* \otimes \mathcal{K})$) has one global section $dz(\frac{1}{z-z_1} - \frac{1}{z-z_2})$. Consider the moduli space $\mathcal{M}$ of holomorphic bundles E of rank n on Σ_{node} with a fixed trivialization at the point z_3. It means that

$$T_E\mathcal{M} = H^1(End(E) \otimes \mathcal{O}(-p)).$$

We restrict to the principal cell of this moduli space which corresponds to the space of equivalence classes of matrices Λ with different eigenvalues. The cotangent space is the space of holomorphic sections of $End^*(E) \otimes \mathcal{K} \otimes \mathcal{O}(p)$. Such sections are matrix-valued functions of z of the form

$$\Phi(z) = \left(\frac{\Phi_1}{z - z_1} - \frac{\Phi_2}{z - z_2} + \frac{\Phi_3}{z - z_3} \right) dz$$

such that

$$\Phi_1\Lambda = \Lambda\Phi_2 \quad \text{and} \quad \Phi_1 - \Phi_2 + \Phi_3 = 0.$$

CM suite

The phase space is parameterized by $U \in GL(n)$ which is the trivialization at z_3 the matrix Λ which characterize the projective module over the structure ring $\mathcal{O}(\Sigma_{mode})$, the residues of the Higgs field Φ_i. In these coordinates the canonical symplectic form on $\mathcal{T}^*\mathcal{M}$ can be represented as follows

$$\omega = Tr(d(\Lambda^{-1}\Phi_1) \wedge d\Lambda) + Tr(d(U^{-1}\Phi_3) \wedge dU)$$

After the hamiltonian reduction with respect to the right action on $GL(n)$ on U and the adjoint action of $GL(n)$ on Φ_i, Λ one obtains the space parametrized by $(\Phi_3)_{ij} = f_{ij}, i \neq j$; the eigenvalues e^{2x_i} of the matrix Λ and the diagonal elements of the matrix $(\Phi_1)_{ii} = p_i$ with the following Poisson bracket

$$\{x_i, p_j\} = \delta_{ij}, \qquad \{f_{ij}, f_{kl}\} = \delta_{jk} f_{il} - \delta_{il} f_{kj}$$

The CM hamiltonian can be obtained as coefficient of $Tr\Phi^2(z)$ at $1/(z - z_1)^2$

$$H = Tr\Phi_1^2 = \sum_{i=1}^{n} p_i^2 - 4 \sum_{i \neq j} \frac{f_{ij} f_{ji}}{\sinh^2(x_i - x_j)}$$

Lax operator

The Gaudin model can be considered as the generalized Hitchin system on curves with marked points. One should take the curve $\Sigma = \mathbb{C}P^1$ with N marked points $z_1, \ldots, z_N$. The Higgs field in this case is $\Phi = L(z)dz$ where

$$L(z) = \sum_{i=1\ldots N} \frac{\Phi_i}{z - z_i}. \tag{1}$$

The expression $L(z)$ is traditionally called the Lax operator due to its crucial role in the context of commutation representation of the corresponding dynamics:

$$\dot{L} = [M, L]$$

for some matrix valued function M.
The residues of the Gaudin Lax operator Φ_i are the $n \times n$-matrices with values in $\mathfrak{gl}_n \oplus \ldots \oplus \mathfrak{gl}_n$ such that $(\Phi_i)_{kl}$ is the kl-th generator of the i-th copy of $\mathfrak{gl}_n$. We interpret the Lie algebra generators as functions on the dual space $\mathfrak{gl}_n^*$. The symmetric algebra $S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*]$ is endowed with the Kirillov Poisson bracket whiten in coordinates as follows:

$$\{(\Phi_i)_{kl}, (\Phi_j)_{mn}\} = \delta_{ij}(\delta_{lm}(\Phi_i)_{kn} - \delta_{nk}(\Phi_i)_{ml}).$$

R-matrix bracket

Let us introduce some notation:

- $\{e_i\}$ be a standard basis of $\mathbb{C}^n$ and $\{E_{ij}\}$ be a standard basis of $End(\mathbb{C}^n)$, that is $E_{ij}e_k = \delta_k^j e_i$;
- $e_{ij}^{(s)}$ are generators of the s-th copy of $\mathfrak{gl}_n \subset \oplus^N \mathfrak{gl}_n$.

Then the Lax operator can be represented in the form

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}$$

The Poisson structure can be summarized in the so-called R-matrix form:

$$\{L(z) \otimes L(u)\} = [R_{12}(z-u), L(z) \otimes 1 + 1 \otimes L(u)] \in End(\mathbb{C}^n)^{\otimes 2} \otimes S(\mathfrak{gl}_n)^{\otimes N},$$

with the classical Yang R-matrix

$$R(z) = \frac{P_{12}}{z} \qquad P_{12} v_1 \otimes v_2 = v_2 \otimes v_1 \qquad P_{12} = \sum_{ij} E_{ij} \otimes E_{ji}$$

Integrals

Characteristic polynomial

$$det(L(z)-\lambda)=0=\sum_{k=0}^{n} I_k(z)\lambda^{n-k}$$

Alternative basis of symmetric functions:

$$J_k(z)=TrL^k(z) \qquad k=1,\dots,n.$$

$$H_{2,k}=Res_{z=z_k}TrL^2(z)=\sum_{j\neq k}\frac{2Tr\Phi_k\Phi_j}{(z_k-z_j)}=2\sum_{j\neq k}\frac{\sum_{lm}e^{(k)}_{lm}e^{(j)}_{ml}}{z_k-z_j}$$

Proposition

The coefficients of the characteristic polynomial of $L(z)$ commute with respect to the Kirillov Poisson bracket

$$\{I_k(z),I_m(u)\}=0$$

Proof

Let $L_1(z) = L(z) \otimes 1$ and $L_2(u) = 1 \otimes L(u)$.

$$
\begin{aligned}
\{J_k(z), J_m(u)\} &= Tr_{12}\{L^k(z) \otimes L^m(u)\} \\
&= Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) \{L(z) \otimes L(u)\} L_1^{k-i-1}(z) L_2^{m-j-1}(u) \\
&= Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i}(z) L_2^{m-j-1}(u) \qquad (2) \\
&+ Tr_{12} \sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j}(u) \\
&- Tr_{12} \sum_{ij} L_1^{i+1}(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u) \qquad (3) \\
&- Tr_{12} \sum_{ij} L_1^i(z) L_2^{j+1}(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u)
\end{aligned}
$$

$$
2 + 3 = Tr_{12}[\sum_{ij} L_1^i(z) L_2^j(u) R_{12}(z-u) L_1^{k-i-1}(z) L_2^{m-j-1}(u), L_1(z)] = 0
$$

Separated variables

In the $\mathfrak{sl}_2$ case there is a SOV procedure: The Lax operator is defined be the formula:

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & -A(z) \end{pmatrix} \qquad det(L(z) - \lambda) = R(z, \lambda, h_1, \ldots, h_d)$$

Let us define the variables y_j as the zeroes of $C(z)$ and $w_j = A(y_j)$. They are canonical: $\{y_i, w_j\} = \delta_{ij}$ Let us consider the generating function $S(I, y)$ for the canonical transformation from y_j, w_j to the action-angle variables I_j, ϕ_j

$$w_j = \partial_{y_j} S \qquad \phi_j = \partial_{I_j} S$$

Due to the fact that the point (y_j, w_j) it the point of the spectral curve and the fact that the action variables are function on Hamiltonians one obtains the separation of S into the product

$$S(I, y_1, \ldots, y_d) = \prod_i s(I, y_i)$$

where the function $s(I, z)$ solves the equation

$$R(z, \partial_z s, h_1, \ldots, h_d) = 0$$

Correspondence

$$\begin{array}{rcl} \text{Poisson algebra } (\cdot,\{\circ,\circ\}) & \Leftrightarrow & \text{Associative algebra } (*) \\ \mathcal{A}_{cl} & & \mathcal{A} \simeq \mathcal{A}_{cl}[[\hbar]] \text{ as linear space} \end{array}$$

$$\begin{aligned} a * b &= a \cdot b + O(\hbar) \\ a * b - b * a &= \hbar\{a, b\} + O(\hbar^2) \end{aligned}$$

Let the classical algebra be $S(\mathfrak{gl}_n)$. There is a canonical quantization: let $U_\hbar(\mathfrak{gl}_n)$ be the slightly deformed universal enveloping algebra

$$U_\hbar(\mathfrak{gl}_n) = T^*(\mathfrak{gl}_n)[[\hbar]]/\{x \otimes y - y \otimes x - \hbar[x, y]\}$$

The classical limit is the limit $\hbar \to 0$.
An integrable system is a pair of Poisson algebra $\mathcal{A}_{cl}$ with the Poisson commutative subalgebra $\mathcal{H}_{cl}$ of an appropriate dimension $dim(Spec(\mathcal{H}_{cl})) = 1/2 dim(Spec(\mathcal{A}_{cl}))$.

$$\mathcal{H}_{cl} \subset \mathcal{A}_{cl} \Leftrightarrow \mathcal{H} \subset \mathcal{A}$$

such that

- $\mathcal{A} \simeq \mathcal{A}_{cl}[[\hbar]]$ as linear spaces, the map $lim : \mathcal{A} \to \mathcal{A}_{cl}$ is called the semiclassical limit;
- $\mathcal{H}$ is commutative;
- $lim : \mathcal{H} \simeq \mathcal{H}_{cl}$

Realization

On can consider just $U(\mathfrak{gl}_n)$ which is filtered by the degree $\{\mathcal{F}_i\}$ with the map to its associated graded algebra with the induced Poisson structure:

$$U(\mathfrak{gl}_n) \to Gr(U(\mathfrak{gl}_n)) = \oplus_i \mathcal{F}_i/\mathcal{F}_{i-1} = S(\mathfrak{gl}_n) \tag{4}$$

This means that for the generators $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$:

$$a * b = a \cdot b \mod \mathcal{F}_{i+j-1} \qquad a * b - b * a = \{a, b\} \mod \mathcal{F}_{i+j-2}$$

Classical part

$$\begin{array}{lcl} \mathcal{A}_{cl} & = & S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*] \\ \mathcal{H}_{cl} & - & \text{the subalgebra generated by the Gaudin hamiltonians} \end{array}$$

Quantum part

$$\mathcal{A} = U(\mathfrak{gl}_n)^{\otimes N} \qquad \mathcal{H} - ?$$

"Quantum" determinant

Let $B = \sum_{ij} E_{ij} \otimes B_{ij}$ be a matrix those elements do not commute $B_{ij} \in A$, the determinant of such a matrix is defined as follows

$$det(B) = \frac{1}{n!} \sum_{\tau,\sigma \in \Sigma_n} (-1)^{\tau\sigma} B_{\tau(1),\sigma(1)} \dots B_{\tau(n),\sigma(n)}$$

This definition coincide with the classical determinant for a matrix with commutative elements. Let A_n be the operator of antisimmetrization in $(\mathbb{C}^n)^{\otimes n}$

$$A_n v_1 \otimes \dots \otimes v_n = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(n)}.$$

Then

$$det(B) = Tr_{1\dots n} A_n B_1 \dots B_n$$

Here B_k is the following element of $End(\mathbb{C}^n)^{\otimes n} \otimes A$

$$B_k = \sum_{ij} 1 \otimes \dots \otimes \underbrace{E_{ij}}_{k} \otimes \dots \otimes 1 \otimes B_{ij}$$

Quantum spectral curve

Consider the quantum Lax operator:

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

Here $L(z)$ is a rational function in z with values in $End(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$. We define the quantum characteristic polynomial of the quantum Lax operator

$$det(L(z) - \partial_z) = \sum_{k=0}^{n} QI_k(z)\partial_z^{n-k} \quad (5)$$

Theorem

The coefficients $QI_k(z)$ commute

$$[QI_k(z), QI_m(u)] = 0$$

they quantize the classical Hamiltonians for the Gaudin model.

Yangian

The generators: $t_{ij}^{(k)}$ $i = 1, \dots, n$; $j = 1, \dots, n$; $k = 1, \dots, \infty$. Defining conditions can be formulated in terms of the following generating function:

$$T(z, \epsilon) = \sum_{i,j} E_{ij} \otimes t_{ij}(z, \epsilon), \qquad t_{ij}(z, \epsilon) = \delta_{ij} + \sum_{k} t_{ij}^{(k)} \epsilon^k z^{-k},$$

$$R(z - u, \epsilon) T_1(z, \epsilon) T_2(u, \epsilon) = T_2(u, \epsilon) T_1(z, \epsilon) R(z - u, \epsilon) \quad R(z) = 1 - \frac{\epsilon}{z} P_{12}$$

where $T_{1,2}$ take values in $End(\mathbb{C}^n)^{\otimes 2} \otimes Y(\mathfrak{gl}_n)$

$$T_1(z, \epsilon) = \sum_{i,j} E_{ij} \otimes 1 \otimes t_{ij}(z, \epsilon), \quad T_2(u, \epsilon) = \sum_{i,j} 1 \otimes E_{ij} \otimes t_{ij}(u, \epsilon).$$

Comultiplication homomorphism: $\Delta : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)^{\otimes 2}$:

$$(1 \otimes \Delta) T(z, \epsilon) = T^1(z, \epsilon) T^2(z, \epsilon) \qquad \Delta t_{ij}(z, \epsilon) = \sum_{k} t_{ik}(z, \epsilon) \otimes t_{kj}(z, \epsilon)$$

where $T^{1,2}$ take values in $End(\mathbb{C}^n) \otimes Y(\mathfrak{gl}_n)^{\otimes 2}$

$$T^1(z, \epsilon) = \sum_{i,j} E_{ij} \otimes t_{ij}(z, \epsilon) \otimes 1, \quad T^2(z, \epsilon) = \sum_{i,j} E_{ij} \otimes 1 \otimes t_{ij}(z, \epsilon).$$

Bethe subalgebra

[Molev A. I. Yangians and their applications. math.QA/0211288.]

Let us introduce the notations

$$T_m(z,\epsilon) = \sum_{ij} 1 \otimes \ldots \otimes 1 \otimes \overset{m}{E_{ij}} \otimes 1 \otimes \ldots \otimes 1 \otimes t_{ij}(z,\epsilon)$$

Then the following expressions

$$\tau_k(z,\epsilon) = TrA_n T_1(z,\epsilon) T_2(z-\epsilon,\epsilon) \ldots T_k(z-\epsilon(k-1),\epsilon) C_{k+1} \ldots C_n \quad k = 1, \ldots n$$

produce a commutative subalgebra in $Y(\mathfrak{gl}_n)$ for general C

$$[\tau_i(z,\epsilon), \tau_j(u,\epsilon)] = 0$$

There is the "evaluation representation" $\rho_{z_0} : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$

$$T_{z_0}(u,\epsilon) = 1 + \frac{\epsilon}{z-z_0} \sum_{i,j} E_{ij} \otimes e_{ij} \overset{def}{=}: 1 + \frac{\epsilon\Phi}{z-z_0} \tag{6}$$

Sketch of the proof

Fusion: the image of $T(z)$ by $\rho_{z_1} \otimes \ldots \otimes \rho_{z_N} \Delta^{N-1}$ is of the form

$$T^{\aleph}(u) = T^1_{z_1}(z) \ldots T^N_{z_N}(z) \in End(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$$

The image of the Bethe subalgebra can be organized into a generating function:

$$\begin{aligned} Q(z,\epsilon) &= TrA_n(e^{-\epsilon\partial_z} T^{\aleph}_1(z,\epsilon) - 1)(e^{-\epsilon\partial_z} T^{\aleph}_2(z,\epsilon) - 1)\ldots(e^{-\epsilon\partial_z} T^{\aleph}_n(z,\epsilon) - 1) \\ &= \sum_{j=0}^{n} \tau_j(z-\epsilon,\epsilon)(-1)^{n-j} C_n^j e^{-j\epsilon\partial_z} \\ &= det(e^{-\epsilon\partial_z} T^{\aleph}(z,\epsilon) - 1) \end{aligned}$$

The highest term of this expression in ϵ is of the form

$$det(e^{-\epsilon\partial_z} T^{\aleph}(z,\epsilon) - 1) = \epsilon^n det(L(z) - \partial_z) + O(\epsilon^{n+1})$$

due to the expansion:

$$e^{-\epsilon\partial_z} T^{\aleph}(z) - 1 = \epsilon(L(z) - \partial_z) + O(\epsilon^2).$$

$\mathfrak{sl}_2$ explicit form

Let us consider the Gaudin model for the $\mathfrak{sl}_2$ case

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \sum_i \frac{\Phi_i}{z - z_i}$$

where

$$\Phi_i = \begin{pmatrix} h_i/2 & e_i \\ f_i & -h_i/2 \end{pmatrix}$$

The quantum characteristic polynomial:

$$det(L(z) - \partial_z) = \partial_z^2 - \frac{1}{2}\sum_i \frac{c_i^{(2)}}{(z - z_i)^2} - \sum_i \frac{H_i}{z - z_i}$$

The Gaudin Hamiltonians are the residues

$$H_i = \sum_{i \neq j} \frac{h_i h_j/2 + e_i f_j + e_j f_i}{z_i - z_j}$$

Bethe vectors

Let us consider the $\mathfrak{sl}_2$ Gaudin model and fix the representation $V_\lambda = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$ where V_{λ_i} is the λ_i-highest weight irreducible representation.

Bethe ansatz

$$\Omega = \prod_{j=1}^{M} C(\mu_j)|vac>$$

is an eigenvector if the parameters μ_j satisfy the system of algebraic equation (Bethe system)

$$-\frac{1}{2}\sum_i \frac{\lambda_i}{\mu_j - z_i} + \sum_{k\neq j} \frac{1}{\mu_j - \mu_k} = 0 \quad j = 1, \ldots, M \tag{7}$$

The eigenvalues of H_i on Ω are given by

$$H_i^{\Omega} = -\lambda_i \left(\sum_j \frac{1}{z_i - \mu_j} - \frac{1}{2}\sum_{j\neq i} \frac{\lambda_j}{z_i - z_j} \right)$$

return

Proof

In the considered case the QCP takes the form:

$$det(L(z)-\partial_z)=\partial_z^2-A^2(z)-C(z)B(z)+A'(z)=\partial_z^2-H(z)$$

One has the commutation relations:

$$[A(z),B(z)]=-B'(z)\qquad [A(z),C(u)]=\frac{1}{z-u}(C(z)-C(u))$$

$$[A(z),C(z)]=C'(z)\qquad [B(z),C(u)]=\frac{2}{u-z}(A(z)-A(u))$$

Using this relations and the condition:

$$H(z)|vac>=\left(\frac{1}{4}(\sum_i\frac{\lambda_i}{z-z_i})^2-\frac{1}{2}\sum_i\frac{\lambda_i}{(z-z_i)^2}\right)|vac>=h_0(z)|vac>$$

one obtains:

$$\begin{aligned} H(z)\Omega &= \left(h_0(z)+2\sum_{j=1}^M\frac{1}{\mu_j-z}A(z)+\sum_{j\neq k}\frac{1}{(\mu_j-z)(\mu_k-z)}\right)\Omega \\ &+ 2C(z)\sum_{j=1}^M\frac{1}{z-\mu_j}\prod_{l\neq j}C(\mu_l)\left(\sum_{k\neq j}\frac{1}{\mu_k-\mu_j}+A(\mu_j)\right) \end{aligned}$$

Quantum separated variables

Let us consider the $\mathfrak{sl}_2$ Gaudin model and fix the representation $V_\lambda = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$ where V_{λ_i} is the λ_i-highest weight module. Let us realize V_{λ_i} as the factor space of the Verma module $\mathbb{C}[t_i]/t_i^{\lambda_i+1}$ such that the generators of $\mathfrak{sl}_2$ take the form:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \lambda_s, e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \lambda_s \frac{\partial}{\partial t_s}, f^{(s)} = t_s.$$

Let us introduce the separated variables y_j be the formula:

$$C(z) = C_0 \frac{\prod_j (z - y_j)}{\prod_i (z - z_i)}$$

They are elements of an algebraic extension of the field $\mathbb{C}[t_1, \ldots, t_N]$.
Let now Ω be the common eigenvector of Gaudin Hamiltonians in the tensor product of Verma modules $\mathbb{C}[t_1, \ldots, t_N]$

$$H(z)\Omega = h(z)\Omega \tag{8}$$

Consider both parts of 8 as a rational function on z placing z on the left and substituting $z = y_j$ one obtains:

$$\begin{aligned} H(y_j) &= A^2(y_j) - A'(y_j) \\ &= \frac{1}{4}\sum_{i,k} \frac{1}{(y_j - z_i)(y_j - z_k)} h_i h_k + \frac{1}{2}\sum_k \frac{1}{(y_j - z_k)^2} h_k \end{aligned}$$

Using the definition of the separated variables one has:

$$\partial_{y_j} = \sum_k \frac{\partial t_k}{\partial y_j}\partial_{t_k} = \sum_k \frac{t_k}{y_j - z_k}\partial_{t_k}$$

Substituting all this we get:

$$\left(-\partial_{y_j} + \frac{1}{2}\sum_k \frac{\lambda_k}{y_j - z_k}\right)^2 \Omega = h(y_j)\Omega$$

Hence the common eigenfunction factorizes:

$$\Omega = \prod_j \omega(y_j)$$

such that the function $\widetilde{\omega}(z) = \prod_i (z - z_i)^{-\lambda_i/2}\omega(z)$ satisfies an equation:

$$(\partial_z^2 - h(z))\widetilde{\omega}(z) = 0$$

Monodromy

The proposed arguments demonstrate that the spectrum description for the quantum model is related to the special differential operators, those specialty is some vanishing condition of the monodromy. In the $\mathfrak{sl}_2$ case we obtained: if Ω is a Bethe vector with eigenvalues H_i^Ω then the equation

$$\left(\partial^2 - \frac{1}{4}\sum_i \frac{\lambda_i(\lambda_i+2)}{(z-z_i)^2} - \sum_i \frac{H_i^\Omega}{z-z_i}\right)\Psi(z) = 0 \tag{9}$$

has solution of the form

$$\Psi(z) = \prod_i (z-z_i)^{-\lambda_i/2}\prod_j (z-\mu_j)$$

where μ_j satisfy the system of Bethe equations.
This observation is summarized by the Mukhin, Tarasov, Varchenko theorem

Theorem (math.AG/0512299)

There is a one-to-one correspondence between the differential equation of the form 9 with the monodromy ± 1 and the common eigenvectors for the Gaudin model.

This is generalized to the $\mathfrak{sl}_n$ case. return

AKS

This construction is related to the Adler-Kostant-Symes scheme which is in the origin of the large class of commutative subalgebras. Let $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ be a finite dimensional Lie algebra which is a direct sum of two its Lie subalgebras. One has an isomorphism of linear spaces related to some normal ordering

$$\phi : U(\mathfrak{g}) \to U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-)$$

We introduce the opposite Lie algebra structure on the linear space $\mathfrak{g}_-$ defined by $-\{\circ, \circ\}$. The corresponding universal enveloping algebras can be identified as linear spaces with respect to some PBW basis:

$$U(\mathfrak{g}_-^{op}) \simeq U(\mathfrak{g}_-).$$

Lemma

The center $Z \subset U(\mathfrak{g})$ maps to a commutative subalgebra in $U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_-^{op})$ via ϕ.

Proof

Let us denote the commutator in $U(\mathfrak{g}_+)\otimes U(\mathfrak{g}_-^{op})$ by $[*,*]_R$. Let c_1, c_2 be two central elements in $U(\mathfrak{g})$ taken in the form

$$c_i=\sum_j x_j^{(i)}y_j^{(i)} \qquad x_j^{(i)}\in U(\mathfrak{g}_+),\ y_j^{(i)}\in U(\mathfrak{g}_-)$$

$$\begin{aligned}[\phi(c_1),\phi(c_2)]_R &= [\sum_j x_j^{(1)}y_j^{(1)},\sum_k x_k^{(2)}y_k^{(2)}]_R\\ &= \sum_{j,k}[x_j^{(1)},x_k^{(2)}]_R y_j^{(1)}y_k^{(2)}+x_j^{(1)}x_k^{(2)}[y_j^{(1)},y_k^{(2)}]_R\end{aligned}$$

Due to the definition of the algebraic structure

$$[x_j^{(1)},x_k^{(2)}]_R=[x_j^{(1)},x_k^{(2)}] \qquad [y_j^{(1)},y_k^{(2)}]_R=-[y_j^{(1)},y_k^{(2)}]$$

$$[\phi(c_1),\phi(c_2)]_R=\sum_k[c_1,x_k^{(2)}]y_k^{(2)}-\sum_j x_j^{(1)}[y_j^{(1)},c_2]$$

which is zero due to the centrality of c_1, c_2 $\square$

Center of $U_c(\widehat{\mathfrak{sl}_n})$

Consider $U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n^{op}[t])$ and the corresponding Lax operator:

$$L_{full}(z) = \sum_{i=-\infty,\infty} \Phi_i z^{-i-1}$$

The commutation relations can be represented in the same form as for the Gaudin Lax operator

$$[L^1_{full}(z), L^2_{full}(u)] = [\frac{P}{z-u}, L^1_{full}(z) + L^2_{full}(u)]$$

Theorem

The center of $U_c(\widehat{\mathfrak{gl}_n})$ is isomorphic as a commutative algebra to the commutative subalgebra in $U(\mathfrak{gl}_n[t^{-1}] \oplus t\mathfrak{gl}_n^{op}[t])$ defined by the the coefficients of the characteristic polynomial $det(L_{full}(z) - \partial_z)$. The isomorphism is given by the map I,

$$I : U(\mathfrak{gl}_n[t^{-1}]) \otimes U(t\mathfrak{gl}_n^{op}[t]) \to U_c(\widehat{\mathfrak{gl}_n}), \qquad I : h_1 \otimes h_2 \to h_1 h_2, \tag{10}$$

[B. Feigin, E. Frenkel, Int. J. Mod. Phys. A7, Suppl. 1A 1992, 197-215]

Beilinson-Drinfeld scheme

Preprint 1991: Quantization of Hitchin's integrable system and Heche eigensheaves.

Let Σ be a connected smooth projective curve over $\mathbb{C}$ of genus $g > 1$, G - a semisimple Lie group, $\mathfrak{g}$ its Lie algebra. Bun_G is the moduli stack of G-bundles on Σ. Main results:

- There is a commutative ring of differential operators $\mathfrak{z}(\Sigma)$ defined on the canonical bundle K_{Bun_G}, the symbol map produces the commutative subalgebra of classical Hitchin hamiltonians on T^*Bun_G.
- The spectrum of $\mathfrak{z}$ identifies canonically with the moduli of ${}^L\mathfrak{g}$-opers.(If $G = SL_2$ then a ${}^L\mathfrak{g}$ is just the Sturm-Liouville operator on Σ.)
- To each ${}^L\mathfrak{g}$-oper one assigns a D-module on Bun_G specializing the values of Hitchin hamiltonians. This D-module is an eigensheaf for the Hecke algebra whose eigenvalue is the corresponding ${}^L\mathfrak{g}$-oper.

The origin of this commutative algebra is the center of $U_c(\widehat{\mathfrak{g}})$ whose action on the loop group can be lifted to the moduli space due to the following realizations of $Bun_G(\Sigma)$:

$$Bun_G(\Sigma) \simeq G(F)\backslash G(\mathcal{A}_F)/G(\mathcal{O}_F) \simeq G_{in}\backslash G[[z, z^{-1}]/G_{out}$$

where G_{in} and G_{out} correspond to some covering of the curve Σ by two open sets: U_{in} - small disk around a point x_0 with the local parameter z and $U_{out} = \Sigma\backslash x_0$.

Number field

Let F be a number field (a finite extension of $\mathbb{Q}$), $\bar{F}$ - its maximal algebraic extension, F^{ab} - its maximal abelian extension. For $F \subset F'$ the Galois group is

$$\begin{aligned} Gal(F', F) &= \{\sigma \in Aut(F') : \sigma(x) = x \ \ \forall x \in F\} \\ Gal(\mathbb{C}, \mathbb{R}) &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

Abelian reciprocity law

$$Gal(F^{ab}, F) \simeq \text{The group of connected components of } F^{\times} \backslash \mathcal{A}_F^{\times}$$

***n*-dimensional generalization**

$$Rep_n(Gal(\bar{F}, F)) \Leftrightarrow Rep(GL_n(\mathcal{A}_F)) \text{ in functions on } GL_n(F) \backslash GL_n(\mathcal{A}_F)$$

Why the right hand side is called automorphic

$$f((az+b)/(cz+d)) = \chi(a)(cz+d)^k f(z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$$SL_2(\mathbb{R})/SL_2(\mathbb{Z}) \simeq K \backslash GL_2(\mathcal{A}_{\mathbb{Q}})/GL_2(\mathbb{Q})$$

Cases F:

- number field
- field of functions on an algebraic curve over a finite field F_q (Laforgue)
- field of function on an algebraic curve over the field $\mathbb{C}$

Transformations over $\mathbb{C}$:
Galois side

$$Rep_n(Gal) \Rightarrow \text{ flat connections on rank } n \text{ bundles}$$

Automorphic side

$$\text{Hitchin } D\text{-module over } \; GL(F)\backslash GL(\mathcal{A}_F)/GL(\mathcal{O}_F) \simeq Bun_n(\Sigma)$$

New features:

$$\text{Hitchin } D\text{-module} \overset{FF,BD}{\Leftrightarrow} \text{Character } \chi \text{ on } \mathfrak{z}(U_c(\widehat{\mathfrak{g}})) \overset{CT}{\Leftrightarrow} \chi det(L_{full} - \partial_z)$$

return

Matrix monodromy condition

Monodromy Langlands

Let us consider the Fuchsian system

$$(\partial_z - A(z))\Psi(z) = 0 \qquad A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \sum_{i=1}^{k} \frac{A_i}{z - z_i}$$

with additional conditions

$$Tr(A_i) = 0; \quad Det(A_i) = -d_i^2; \quad \sum_i A_i = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}.$$

This system is related with the following Sturm-Liouville operator

$$\Phi'' + U\Phi = 0 \quad \Phi = \psi_1/\sqrt{a_{12}}$$

where the potential is given by the formula

$$U = \sum_{j=1}^{k-2} \frac{-3/4}{(z - w_j)^2} + \sum_{i=1}^{k} \frac{1/4 + detA_i}{(z - z_i)^2} + \sum_{j=1}^{k-2} \frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k} \frac{H_{z_i}}{z - z_i}$$

and the points w_j are defined by the condition

$$a_{12}(z) = \frac{c\prod_{j=1}^{k-2}(z - w_j)}{\prod_{i=1}^{k}(z - z_i)}$$

Theorem, math-ph 0802.0383

The equation

$$(\partial_z - A(z))\Psi = 0 \tag{11}$$

has solutions of the form

$$\Psi = \prod_{i=1}^{k}(z - z_i)^{-d_i} \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix} \tag{12}$$

$$\phi_1 = \prod_{j=1}^{M}(z - \gamma_j); \quad \phi_2/\phi_1 = \sum_{j=1}^{M} \frac{\alpha_j}{z - \gamma_j} \tag{13}$$

iff the set of numbers γ_i where $i = 1, \ldots, M$ satisfy the system of Bethe equations with parameters: the set of poles is $z_1, \ldots, z_k$ and $w_1, \ldots, w_{k-2}$ with the highest weights $2d_1 - 1, \ldots, 2d_k - 1$ and $1, \ldots, 1$ correspondingly.

bethe

Schlesinger transformation

Action on bundles

Consider C - a curve, E a holomorphic bundle on C, $\mathcal{F}$ - the corresponding sheaf, $x \in C$ and $l \in E_x^*$ then the lower Hecke transformation $T_{(x,l)}E$ is defined in the language of sheaves as the subsheaf $\mathcal{F}' = \{s \in \mathcal{F} : (s(x), l) = 0\}$

In terms of gluing functions, for example the $T_{0,l}$ of the trivial rank 2 bundle on $\mathbb{C}P^1$ produces a degree 1 bundle given by the gluing function

$$\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$

Action on connections

A connection is the map of sheaves

$$\Delta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1$$

The Schlesinger action can be extended to the space of connection preserving the $Ann_l = \{v \in \mathcal{F}_x :< l, v >= 0\}$

$$\Delta_x : Ann_l \to Ann_l \otimes \Omega_x^1$$

For our particular case we consider compositions of pairs of Hecke transformations located at singular points $z_i,\ z_j$ of the connection and preserving the trivial rank 2 bundle. We consider the covering $U_0,\ U_\infty$ of $\mathbb{C}P^1$ where U_0 is an open disk containing $z_i,\ z_j$ and $U_\infty = \mathbb{C}P^1 \backslash \{z_1, z_2\}$. In terms of gluing functions this is represented by $G_{ij}(z) = G_i(z)G_j^{-1}(z)$ where

$$G_s(z) = G_s \begin{pmatrix} z - z_s & 0 \\ 0 & 1 \end{pmatrix} G_s^{-1}$$

the G_s are chosen by the condition

$$A_s = G_s \begin{pmatrix} \pm d_s & 0 \\ 0 & \mp d_s \end{pmatrix} G_s^{-1} \quad G_s = \begin{pmatrix} 1 & x_s \\ y_s & 1 \end{pmatrix}$$

To obtain the action on connections in the trivial rank 2 bundle one has to perform the decomposition $G_{ij}(z) = G_0(z)G_\infty(z)$ where $G_0(z),\ G_\infty(z)$ are invertible respectively in $U_0,\ U_\infty$.

$$G_\infty(z) = \begin{pmatrix} \frac{z(1-x_jy_i)(1-x_jy_j)-x_j(y_i-2y_j-x_jy_iy_j)}{(1-2x_jy_i+x_jy_j)(1-x_jy_i)(1-x_jy_j)(z-1)} & -\frac{x_j}{(1-x_jy_i)(1-x_jy_j)(z-1)} \\ \frac{y_i-2y_j+x_jy_iy_j}{(1-x_jy_i)(1-2x_jy_i+x_jy_j)} & \frac{1}{1-x_jy_i} \end{pmatrix}.$$

Then one should realize the gauge transformation with $G_\infty(z)$

$$A \mapsto G_\infty A G_\infty^{-1} + \partial_z G_\infty G_\infty^{-1}$$

The local consideration shows that the eigenvalues of the residues A_i transform depending on the choice of the subspace and the type (upper or lower) of the Schlesinger transformations correspondingly to the following table

$$\begin{array}{ccc}
(\dots,\lambda_i,\dots,\lambda_j,\dots) & \longmapsto & (\dots,\lambda_i+1,\dots,\lambda_j-1,\dots), \\
(\dots,\lambda_i,\dots,\lambda_j,\dots) & \longmapsto & (\dots,\lambda_i+1,\dots,\lambda_j+1,\dots), \\
(\dots,\lambda_i,\dots,\lambda_j,\dots) & \longmapsto & (\dots,\lambda_i-1,\dots,\lambda_j-1,\dots), \\
(\dots,\lambda_i,\dots,\lambda_j,\dots) & \longmapsto & (\dots,\lambda_i-1,\dots,\lambda_j+1,\dots).
\end{array}$$