Travaux mathématiques

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Special issue
based on the
School GEOQUANT at the ICMAT
Madrid, Spain, September 2015

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Editor
Martin Schlichenmaier

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Geometry and Quantization

Lectures presented

at the School GEOQUANT

September 2015

ICMAT (Instituto de Ciencias Matemáticas)

Madrid, Spain

Edited by

Martin Schlichenmaier

International School and Conference on Geometry and Quantization (GEOQUANT 2015)
ICMAT, Campus de Cantoblanco, Madrid, Spain,
September 7-18, 2015,

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HORIZON 2020

Foreword

In September 2015 the *International School and Conference on Geometry and Quantization (GEOQUANT)* took place at the ICMAT (Instituto de Ciencias Matemáticas). The scientific topics discussed in Madrid were

- concepts of differential and complex geometry arising in quantization
- relations between quantization and the geometry of moduli spaces, in particular of the moduli space of Higgs bundles
- algebraic aspects of quantization, in particular, infinite-dimensional Lie algebras and groups and their representations
- asymptotic geometric analysis
- relations with modern theoretical physics
- non-commutative quantum field theory.

The activity lasted for two weeks. The first week was a school consisting of four lecture courses aiming at the newcomer to the field. The second week was a scientific conference.

This volume of the *Travaux Mathématiques* is mainly based on the school. The focus on the 2015 Geoquant school was on the quantization of moduli spaces of bundles, Higgs bundles, Hitchin systems, wall crossing, hyperKähler geometry, quantum gravity, cluster algebras, and quantum Hall effect. Those topics are clearly in the center of the current interest in the field. The aim of this volume is to give an introduction to some of the hot topics of ongoing research in the field. Furthermore, it was the desire of the participants to have some written material of the courses available. We asked the lecturers whether they would be able to produce such a write-up of their lectures. We are happy that most of them could manage to contribute. In addition, we ask for additional contribution related to the topics of the school and complementing them. For a first orientation on the content of the lectures we recommend the collection of their summaries given further down.

There were further solicited articles related to the topics. They will appear in a forthcoming volume of the *Travaux Mathématiques*. It will be dedicated to contributions originating from the *Centre for Quantum Geometry of Moduli Spaces (QGM)*, Aarhus, Denmark (director Jørgen Ellegaard Andersen).

The organizers thank all lecturers contributing to the success of the school and conference in an essential manner. Furthermore, we thank the participants for their active role and for their very positive feedback. We thank very much the ICMAT under its former director Manuel de Leon for its hospitality. All participants enjoyed very much the place, the surrounding, and the facilities. Last, but not least, we thank the following institutions for generous financial support: ICMAT, University of Luxembourg, Foundation Compositio Mathematica, and the EU COST research network QSPACE (Quantum Structure of Space Time).

The Geoquant activity is an international scientific event, highly respected and well-known in the community. The current event was the 6th in a biannual series. The previous school/conferences were held in: (1) Japan (Tokyo, Nagoya) (2005), (2) Russia (Moscow) (2007), (3) Luxembourg (Luxembourg City) (2009), (4) China (Beijing, Tianjin) (2011), and (5) Austria (Vienna) (2013).

Martin Schlichenmaier (for the organisers)

Collection of Summaries

Mario Garcia-Fernandez: Lectures on the Strominger system

These notes give an introduction to the Strominger system of partial differential equations, and are based on lectures given in September 2015 at the GEOQUANT School, held at the Institute of Mathematical Sciences (ICMAT) in Madrid. We describe the links with the theory of balanced metrics in hermitian geometry, the Hermite-Yang-Mills equations, and its origins in physics, that we illustrate with many examples. We also cover some recent developments in the moduli problem and the interrelation of the Strominger system with generalized geometry, via the cohomological notion of string class.

Semyon Klevtsov: Geometry and large N limits in Laughlin states

In these notes I survey geometric aspects of the lowest Landau level wave functions, integer quantum Hall state and Laughlin states on compact Riemann surfaces. In particular, I review geometric adiabatic transport on the moduli spaces, derivation of the electromagnetic and gravitational anomalies, Chern-Simons theory and adiabatic phase, and the relation to holomorphic line bundles, Quillen metric, regularized spectral determinants, bosonisation formulas on Riemann surfaces and asymptotic expansion of the Bergman kernel.

Tomoki Nakanishi and Dylan Rupel: Companion cluster algebras to a generalized cluster algebra

We study the c-vectors, g-vectors, and F-polynomials for generalized cluster algebras satisfying a normalization condition and a power condition recovering classical recursions and separation of additions formulas. We establish a relationship between the c-vectors, g-vectors, and F-polynomials of such a generalized cluster algebra and its (left- and right-) companion cluster algebras. Our main result states that the cluster variables and coefficients of the (left- and right-) companion cluster algebras can be recovered via a specialization of the F-polynomials.

David Alfaya and Tomas L. Gómez: On the Torelli theorem for Deligne-Hitchin moduli spaces

We prove a Torelli theorem for the parabolic Deligne-Hitchin moduli space, and compare it with previous Torelli theorems for non-parabolic Deligne-Hitchin moduli spaces.

Indranil Biswas, Ugo Bruzzo, Beatriz Graña Otero, and Alessio Lo Giudice: Yang–Mills–Higgs connections on Calabi–Yau manifolds, II

In this paper we study Higgs and co–Higgs G–bundles on compact Kähler manifolds X. Our main results are:

1. If X is Calabi–Yau (i.e., it has vanishing first Chern class), and (E, θ) is a semistable Higgs or co–Higgs G–bundle on X, then the principal G–bundle E is semistable. In particular, there is a deformation retract of $\mathcal{M}_H(G)$ onto $\mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli space of semistable principal G–bundles with vanishing rational Chern classes on X, and analogously, $\mathcal{M}_H(G)$ is the moduli space of semistable principal Higgs G–bundles with vanishing rational Chern classes.

2. Calabi–Yau manifolds are characterized as those compact Kähler manifolds whose tangent bundle is semistable for every Kähler class, and have the following property: if (E, θ) is a semistable Higgs or co–Higgs vector bundle, then E is semistable.

Peter B. Gothen: Hitchin Pairs for non-compact real Lie groups

Hitchin pairs on Riemann surfaces are generalizations of Higgs bundles, allowing the Higgs field to be twisted by an arbitrary line bundle. We consider this generalization in the context of G-Higgs bundles for a real reductive Lie group G. We outline the basic theory and review some selected results, including recent results by Nozad and the author on Hitchin pairs for the unitary group of indefinite signature $\mathrm{U}(p, q)$.

André Oliveira: Quadric bundles applied to non-maximal Higgs bundles

We present a survey on the moduli spaces of rank 2 quadric bundles over a compact Riemann surface X. These are objects which generalise orthogonal bundles and which naturally occur through the study of the connected components of the moduli spaces of Higgs bundles over X for the real symplectic group $\mathrm{Sp}(4, \mathbb{R})$, with non-maximal Toledo invariant. Hence they are also related with the moduli space of representations of $\pi_1(X)$ in $\mathrm{Sp}(4, \mathbb{R})$. We explain this motivation in some detail.

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Lectures on the Strominger system

by Mario Garcia-Fernandez

Abstract

These notes give an introduction to the Strominger system of partial differential equations, and are based on lectures given in September 2015 at the GEOQUANT School, held at the Institute of Mathematical Sciences (ICMAT) in Madrid. We describe the links with the theory of balanced metrics in hermitian geometry, the Hermite-Yang-Mills equations, and its origins in physics, that we illustrate with many examples. We also cover some recent developments in the moduli problem and the interrelation of the Strominger system with generalized geometry, via the cohomological notion of string class.

1 Introduction

The Strominger system of partial differential equations has its origins in supergravity in physics [64, 93], and it was first considered in the mathematics literature in a seminal paper by Li and Yau [76]. The mathematical study of this PDE has been proposed by Yau as a natural generalization of the Calabi problem for non-kählerian complex manifolds [104], and also in relation to *Reid's fantasy* on the moduli space of projective Calabi-Yau threefolds [88]. There is a conjectural relation between the Strominger system and conformal field theory, which arises in a certain physical limit in compactifications of the heterotic string theory.

In complex dimensions one and two, solutions of the Strominger system are given (after conformal re-scaling) by polystable holomorphic vector bundles and Kähler Ricci flat metrics. In dimension three, the existence and uniqueness problem for the Strominger system is still open, and it is the object of much current investigation (see Section 5.2). The existence of solutions has been conjectured by Yau under natural assumptions [105] (see Conjecture 5.10).

The main obstacle to prove the existence of solutions in complex dimension three and higher is an intricate equation for 4-forms

$$dd^c\omega = \operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A, \tag{1.1}$$

coupling the Kähler form ω of a (conformally) balanced hermitian metric on a complex manifold X with a pair of Hermite-Yang-Mills connections ∇ and A.

This subtle condition – which arises in the quantization of the sigma model for the heterotic string – was studied by Freed [38] and Witten [101] in the context of index theory for Dirac operators, and more recently it has appeared in the topological theory of *string structures* [15, 87, 91] and in generalized geometry [12, 47, 48]. Despite these important topological and geometric insights, to the present day we have a very poor understanding of equation (1.1) from an analytical point of view.

In close relation to the existence problem, an important object in the theory of the Strominger system is the moduli space of solutions. The moduli problem for the Strominger system is largely unexplored, and only in recent years there has been progress in the understanding of its geometry [6, 24, 48, 49]. From a physical perspective, it can be regarded as a first approximation to the moduli space of 2-dimensional (0,2)-superconformal field theories, and is expected to host a generalization of mirror symmetry [104].

Organization:

These lecture notes intend to give an introduction to the theory of the Strominger system, going from classical hermitian geometry to the physical origins of the equations, and its many legs in geometric analysis, algebraic geometry, topology, and generalized geometry. Hopefully, this manuscript also serves as a guide to the vast literature in the topic.

In Section 2 we give an introduction to the theory of balanced metrics in hermitian geometry (in the sense of Michelson [79]). In Section 3 we study balanced metrics in non-kählerian Calabi-Yau manifolds and introduce the dilatino equation, one of the building blocks of the Strominger system. In Section 4 we go through the theory of Hermite-Einstein metrics on balanced manifolds, and its relation with slope stability. Section 5 is devoted to the definition of the Strominger system and the existence of solutions. In Section 5.2 we discuss several methods to find solutions of the equations, that we illustrate with examples, and comment on Yau's Conjecture for the Strominger system (Conjecture 5.10).

In Section 6 we review the physical origins of the Strominger system in string theory. This provides an important motivation for its study, and reveals the links of the Strominger system with conformal field theory, and the theory of string structures. For the physical jargon, we refer to the Glossary in [26]. Finally, in Section 7 we consider recent developments in the geometry of the Strominger system, based on joint work of the author with Rubio and Tipler [48]. As we will see, the interplay of the Strominger system with the notion of string class [87] leads naturally to an interesting relation with Hitchin's theory of generalized geometry [63], that we discuss in the context of the moduli problem in Section 7.2.

Acknowledgements: I would like to thank Bjorn Andreas – who introduced me to this topic –, Luis Álvarez-Cónsul, Xenia de la Ossa, Antonio Otal, Roberto

Rubio, Eirik Svanes, Carl Tipler, and Luis Ugarte for useful discussions and comments about the manuscript. I thank the organizers of GEOQUANT 2015 for the invitation to give this lecture course, and for their patience with the final version of this manuscript.

2 Special metrics in hermitian geometry

2.1 Kähler, balanced, and Gauduchon metrics

Let X be a compact complex manifold of dimension n, with underlying smooth manifold M. A hermitian metric on X is a riemannian metric g on M such that $g(J\cdot, J\cdot) = g$, where $J\colon TM \to TM$ denotes the almost complex structure determined by X. Denote by Ω^k (resp. $\Omega^k_{\mathbb{C}}$) the space of real (resp. complex) smooth k-forms on M. Denote by $\Omega^{p,q} \subset \Omega^{p+q}_{\mathbb{C}}$ the space of smooth complex $(p+q)$-forms on X of type (p,q). Note that $\Omega^{p,q}$ belongs to the eigenspace of $\Omega^{p+q}_{\mathbb{C}}$ with eigenvalue i^{q-p} with respect to the endomorphism

$$\alpha \to (-1)^{p+q}\alpha(J\cdot, \ldots, J\cdot).$$

Associated to g there is a canonical non-degenerate $(1,1)$-form $\omega \in \Omega^{1,1}$, defined by

$$\omega(V,W) = g(JV,W)$$

for any pair of vector fields V, W on M. The 2-form ω is called the *Kähler form* of the hermitian manifold (X, g).

By integrability of the almost complex structure, we have the decomposition of the exterior differential $d = \partial + \bar{\partial}$ acting on $\Omega^{p,q}$, where ∂ and $\bar{\partial}$ are given by projection

$$\partial\colon \Omega^{p,q} \to \Omega^{p+1,q}, \qquad \bar{\partial}\colon \Omega^{p,q} \to \Omega^{p,q+1}.$$

Consider the operator $d^c = i(\bar{\partial} - \partial)$ acting on forms $\Omega^k_{\mathbb{C}}$. We have the following special types of hermitian structures.

Definition 2.1. A hermitian metric g on X is

1. Kähler if $d\omega = 0$,
2. balanced if $d\omega^{n-1} = 0$,
3. Gauduchon if $dd^c(\omega^{n-1}) = 0$.

Remark 2.2. There are other important notions of special hermitian metrics (see e.g. [37, 86]), such as pluriclosed metrics or astheno-Kähler metrics (given by the conditions $dd^c\omega = 0$ and $dd^c(\omega^{n-2}) = 0$, respectively), but their study goes beyond the scope of the present notes.

The Kähler condition for g is equivalent to $\nabla^g J = 0$, where ∇^g denotes the Levi-Civita connection of the riemannian metric g (see [54, Section 1.1]). Using that $dd^c = -d^c d$, we have a simple chain of implications:

$$\text{Kähler} \Rightarrow \text{balanced} \Rightarrow \text{Gauduchon}.$$

We say that a complex manifold is *kählerian* (respectively *balanced*) if it admits a Kähler (respectively *balanced*) metric. The existence of Kähler and balanced metrics in a compact complex manifold is a delicate question [60, 79]. In complex dimension two, the conditions of being balanced and kählerian are both equivalent to the first Betti number of X being even (see e.g. [77, Th. 1.2.3]). Thus, there are complex surfaces — such as the Hopf surfaces — which carry no Kähler metric (see Example 2.10). However, in all dimensions $n \geqslant 3$ there exist compact balanced manifolds which are not kählerian. This is true, for example, for certain complex nilmanifolds (see Example 2.5). In contrast, due to a theorem of Gauduchon [50] every compact complex manifold admits a Gauduchon metric.

Theorem 2.3 ([50]). *Every hermitian metric on X is conformal to a Gauduchon metric, uniquely up to scaling when $n > 1$.*

A large class of kählerian complex manifolds is given by the projective algebraic manifolds. This follows from the basic fact that the kählerian property is inherited by holomorphic immersions, that is, if X is kählerian and there exists a holomorphic immersion $f\colon Y \to X$, then Y is kählerian (by pull-back of a Kähler metric on X).

Example 2.4. Any closed complex submanifold of $\mathbb{CP}^N$ is kählerian. Recall that any closed analytic submanifold $X \subset \mathbb{CP}^N$ is algebraic, by Chow's Theorem.

Basic examples of balanced manifolds which are not kählerian can be found among complex parallelizable manifolds. Compact complex parallelizable manifolds were characterized by Wang [100], who proved that all arise as a quotient of a complex unimodular Lie group G by a discrete subgroup Γ. They are, in general, non-kählerian: such a manifold is kählerian if and only if it is a torus [100, p.776]. Using Wang's characterization, Abbena and Grassi [1] showed that all parallelizable complex manifolds are balanced. In fact, any right invariant hermitian metric on G is balanced and this induces a balanced metric on the manifold G/Γ [1, Theorem 3.5]. We discuss a concrete example on the Iwasawa manifold, due to Gray (see e.g. [51, p. 120]).

Example 2.5. Let $G \subset \mathrm{SL}(3,\mathbb{C})$ be the non-abelian group given by elements of the form

$$\begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SL}(3,\mathbb{C}), \tag{2.1}$$

for z_1, z_2, z_3 arbitrary complex numbers. Let $\Gamma \subset G$ be the subgroup whose entries are given by Gaussian numbers

$$\Gamma = G \cap \mathrm{SL}(3, \mathbb{Z}[i]).$$

The quotient $X = G/\Gamma$ is a compact complex parallelizable manifold of dimension 3. The holomorphic cotangent bundle T^*X can be trivialized explicitly in terms of three holomorphic 1-forms θ_1, θ_2, θ_3, locally given by

$$\theta_1 = dz_1, \qquad \theta_2 = dz_2 - z_3 dz_1, \qquad \theta_3 = dz_3,$$

satisfying the relations

$$d\theta_1 = d\theta_3 = 0, \qquad d\theta_2 = \theta_1 \wedge \theta_3.$$

Since $d\theta_2 = \partial\theta_2$ is a non-vanishing exact holomorphic $(2,0)$-form, X is not kählerian (by the $\partial\bar{\partial}$-lemma, any such form vanishes in a Kähler manifold). To show that X is balanced, we can take the positive $(1,1)$-form

$$\omega = i(\theta_1 \wedge \overline{\theta}_1 + \theta_2 \wedge \overline{\theta}_2 + \theta_3 \wedge \overline{\theta}_3), \tag{2.2}$$

which defines a hermitian metric $g = \omega(\cdot, J\cdot)$ on X. It can be readily checked that $d\omega \wedge \omega = 0$.

We introduce next an important 1-form canonically associated to any hermitian structure, that we use to give a characterization of the balanced and Gauduchon conditions..

Definition 2.6. The *Lee form* of a hermitian metric g on X is the 1-form $\theta_\omega \in \Omega^1$ defined by

$$\theta_\omega = J d^* \omega. \tag{2.3}$$

Here, $d^* = - * d *$ is the adjoint of the exterior differential d for the hermitian metric g, where $*$ denotes the (riemannian) *Hodge star operator* of g. Alternatively, using the operator

$$\Lambda_\omega : \Omega^k \longrightarrow \Omega^{k-2} \colon \psi \longmapsto \iota_{\omega^\sharp}(\psi), \tag{2.4}$$

where $\sharp$ is the operator acting on k-forms induced by the symplectic duality $\sharp\colon T^*X \to TX$ and ι denotes the contraction operator, we have

$$\theta_\omega = \Lambda_\omega d\omega.$$

A different way of defining the Lee form is via the equation

$$d\omega^{n-1} = \theta_\omega \wedge \omega^{n-1},$$

which in particular implies $\Lambda_\omega d\theta_\omega = 0$. From the previous formula we can deduce the change of the Lee form under conformal transformations: if $\tilde{\omega} = e^\phi \omega$ for a smooth function $\phi \in C^\infty(X)$ then

$$\theta_{\tilde{\omega}} = \theta_\omega + (n-1)d\phi. \tag{2.5}$$

Note that $d\theta_\omega$ is a conformal invariant (when θ_ω is closed, so is $[\theta_\omega] \in H^1(X,\mathbb{R})$).

Proposition 2.7. *A hermitian metric g on X is*

1. *balanced if and only if $\theta_\omega = 0$,*
2. *Gauduchon if and only if $d^*\theta_\omega = 0$.*

Proof. The first part follows from the equality $*\omega = \frac{\omega^{n-1}}{(n-1)!}$, since by definition $d^* = - * d *$. As for the second part, the statement follows from

$$d^*\theta_\omega = - * dJd * \omega = - * Jd^c d * \omega,$$

where we have used that $d^c = JdJ^{-1}$, and the algebraic identities $*J = J*$, $*^2\alpha^k = (-1)^k\alpha^k$ for any $\alpha^k \in \Omega^k$. □

2.2 Balanced manifolds

In this section we study general properties of balanced manifolds, which enable to construct a large class of examples, and also to identify complex manifolds which are not balanced. The guiding principle is that the balanced property for a complex manifold is, in a sense, dual to the kählerian property. This can be readily observed from Proposition 2.7, which implies that the balanced condition for a hermitian metric is equivalent to the Kähler form being co-closed

$$d^*\omega = 0.$$

As mentioned in Example 2.4, the kählerian property of complex manifolds is inherited by holomorphic immersions. Balanced manifolds satisfy a dual 'functorial property', in terms of proper holomorphic submersions. Thus, the Kähler property is induced on sub-objects and the balanced property projects to quotient objects.

Proposition 2.8 ([79])**.** *Let X and Y be compact complex manifolds. Then*

1. *If X and Y are balanced, then $X \times Y$ is balanced.*
2. *Let $f\colon X \to Y$ be a proper holomorphic submersion. If X is balanced then Y is balanced.*

Proof. Let n and m denote the complex dimensions of X and Y, respectively. To prove part 1, we simply note that if ω_X and ω_Y are (the Kähler forms of) balanced metrics on X and Y, respectively, then

$$(\omega_X + \omega_Y)^{n+m-1} = \binom{n-1}{m+n-1}\omega_X^{n-1} \wedge \omega_Y^m + \binom{m-1}{m+n-1}\omega_X^n \wedge \omega_Y^{m-1},$$

for the product hermitian structure on $X \times Y$.

As for part 2, let ω_X be a balanced metric on X and consider the closed $(n-1, n-1)$-form $\tau_X = \omega_X^{n-1}$. Since f is proper, there exists a closed $2m-2$-form $\tau_Y = f_*\tau_X$ on Y, given by integration along the fibres. The fibres are complex, and hence τ_Y is actually a form of type $(m-1, m-1)$. Furthermore, it can be checked that it is positive, in the sense that it induces a positive form on every complex hyperplane in TY. A linear algebra argument (see [79, p. 280]) shows now that τ_Y admits an $(m-1)$-th positive root, that is, a positive $(1,1)$-form ω_Y on Y, such that $\omega_Y^{m-1} = \tau_Y$. □

The converse of part 2 of Proposition 2.8 is not true. To see a counterexample, we need a basic cohomological property of balanced manifolds. This shall be compared with the fact that, on a kälerian manifold, no compact complex curve can be trivial in homology.

Proposition 2.9 ([79]). *Let X be a compact complex balanced manifold of dimension n. Then every compact complex subvariety of codimension* 1 *represents a non-zero class in $H_{2n-2}(X, \mathbb{R})$.*

The proof of Proposition 2.9 follows easily by integration of the closed form ω^{n-1} along the subvariety, for any balanced metric on X. This property of codimension 1 subvarieties of a balanced manifold was strengthened by Michelson in terms of currents, whereby he provided a cohomological characterization of those compact complex manifolds which admit balanced metrics [79, Th. A]. We will not comment further on this more sophisticated notion, since Proposition 2.9 is enough to give first examples of complex manifolds which are not balanced.

Example 2.10 ([18]). Consider $\hat{X} = \mathbb{C}^{p+1}\backslash\{0\} \times \mathbb{C}^{q+1}\backslash\{0\}$ and the abelian group $\mathbb{C}$ acting on $\hat{X}$ by

$$t \cdot (x, y) = (e^t x, e^{\alpha t} y),$$

for $\alpha \in \mathbb{C}\backslash\mathbb{R}$. The *Calabi-Eckmann manifold* $X = \hat{X}/\mathbb{C}$, is diffeomorphic to $S^{2p+1} \times S^{2q+1}$, and admits a natural holomorphic fibration structure

$$f \colon X \to \mathbb{CP}^p \times \mathbb{CP}^q$$

induced by the product of the Hopf mappings $S^{2k+1} \to \mathbb{CP}^k$, for $k = p, q$. Hence, X admits plenty of compact complex submanifolds of codimension-one. Since the homology is zero in dimension $2p + 2q$, we see that these manifolds support no balanced metrics if $p + q > 0$. Note that p is proper and $\mathbb{CP}^p \times \mathbb{CP}^q$ is kählerian, and hence the converse of part 2 in Proposition 2.8 does not hold.

The next result, also due to Michelson, provides a weak converse of part 2 in Proposition 2.8.

Theorem 2.11 ([79]). *Let X be a compact complex connected manifold. Suppose that X admits a holomorphic map $f\colon X \to C$ onto a complex curve C, with a cross section. If the non-singular fibres of f are balanced, then X is balanced.*

Note that the existence of the cross section implies that the pull-back $f^*[C]$ of the fundamental class $[C]$ of C is non-trivial in $H^2(X,\mathbb{R})$, and therefore no regular fibre can be homologous to zero (since its class in homology is the Poincare dual of $[C]$). To illustrate this result, consider the Hopf surface $S^1 \times S^3$ – regarded as a Calabi-Eckmann manifold in Example 2.10 –. This manifold admits a holomorphic submersion $S^1 \times S^3 \to \mathbb{CP}^1$ with Kähler fibres (given by elliptic curves) but it is not balanced, because the fibres are homologically trivial.

The following example, which fulfils the hypothesis of Theorem 2.11, is due to Calabi.

Example 2.12 ([17]). Let C be a compact Riemann surface and $\iota\colon \tilde{C} \to \mathbb{R}^3$ a conformal minimal immersion, where $\tilde{C}$ denotes the universal cover of C. Considering the product immersion

$$\iota \times \mathrm{Id}\colon \tilde{C} \times \mathbb{R}^4 \to \mathbb{R}^3 \oplus \mathbb{R}^4 = \mathbb{R}^7,$$

the 6-dimensional submanifold $\iota(\tilde{C} \times \mathbb{R}^4)$ inherits an integrable complex structure induced by Cayley multiplication (where $\mathbb{R}^7$ is regarded as the imaginary octonions). This complex structure is invariant under covering transformations on $\tilde{C}$ and translations on $\mathbb{R}^4$, and satisfies that the natural inclusions $\tilde{C} \times \{x\} \to \tilde{C} \times \mathbb{R}^4$ are holomorphic for any $x \in \mathbb{R}^4$. Hence for any lattice $\Lambda \subset \mathbb{R}^4$ we can produce a compact quotient manifold X_Λ which admits a holomorphic projection $f\colon X_\Lambda \to C$. The fibres of this map are complex tori and the map f has holomorphic cross sections (induced by the inclusions $\tilde{C} \times \{x\} \to \tilde{C} \times \mathbb{R}^4$). By Theorem 2.11, X_Λ is a balanced manifold. Calabi proved in [17] that the manifolds X_Λ are not kählerian.

Further examples of balanced holomorphic fibrations (which do not satisfy the hypothesis of Theorem 2.11) are given by twistor spaces of four-dimensional riemannian manifolds. Given an oriented riemannian 4-manifold N, there is associated twistor space T, given by an S^2-bundle over N. The fibre of T over a point x in N is the sphere of all orthogonal almost complex structure on T_xN compatible with the orientation. The twistor space T has a canonical almost complex structure, which is integrable if and only if N is self-dual [11]. Furthermore, there is a natural balanced metric on T. As shown by Hitchin [62], the only compact twistor spaces which are Kähler are those associated S^4 and $\mathbb{CP}^2$.

The next result is due to Alessandrini and Bassanelli, and proves that the property of being balanced is a birational invariant. Due to a counterexample of Hironaka [61], this is not true for Kähler manifolds, and thus the existence of balanced metrics is a more robust property than the kählerian condition for a compact complex manifold.

Theorem 2.13 ([3, 4])**.** *Suppose X and Y are compact complex manifolds. Let $f\colon X \to Y$ be a modification. Then X is balanced if and only if Y is balanced.*

A modification $f\colon X \to Y$ is a holomorphic map such that there exists a complex submanifold $N \subset Y$ of codimension at least two and a biholomorphism $f\colon X\backslash f^{-1}(N) \to Y\backslash N$ given by restriction. As a corollary of the previous result, any compact complex manifold of Fujiki class $\mathcal{C}$ is balanced, since, by definition, it is bimeromorphic to a Kähler manifold.

We finish this section with some comments on the behaviour of the kählerian and balanced properties under deformations of complex structure. Kodaira and Spencer proved that any small deformation of a compact Kähler manifold is again a Kähler manifold [73, Th. 15]. Unlike for kählerian manifolds, the existence of balanced metrics on a compact complex manifold is not an open condition under small deformations of the complex structure. This was shown explicitly in [2, Proposition 4.1], for the Iwasawa manifold endowed with the holomorphically parallelizable complex structure (see Example 2.5). As shown in [9, 46, 102], a balanced analogue of the stability result of Kodaira and Spencer requires a further assumption on the variation of Bott-Chern cohomology of the complex manifold. In particular, Wu proved in [102, Th. 5.13] that small deformations of compact balanced manifolds satisfying the $\partial\bar{\partial}$-lemma still admit balanced metrics.

Hironaka's Example [61] mentioned above shows in particular that the Kähler property is not closed under deformations of complex structure. Recently, Ceballos, Otal, Ugarte and Villacampa [20] proved the analogue result for balanced manifolds, that is, that the balanced property is not closed under holomorphic deformations. This result has been strengthened in [36], via the construction of a family over a disk whose generic element is a balanced manifold satisfying the $\partial\bar{\partial}$-lemma and whose central fibre is not balanced.

3 Balanced metrics on Calabi-Yau manifolds

3.1 The dilatino equation

We introduce next an equation – for a hermitian metric on a complex manifold with trivial canonical bundle – which is closely related to the balanced condition, and constitutes one of the building blocks of the Strominger system. We will use the following notation throughout.

Definition 3.1. A Calabi-Yau n-fold is a pair (X, Ω), given by a complex manifold X of dimension n and a non-vanishing holomorphic global section Ω of the canonical bundle $K_X = \Lambda^n T^*X$.

We should stress that in the previous definition we do not require X to be kählerian. To introduce the dilatino equation, given a hermitian metric g on X we will denote by $\|\Omega\|_\omega$ the norm of Ω, given explicitly by

$$\|\Omega\|_\omega^2 \frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} i^n \Omega \wedge \overline{\Omega}. \tag{3.1}$$

Definition 3.2. The *dilatino equation*, for a hermitian metric g on (X, Ω), is

$$d^*\omega = d^c \log \|\Omega\|_\omega. \tag{3.2}$$

When $n = 1$, g is necessarily Kähler and the Calabi-Yau assumption implies that X is an elliptic curve. Then, (3.2) reduces to $d \log \|\Omega\|_\omega = 0$, which is equivalent to ω being Ricci-flat. We assume $n > 1$ in the sequel.

The next result shows that the existence of solutions of (3.2) is equivalent to the existence of a Kähler Ricci-flat metric, if $n = 2$, and to the existence of a balanced metric when $n \geqslant 3$. The proof relies on the observation by Li and Yau [76] that the dilatino equation is equivalent to the *conformally balanced equation*

$$d(\|\Omega\|_\omega \omega^{n-1}) = 0. \tag{3.3}$$

Proposition 3.3. *Let σ be a hermitian conformal class on (X, Ω). Then*

1. *if $n = 2$, then σ admits a solution of (3.2) if and only if all $g \in \sigma$ is a solution, if and only there exists a Kähler Ricci-flat metric on σ.*
2. *if $n \geqslant 3$ then σ admits a solution of (3.2) if and only if σ admits a balanced metric.*

If X is compact, then there exists at most one balanced metric on σ up to homothety.

Proof. We sketch the proof and leave the details for the reader. First we use that (3.2) is equivalent to

$$\theta_\omega = -d \log \|\Omega\|_\omega,$$

(in particular θ_ω needs to be exact if there exists a solution). Thus, using (2.5) the dilatino equation is equivalent to (3.3), which holds if and only if $\tilde{\omega} = \|\Omega\|_\omega^{\frac{1}{n-1}} \omega$ is balanced. Conversely, for $n \geqslant 3$, $\tilde{\omega}$ is balanced if and only if $\omega = \|\Omega\|_{\tilde{\omega}}^{\frac{-2}{n-2}} \tilde{\omega}$ solves the dilatino equation. The existence part of the statement follows from the change of the $(n-1, n-1)$-form $\|\Omega\|_\omega \omega^{n-1}$ under a conformal transformation of the metric. The uniqueness part follows by direct application of Gauduchon's Theorem [52, Th. I.14]. □

The equivalence between (3.2) and (3.3) implies that any solution of the dilatino equation has an associated class in cohomology. To be more precise, consider the Bott-Chern cohomology of X, defined by

$$H^{p,q}_{BC}(X) = \frac{\operatorname{Ker}(d\colon \Omega^{p,q} \to \Omega^{p+1,q} \oplus \Omega^{p,q+1})}{\operatorname{Im}(dd^c\colon \Omega^{p-1,q-1} \to \Omega^{p,q})}.$$

Since d and dd^c are real operators, the cohomology groups of bi-degree (p,p) have a natural real structure

$$H^{p,p}_{BC}(X,\mathbb{R}) \subset H^{p,p}_{BC}(X).$$

Definition 3.4. Given a solution ω of the dilatino equation (3.2), its *balanced class* is defined by

$$[\|\Omega\|_\omega \omega^{n-1}] \in H^{n-1,n-1}_{BC}(X,\mathbb{R}). \tag{3.4}$$

Remark 3.5. Given a balanced metric $\tilde{\omega}$, there are infinitely many such metrics with the same balanced class $[\tilde{\omega}^{n-1}]$, as we can always take a real form $\varphi \in \Omega^{n-2,n-2}$ and deform $\tilde{\omega}^{n-1}$ by

$$\Psi = \tilde{\omega}^{n-1} + dd^c\varphi.$$

For φ small we have that Ψ is positive, and thus its $(n-1)$-th root is a balanced metric [79].

A solution of the dilatino equation has an alternative interpretation, in terms of a connection with skew-torsion and restricted holonomy. Let X be a complex manifold, with underlying smooth manifold M. A hermitian connection on (X,g) is a linear connection on TM such that $\nabla J = 0$ and $\nabla g = 0$, where J is the almost complex on M determined by X. Gauduchon observed in [53] that the *Bismut connection* [13]

$$\nabla^B = \nabla^g - \frac{1}{2} g^{-1} d^c\omega. \tag{3.5}$$

is the unique hermitian connection on X with skew-symmetric torsion. Here we regard the metric as an isomorphism $g\colon TM \to T^*M$ and $g^{-1}d^c\omega$ as 1-form with values on the endomorphisms of TM. Note that the torsion of ∇^B is given by

$$gT_{\nabla^B} = -d^c\omega \in \Omega^3.$$

Proposition 3.6. *Let (X,Ω) be a Calabi-Yau manifold endowed with a hermitian metric g. If g is a solution of the dilatino equation* (3.2)*, then the Bismut connection ∇^B has restricted holonomy in the special unitary group*

$$hol(\nabla^B) \subset SU(n). \tag{3.6}$$

The converse is true if X is compact.

For the proof, we need a formula for the unitary connection induced by ∇^B on K_X due to Gauduchon [53, Eq. (2.7.6)]. Let ∇^C be the Chern connection of the hermitian metric on K_X induced by g. Then,

$$\nabla^B = \nabla^C - i d^* \omega \otimes \mathrm{Id}. \tag{3.7}$$

Recall that ∇^C is uniquely determined by the property $(\nabla^C)^{0,1} = \bar{\partial}$, where $\bar{\partial}$ is the canonical Dolbeault operator on the holomorphic line bundle K_X. Since X is Calabi-Yau, we have an explicit formula for ∇^C in (3.7) in the holomorphic trivialization of K_X given by Ω, namely,

$$\nabla^C = d + 2\partial \log \|\Omega\|_\omega.$$

Proof of Proposition 3.6. Let ψ be a smooth section of K_X. Then, there exists $f \in C^\infty(X, \mathbb{C})$ a smooth complex-valued function on X such that $\psi = e^f \Omega$. Applying (3.7) we have

$$\nabla^B(\psi) = (df + 2\partial \log \|\Omega\|_\omega - i d^* \omega) \otimes \psi. \tag{3.8}$$

Assume that ω is a solution of (3.2). Then, by the previous equation

$$\nabla^B(\psi) = (df + d \log \|\Omega\|_\omega) \otimes \psi$$

and we can set $f = -\log \|\Omega\|_\omega$ to obtain a parallel section $\psi = \|\Omega\|_\omega^{-1} \Omega$.

For the converse, if ψ is parallel with respect to ∇^B, applying (3.8) we obtain

$$df = i d^* \omega - 2\partial \log \|\Omega\|_\omega.$$

Further, ψ must have constant norm $\|\psi\|_\omega = t \in \mathbb{R}_{>0}$ and it follows from (3.1) that

$$f + \overline{f} = 2 \log t - 2 \log \|\Omega\|_\omega.$$

Hence, setting $\phi = f - \overline{f}$ we obtain

$$d\phi = d^* \omega - d^c \log \|\Omega\|_\omega.$$

It suffices to prove that ϕ is constant. For this, we define $\tilde{\omega} = \|\Omega\|_\omega^{\frac{1}{n-1}} \omega$ and note that

$$d\phi = -J\theta_{\tilde{\omega}} = d^{\tilde{*}} \tilde{\omega},$$

by the behaviour of the Lee under conformal rescaling (2.5). Applying now the operator $d^{\tilde{*}}$ on both sides of the equation $\Delta_{\tilde{\omega}} \phi = d^{\tilde{*}} d\phi = 0$, and therefore $d\phi = 0$ since by assumption X is compact. □

As we have just seen, the restriction of the holonomy of the Bismut connection to the special unitary group on a Calabi-Yau manifold is essentially equivalent to the existence of solutions of the dilatino equation (3.2). This equation for the Kähler form of the hermitian structure is strongly reminiscent of the complex Monge-Ampère equation in Kähler geometry. To see this, assume for a moment that X is a domain in $\mathbb{C}^n$, $\Omega = dz_1 \wedge \ldots \wedge dz_n$ and that g is Kähler, with

$$\omega = dd^c\varphi$$

for a smooth function φ. Then $d^*\omega = 0$ and the dilatino equation reduces to the complex Monge-Ampère equation

$$\log\det \partial_i \partial_{\bar{j}}\varphi = t,$$

for a choice of constant t. Relying on Yau's solution of the Calabi Conjecture [103], the previous observation gives a hint that *Calabi-Yau metrics*, that is, Kähler metrics with vanishing Ricci tensor, provide examples of solutions for the dilatino equation.

Example 3.7. Assume that X is compact and kählerian. Then, by Yau's Theorem [103], X admits a unique Kähler Ricci-flat metric g on each Kähler class. Since g is Kähler, the left hand side of (3.2) vanishes. Further, Ricci-flatness implies that the holomorphic volume form Ω has to be parallel [54, Prop. 1.22.6], and therefore g solves (3.2). Note that in this case $d^c\omega = 0$ and hence $\nabla^B = \nabla^C = \nabla^g$.

To find non-Kähler examples of solutions of the dilatino equation we revisit the complex parallelizable manifold in Example 2.5.

Example 3.8. The compact complex manifold $X = G/\Gamma$ in Example 2.5 is parallelizable, and hence admits a trivialization of the canonical bundle. Using the frame $\theta_1, \theta_2, \theta_3$ of T^*X we consider

$$\Omega = \theta_1 \wedge \theta_2 \wedge \theta_3.$$

Then, it can be readily checked that $\|\Omega\|_\omega$ is constant, and hence the balanced metric (2.2) is a solution of the dilatino equation (3.2). We note that the same exact argument works in an arbitrary complex parallelizable manifold.

3.2 Balanced metrics and conifold transitions

We describe next a more sophisticated construction of non-kählerian Calabi-Yau threefolds due to Clemens [22] and Friedman [39] (for a review see [89]), which provides a strong motivation for the study of the Strominger system. These threefolds are typically not birational to kählerian manifolds (class $\mathcal{C}$) as they may have vanishing second Betti number. Thus, Alessandrini-Bassanelli Theorem 2.13 cannot be applied to ensure the existence of balanced metrics. An interesting result due to Fu, Li and Yau [42] shows that the Calabi-Yau manifolds obtained via the Clemens-Friedman construction are balanced.

Let X be a smooth kählerian Calabi-Yau threefold with a collection of mutually disjoint smooth rational curves $C_1, \ldots, C_k$, with normal bundles isomorphic to

$$\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1).$$

Contracting the k rational curves, we obtain a singular Calabi-Yau threefold X_0 with k ordinary double-point singularities $p_1, \ldots, p_k$. Away from the singularities, we have a biholomorphism $X \backslash \bigcup_k C_k \cong X_0 \backslash \{p_1, \ldots, p_k\}$, while a neighbourhood of p_j in X_0 is isomorphic to a neighbourhood of 0 in

$$\{z_1^2 + z_2^2 + z_3^2 + z_4^4 = 0\} \subset \mathbb{C}^4.$$

By results of Friedman, Tian and Kawamata, if the fundamental classes $[C_j] \in H^{2,2}(X, \mathbb{Q})$ satisfy a relation

$$\sum_j n_j [C_j] = 0,$$

with $n_j \neq 0$ for every j, then there exists a family of complex manifolds X_t over a disk $\Delta \subset \mathbb{C}$, such that X_t is a smooth Calabi-Yau threefold for $t \neq 0$, and the central fibre is isomorphic to X_0. For small values of t, the local model for this *smoothing* is isomorphic to a neighbourhood of 0 in

$$\{z_1^2 + z_2^2 + z_3^2 + z_4^4 = t\} \subset \mathbb{C}^4.$$

The construction of X_t from X, typically denoted by a diagram of the form

$$X \to X_0 \dashrightarrow X_t,$$

is known in physics as *Clemens-Friedman conifold transition* (see e.g. [89]). The smooth manifolds X_t satisfy the $\partial\bar{\partial}$-lemma, but they are in general non-kählerian [40]. Explicitly, Friedman observed that $\sharp_k(S^3 \times S^3)$ for any $k \geqslant 2$ can be given a complex structure in this way. The idea is to contract enough rational curves on X so that $H^2(X_t, \mathbb{R}) = 0$ for $t \neq 0$.

Example 3.9. Consider the complete intersection $X \subset \mathbb{CP}^4 \times \mathbb{CP}^1$ given by

$$\begin{aligned} (x_2^2 + x_4^2 - x_5^2)y_1 + (x_1^2 + x_3^4 + x_5^4)y_2 &= 0, \\ x_1 y_1 + x_2 y_2 &= 0, \end{aligned}$$

where $x_1, \ldots, x_5$ and y_1, y_2 are coordinates in $\mathbb{CP}^4$ and $\mathbb{CP}^1$, respectively. This is a Calabi-Yau threefold with $b_2(X) = 2$ (see e.g. [89]). Consider the blow up

$$\psi \colon \hat{\mathbb{CP}}^4 \to \mathbb{CP}^4$$

of $\mathbb{CP}^4$ along the plane $\{x_1 = x_2 = 0\}$, whose exceptional divisor is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^2$. Then, X can be regarded as the proper transform of the singular hypersurface $X_0 \subset \mathbb{CP}^4$ containing the given plane, defined by

$$x_2(x_2^2 + x_4^2 - x_5^2) - x_1(x_1^2 + x_3^4 + x_5^4) = 0.$$

Note that X_0 has 16 ordinary double points $p_1, \ldots, p_{16}$, described by

$$x_1 = x_2 = 0, \qquad x_3^4 + x_5^4 = 0, \qquad x_4^2 - x_5^2 = 0,$$

and $\psi^{-1}(p_j) \cong \mathbb{CP}^1$ for all $j = 1, \ldots, 16$. Choosing now a small $t \in \mathbb{C}$, we can consider the smooth quintic hypersurface $X_t \subset \mathbb{CP}^4$ with equation

$$x_2(x_2^2 + x_4^2 - x_5^2) - x_1(x_1^2 + x_3^4 + x_5^4) = t \sum_{i=1}^{5} x_i^2,$$

which defines a smoothing of X_0 (the so called *generic quintic*). Note that X_t is also a Calabi-Yau threefold, and we have decreased the Betti number $b_2(X_t) = 1$.

The main result in [42] states that the smoothing X_t in a conifold transition admits a balanced metric, providing first examples of balanced metrics on the complex manifolds $\sharp_k(S^3 \times S^3)$.

Theorem 3.10 ([42]). *For sufficiently small $t \neq 0$, X_t admits a smooth balanced metric.*

Reid speculated [88] that all kählerian Calabi-Yau threefolds (that can be deformed to Moishezon manifolds) are parametrized by a single universal moduli space in which families of smooth Calabi-Yau threefolds of different homotopy types are connected by conifold transitions. In order to develop a metric approach to *Reid's fantasy*, Yau has proposed to study special types of balanced metrics which endow the Calabi-Yau threefolds with a preferred geometry [45, 76]. Note here that, similarly as in Kähler geometry, balanced metrics arise in infinite-dimensional families, each of them parametrized by a class in the *balanced cone* in Bott-Chern cohomology [43] (see Remark 3.5). As we will see in Section 5, a natural way to rigidify balanced metrics is imposing conditions on the torsion 3-form of the Bismut connection $d^c\omega$. An alternative (and in some sense orthogonal) approach in the literature is to fix the volume form $\omega^n/n!$ of the balanced metric [94]. In particular, this gives special solutions of the dilatino equation, where the left and right hand side of (3.2) vanish independently.

4 Hermite-Einstein metrics on balanced manifolds

4.1 The Hermite-Einstein equation and stability

Let X be a compact complex manifold of dimension n endowed with a balanced metric g. Let $\mathcal{E}$ be a holomorphic vector bundle over X of rank r, with underlying smooth complex vector bundle E. We will denote by $\Omega^{p,q}(E)$ the space of E-valued (p,q)-forms on X. The holomorphic structure on E given by $\mathcal{E}$ is equivalent to a Dolbeault operator

$$\bar{\partial}_{\mathcal{E}} \colon \Omega^0(E) \to \Omega^{0,1}(E)$$

satisfying the integrability condition $\bar{\partial}_{\mathcal{E}}^2 = 0$ (see e.g. [54]).

Given a hermitian metric h on E, there is an associated unitary *Chern connection* A compatible with the holomorphic structure $\mathcal{E}$, uniquely defined by the properties

$$\begin{aligned} d_A^{0,1} &= \bar{\partial}_{\mathcal{E}}, \\ d(s,t)_h &= (d_A s, t)_h + (s, d_A t)_h, \end{aligned}$$

for $s, t \in \Omega^0(E)$. Here, $d_A \colon \Omega^0(E) \to \Omega^1(E)$ is the covariant derivative defined by A. In a local holomorphic frame $\{e_j\}_{j=1}^r$, the Chern connection is given by the matrix-valued $(1,0)$-form

$$h^{-1}\partial h,$$

where $h_{ij} = (e_j, e_i)_h$. The curvature of the Chern connection is a $(1,1)$-form with values in the skew-hermitian endomorphisms $\mathrm{End}(E,h)$ of E, defined by

$$F_h = d_A^2 \in \Omega^{1,1}(\mathrm{End}(E,h)).$$

In a local holomorphic frame, we have the formula

$$F_h = \bar{\partial}(h^{-1}\partial h).. \tag{4.1}$$

Definition 4.1. The Hermite-Einstein equation, for a hermitian metric h on E, is

$$i\Lambda_\omega F_h = \lambda \mathrm{Id}. \tag{4.2}$$

In equation (4.2), Λ_ω is the contraction operator (2.4), Id denotes the identity endomorphism on E, and $\lambda \in \mathbb{R}$ is a real constant. The Hermite-Einstein equation is a non-linear second-order partial differential equation for the hermitian metric h, as it follows from (4.1).

To understand the existence problem for the Hermite-Einstein equation, the first basic observation is that, in order for a solution to exist, the constant $\lambda \in \mathbb{R}$

must take a specific value fixed by the topology of (X, g) and E, in the following sense. Consider the balanced class of ω^{n-1} in de Rham cohomology

$$\tau := [\omega^{n-1}] \in H^{2n-2}(X, \mathbb{R}), \tag{4.3}$$

determined by the balanced metric g. Recall that the first Chern class $c_1(E)$ of E is represented by

$$c_1(E) = [i \operatorname{tr} F_h / 2\pi] \in H^2(X, \mathbb{Z}),$$

for any choice of hermitian metric h.

Definition 4.2. The τ-degree of E is

$$\deg_\tau(E) = c_1(E) \cdot \tau, \tag{4.4}$$

where $c_1(E) \cdot \tau \in H^{2n}(X, \mathbb{R}) \cong \mathbb{R}$ denotes the cup product in cohomology.

Taking the trace in (4.2) and integrating against $\frac{\omega^n}{n!}$ we obtain

$$\lambda = \frac{2\pi}{(n-1)!} \frac{\deg_\tau(E)}{r \mathrm{Vol}_\omega}, \tag{4.5}$$

where $\mathrm{Vol}_\omega = \int_X \frac{\omega^n}{n!}$ is the volume. When E is a line bundle, the Hermite-Einstein equation can always be solved for this value of λ.

Proposition 4.3. *For a holomorphic line bundle $\mathcal{E}$ the Hermite-Einstein equation* (4.2) *admits a unique solution h up to homothety, provided that λ is given by* (4.5)*.*

Proof. Fixing a reference hermitian metric h_0 any other metric h on the line bundle is given by $h = e^f h_0$ and, using this, equation (4.2) is equivalent to

$$i\Lambda_\omega \bar{\partial}\partial f = \lambda - i\Lambda_\omega F_h. \tag{4.6}$$

By [52, eq. (25)], the balanced condition can be alternatively written as an equality of differential operators on smooth functions on X

$$2i\Lambda_\omega \bar{\partial}\partial = \Delta_\omega := dd^* + d^*d,$$

and therefore $i\Lambda_\omega \bar{\partial}\partial$ is self-adjoint, elliptic, with Kernel given by $\mathbb{R} \subset C^\infty(X)$. By (4.5),

$$\int_X (\lambda - i\Lambda_\omega F_h) \frac{\omega^n}{n!} = 0,$$

so $\lambda - i\Lambda_\omega F_h$ is orthogonal to $\mathbb{R}$ in $C^\infty(X)$. We conclude that (4.6) has a unique smooth solution f with $\int_X f \frac{\omega^n}{n!} = 0$. □

For higher rank bundles, the existence of solutions of the Hermite-Einstein equation relates to an algebraic numerical condition for $\mathcal{E}$ – originally related to the theory of quotients of algebraic varieties by complex reductive Lie groups, known as Geometric Invariant Theory [80]. To state the precise result, we need to extend Definition 4.2 to arbitrary torsion-free coherent sheaves of $\mathcal{O}_X$-modules (for the basic definitions we refer to [72, Section 5-6]). Given such a sheaf $\mathcal{F}$ of rank $r_{\mathcal{F}}$, the determinant of $\mathcal{F}$, defined by

$$\det \mathcal{F} := (\Lambda^{r_{\mathcal{F}}} \mathcal{F})^{**}$$

is a holomorphic line bundle (such that $\mathcal{F} = \det \mathcal{F}$ when $\mathcal{F}$ is torsion-free of rank 1), and we can extend Definition 4.2 setting

$$\deg_\tau(\mathcal{F}) := \deg_\tau(\det \mathcal{F}).$$

We define the τ-slope of a torsion-free coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules by

$$\mu_\tau(\mathcal{F}) = \frac{\deg_\tau(\mathcal{F})}{r_{\mathcal{F}}}. \tag{4.7}$$

Definition 4.4. A torsion-free sheaf $\mathcal{F}$ over X is

1. τ-(semi)*stable* if for every subsheaf $\mathcal{F}' \subset \mathcal{F}$ with $0 < r_{\mathcal{F}'} < r_{\mathcal{F}}$ one has

$$\mu_\tau(\mathcal{F}') < (\leqslant) \mu_\tau(\mathcal{F}),$$

2. τ-*polystable* if $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ with $\mathcal{F}_j$ stable and $\mu_\tau(\mathcal{F}_i) = \mu_\tau(\mathcal{F}_j)$ for all i, j.

When X is projective and g is a Kähler Hodge metric, with Kähler class associated to a hyperplane section of X, this definition coincides with the original definition of slope stability due to Mumford and Takemoto (see e.g. [80]).

We can state now the characterization of the existence of solutions for the Hermite-Einstein equation (4.2).

Theorem 4.5. *There exists a Hermite-Einstein metric h on $\mathcal{E}$ if and only if $\mathcal{E}$ is τ-polystable.*

This result was first proved by Narasimhan and Seshadri in the case of curves [82]. The 'only if part' was proved in higher dimensions by Kobayashi [71]. The 'if part' was proved for algebraic surfaces by Donaldson [29], and for higher dimensional compact Kähler manifolds by Uhlenbeck and Yau [98]. Buchdahl extended Donaldson's result to arbitrary compact complex surfaces in [14], and Li and Yau generalized Uhlenbeck and Yau's theorem to any compact complex hermitian manifold in [75].

Remark 4.6. The statement of Li-Yau Theorem [75] for an arbitrary hermitian metric g uses that the existence of a Hermite-Einstein metric only depends on the conformal class of g (see [77, Lem. 2.1.3]), and therefore g can be assumed to be Gauduchon. As for the stability condition, when g is a Gauduchon metric one defines the g-slope of $\mathcal{E}$ by $\mu_g(\mathcal{E}) = \deg_g(\mathcal{E})/r$, where

$$\deg_g(\mathcal{E}) = \frac{i}{2\pi}\int_X \operatorname{tr} F_h \wedge \omega^{n-1}$$

for a choice of hermitian metric h. The g-degree $\deg_g(\mathcal{E})$ is independent of h by Stokes Theorem, since $dd^c\omega^{n-1} = 0$ and $\operatorname{tr} F_h = \operatorname{tr} F_{h_0} + \bar{\partial}\partial f$ for $h = e^f h_0$. In general, $\deg_g$ is not topological, and depends on the holomorphic structure of $\det \mathcal{E}$ (e.g. for $n = 2$ the g-degree is topological if and only if X is kählerian [77, Cor. 1.3.14]).

On general grounds, an effective check of any of the two equivalent conditions in Theorem 4.5 is a difficult problem. We comment on a class of examples, given by deformations of (essentially) tangent bundles of algebraic Calabi-Yau manifolds. We postpone the examples of solutions of the Hermite-Einstein equations on non-kählerian balanced manifolds to Section 5.2.

Example 4.7. Given a Kähler-Einstein metric g on X, it is easy to check that the induced Chern connection on TX is Hermite-Einstein. In particular, by Yau's solution of the Calabi Conjecture [103], the tangent bundle of any kählerian Calabi-Yau manifold is polystable with respect to any Kähler class, and it is stable if $b_1(X) = 0$. Since being stable is an open condition, any deformation of the tangent bundle of a simply connected Calabi-Yau manifold is stable. As concrete examples (see e.g. [67]): in dimension 2, the tangent bundle of a $K3$ surface has unobstructed deformations and, in dimension 3, the tangent bundle of a simply connected complete intersection Calabi-Yau threefold has unobstructed deformations. For a quintic hypersurface $X \subset \mathbb{CP}^4$, the space of deformations of TX is 224-dimensional. For X the generic quintic (see Example 3.9), Huybrechts has proved that the polystable bundle $TX \oplus \mathcal{O}_X$ admits stable holomorphic deformations [67], with the following property: they have non-trivial restriction to any rational curve of degree one.

It is interesting to compare the previous example with a result by Chuan [21]. Let $X \to X_0 \dashrightarrow X_t$ be a conifold transition between Calabi-Yau threefolds, as in Section 3.2. Let $\mathcal{E}$ be a stable holomorphic vector bundle over X with $c_1(\mathcal{E}) = 0$, and assume that $\mathcal{E}$ is trivial in a neighbourhood of the exceptional rational curves. Assume further the exists a stable bundle $\mathcal{E}_t$ over X_t, given by a deformation of the push-forward of $\mathcal{E}$ along $X \to X_0$. Then, Chuan has proved under this hypothesis that $\mathcal{E}_t$ is stable with respect to the balanced metric constructed by Fu, Li and Yau [42].

4.2 Gauge theory, Kähler reduction and the necessity of stability

In this section we give a (lengthy) geometric proof of the 'only if part' of Theorem 4.5, following [81]. This method of proof is based on the correspondence between symplectic quotients and GIT quotients, given by the Kemp-Ness Theorem [70] and uses some of the ingredients required for the 'if part' of Theorem 4.5. We note that the standard proof of the 'only if part' (see e.g. [77]) – which turns on the principle that "curvature decreases in holomorphic sub-bundles and increases in holomorphic quotients" [10] – is shorter, and does not use any gauge-theoretical methods. Nonetheless, we find the proof of [81] more pedagogical, and better suited for the purposes of these notes. By the end, we discuss briefly the implications of this method for the geometry of the moduli space of stable vector bundles.

Let E be a smooth complex vector bundle of rank r over X. Let $\mathcal{G}^c$ be the gauge group of E, that is, the group of diffeomorphisms of E projecting to the identity on X and $\mathbb{C}$-linear on the fibres. Consider the space $\mathcal{C}$ of Dolbeault operators

$$\bar{\partial}_{\mathcal{E}} \colon \Omega^0(E) \to \Omega^{0,1}(E)$$

on E, which is a complex affine space modelled on $\Omega^{0,1}(\operatorname{End} E)$. Then, $\mathcal{G}^c$ acts on $\mathcal{C}$ by

$$g \cdot \bar{\partial}_{\mathcal{E}} = g \bar{\partial}_{\mathcal{E}} g^{-1},$$

and preserves the (constant) complex structure on $\mathcal{C}$.

Fix now a hermitian metric h on E. Consider the space $\mathcal{A}$ of unitary connections on (E,h), which is an affine space modelled on $\Omega^1(\operatorname{End}(E,h))$. The *unitary gauge group* $\mathcal{G} \subset \mathcal{G}^c$, given by automorphisms of E preserving h, acts on $\mathcal{A}$ by

$$g \cdot d_A = g d_A g^{-1},$$

preserving the symplectic structure

$$\omega_{\mathcal{A}}(a,b) = -\int_X \operatorname{tr}(a \wedge b) \wedge \omega^{n-1} \tag{4.8}$$

for $a, b \in T_A\mathcal{A} = \Omega^1(\operatorname{End}(E,h))$, with $A \in \mathcal{A}$.

There is a real affine bijection between the two infinite-dimensional spaces $\mathcal{A}$ and $\mathcal{C}$, defined by

$$\mathcal{A} \to \mathcal{C} \colon A \to \bar{\partial}_A = (d_A)^{0,1}, \tag{4.9}$$

with inverse given by the Chern connection of h in $\mathcal{E} = (E, \bar{\partial}_{\mathcal{E}})$. Under this bijection, the integrability condition $\bar{\partial}_A^2 = 0$ is equivalent to

$$F_A^{0,2} = 0, \tag{4.10}$$

and the complex structure on $\mathcal{C}$ translates to $a \to Ja$, for J the almost complex structure on X. The symplectic form (4.8) is compatible with this complex structure, and induces a Kähler structure on $\mathcal{C}$.

Using the bijection (4.9) and the Kähler form (4.8), we can now give a geometric interpretation to the Hermite-Einstein equation (4.2). The following observation is due to Atiyah and Bott [10] when X is a Riemann surface, and was generalized by Donaldson [29] to higher dimensional Kähler manifolds. As pointed out by Lübke and Teleman [77, Sec. 5.3], remarkably the construction only needs that g is balanced.

Proposition 4.8. *The $\mathcal{G}$-action on $\mathcal{A}$ is Hamiltonian, with equivariant moment map $\mu\colon \mathcal{A} \to (\operatorname{Lie}\mathcal{G})^*$ given by*

$$\langle \mu(A), \zeta \rangle = -\int_X \operatorname{tr} \zeta(\Lambda_\omega F_A + i\lambda \operatorname{Id})\frac{\omega^n}{n}, \tag{4.11}$$

where $\zeta \in \Omega^0(\operatorname{End}(E,h)) \cong \operatorname{Lie}\mathcal{G}$.

Proof. The $\mathcal{G}$-equivariance follows from $F_{g\cdot A} = gF_A g^{-1}$ for any $g \in \mathcal{G}$. Thus, given $a \in \Omega^1(\operatorname{End}(E,h))$ we need to prove that

$$\langle d\mu(a), \zeta \rangle = \omega_{\mathcal{A}}(Y_\zeta, a)$$

where Y_ζ denotes the infinitesimal action of ζ, given by

$$Y_\zeta(A) = -d_A\zeta.$$

Using $\delta_a F_A = d_A a$ and $d\omega^{n-1} = 0$, integration by parts gives

$$\begin{aligned}\langle d\mu(a), \zeta \rangle &= -\int_X \operatorname{tr} \zeta d_A a \wedge \omega^{n-1} \\ &= \int_X \operatorname{tr} d_A \zeta \wedge a \wedge \omega^{n-1} = \omega_{\mathcal{A}}(Y_\zeta, a).\end{aligned}$$

□

Definition 4.9. A unitary connection $A \in \mathcal{A}$ is called a *Hermite-Yang-Mills connection* if it satisfies $A \in \mu^{-1}(0)$ and $\bar\partial_A^2 = 0$, that is,

$$i\Lambda_\omega F_A = \lambda \operatorname{Id}, \qquad F_A^{0,2} = 0.$$

For a Hermite-Yang-Mills connection A, the metric h on $\mathcal{E} = (E, \bar\partial_A)$ is Hermite-Einstein. Using this fact, we can now prove the following.

Theorem 4.10. *If there exists a Hermite-Einstein metric h on $\mathcal{E}$, then $\mathcal{E}$ is τ-polystable.*

Proof. Let h be a Hermite-Einstein metric on $\mathcal{E}$ and $\mathcal{F} \subset \mathcal{E}$ a coherent subsheaf with $0 < r_{\mathcal{F}} < r$. We can assume that $\mathcal{F}$ is reflexive [77, Prop. 1.4.5]. Using a characterization of reflexive sheaves in terms *weakly holomorphic subbundles* [92, 98] (see also [77, p. 81]), there exists an analytic subset $S \subset X$ of codimension $\geqslant 2$ and $\pi \in L^2_1(\operatorname{End} E)$ such that

$$\pi^* = \pi = \pi^2, \qquad (\mathrm{Id} - \pi)\bar{\partial}_{\mathcal{E}}\pi = 0 \tag{4.12}$$

on $L^1(\operatorname{End} E)$, $\pi_{|X\setminus S}$ is smooth and satisfies (4.12), and

$$\mathcal{F}' = \mathcal{F}_{|X\setminus S} = \operatorname{Im} \pi_{|X\setminus S}$$

is a holomorphic subbundle of $\mathcal{E}' = \mathcal{E}_{|X\setminus S}$. Furthermore, the curvature of $h_{|X\setminus S}$ on $\mathcal{F}'$ defines a closed current on X which represents $-i2\pi c_1(\mathcal{F}) \in H^2(X, \mathbb{C})$.

Using the orthogonal projection π, we can define a *weak element* in the Lie algebra of $\mathcal{G}$ by

$$\zeta = i(\pi - \nu(\mathrm{Id} - \pi)).$$

Associated to ζ, there is a 1-parameter family of (singular) Dolbeault operators

$$\bar{\partial}_{\mathcal{E}_t} = e^{it\zeta} \cdot \bar{\partial}_{\mathcal{E}},$$

and, since $\mathbb{C}^*\mathrm{Id} \subset \mathcal{G}^c$ acts trivially on $\mathcal{C}$, we can assume the normalization

$$\nu = \frac{r_{\mathcal{F}}}{r - r_{\mathcal{F}}}.$$

We want to calculate the *maximal weight*

$$w(\bar{\partial}_{\mathcal{E}}, \zeta) := \lim_{t\to+\infty} \langle \mu(\bar{\partial}_{\mathcal{E}_t}), \zeta\rangle = \lim_{t\to+\infty} -\int_X \operatorname{tr}(\zeta F_{h,\bar{\partial}_{\mathcal{E}_t}})\omega^n/n. \tag{4.13}$$

Note that we can do the calculation in the smooth locus of π, since S is codimension 2. Away from the singularities of π, we have

$$\bar{\partial}_{\mathcal{E}_t} = \pi\bar{\partial}_{\mathcal{E}}\pi + (\mathrm{Id} - \pi)\bar{\partial}_{\mathcal{E}}(\mathrm{Id} - \pi) + e^{-t(1+\nu)}\pi\bar{\partial}_{\mathcal{E}}(\mathrm{Id} - \pi)$$

and therefore

$$\bar{\partial}_{E_\infty} := \lim_{t\to+\infty} \bar{\partial}_{\mathcal{E}_t} = \pi\bar{\partial}_{\mathcal{E}}\pi + (\mathrm{Id} - \pi)\bar{\partial}_{\mathcal{E}}(\mathrm{Id} - \pi),$$

which corresponds to the direct sum

$$(E, \bar{\partial}_{E_\infty})_{|X\setminus S} \cong \mathcal{F}' \oplus \mathcal{E}'/\mathcal{F}'.$$

Using this, we have

$$\begin{aligned} w(\bar{\partial}_{\mathcal{E}}, \zeta) &= -\int_X (i \operatorname{tr} \Lambda_\omega F_{h,\bar{\partial}_{\mathcal{F}'}})\omega^n/n + \nu \int_X (i \operatorname{tr} \Lambda_\omega F_{h,\bar{\partial}_{\mathcal{E}'/\mathcal{F}'}})\omega^n/n \\ &= 2\pi(n-1)!\left(-\deg_\tau(\mathcal{F}) + \frac{r_{\mathcal{F}}}{r - r_{\mathcal{F}}}(\deg_\tau(\mathcal{E}) - \deg_\tau(\mathcal{F}))\right) \\ &= 2\pi(n-1)!\frac{r r_{\mathcal{F}}}{r - r_{\mathcal{F}}}(\mu_\tau(\mathcal{E}) - \mu_\tau(\mathcal{F})). \end{aligned} \tag{4.14}$$

The key point of the proof is the monotonicity of $\langle \mu(\bar{\partial}_{\mathcal{E}_t}), \zeta\rangle$, which follows from the positivity of the Kähler form (4.8),

$$\frac{d}{dt}\langle \mu(\bar{\partial}_{\mathcal{E}_t}), \zeta\rangle = |Y_{\zeta|\bar{\partial}_{\mathcal{E}_t}}|^2 = (1+\nu)^2 e^{-2t(1+\nu)} \|\pi\bar{\partial}_{\mathcal{E}}(\mathrm{Id}-\pi)\|^2_{L_2}$$

combined with the Hermite-Einstein condition, which gives $\langle \mu(\bar{\partial}_{\mathcal{E}}), \zeta\rangle = 0$. Combining these two facts,

$$w(\bar{\partial}_{\mathcal{E}}), \zeta) = \int_0^\infty |Y_{\zeta|e^{it\zeta}u_{\underline{s}}}|^2 dt = \frac{1}{2}(1+\nu)\|\pi\bar{\partial}_{\mathcal{E}}(\mathrm{Id}-\pi)\|^2_{L_2} \geq 0, \tag{4.15}$$

and therefore $\mu_\tau(\mathcal{E}) \geqslant \mu_\tau(\mathcal{F})$ by (4.14). We conclude that $\mathcal{E}$ is semistable.

Suppose now that $\mathcal{E}$ is not stable and that we have an equality in (4.15). Then, $\pi\bar{\partial}_{\mathcal{E}}(\mathrm{Id}-\pi) = 0$ and

$$\bar{\partial}_{\mathcal{E}}(\pi) := \bar{\partial}_{\mathcal{E}}\pi - \pi\bar{\partial}_{\mathcal{E}} = \bar{\partial}_{\mathcal{E}}\pi - \pi\bar{\partial}_{\mathcal{E}}(\pi + (\mathrm{Id}-\pi)) = 0,$$

and therefore π^* is in the Kernel of the elliptic operator $i\Lambda_\omega\bar{\partial}_{\mathcal{E}}\partial_{\mathcal{E}}$ (see [77, Lem. 7.2.3]) and hence it is smooth on X. We conclude that $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are holomorphic vector bundles, and we have an orthogonal decomposition

$$\mathcal{E} \cong \mathcal{F} \oplus \mathcal{E}/\mathcal{F}$$

with $\mu_\tau(\mathcal{E}) = \mu_\tau(\mathcal{F}) = \mu_\tau(\mathcal{E}/\mathcal{F})$. In addition, $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ inherit Hermite-Einstein metrics by restriction. Induction on the rank of $\mathcal{E}$ completes the argument. □

To conclude, we discuss briefly the implications of the existence of the previous infinite-dimensional Kähler structure, for the geometry of the moduli space of stable vector bundles and Hermite-Yang-Mills connections. Let

$$\mathcal{C}^s \subset \mathcal{C}$$

be the subset of integrable Dolbeault operators $\bar{\partial}_{\mathcal{E}}$ which define τ-stable holomorphic vector bundles $\mathcal{E}$. Two integrable Dolbeault operators are in the same $\mathcal{G}^c$-orbit if and only if define isomorphic vector bundles, and therefore the $\mathcal{G}^c$-action preserves $\mathcal{C}^s$. We define the *moduli space of τ-stable vector bundles* by the quotient

$$\mathcal{C}^s/\mathcal{G}^c.$$

This quotient has a natural Hausdorff topology, and can be endowed with a finite dimensional complex analytic structure (which may be non-reduced) [77, Cor. 4.4.4].

Let $\mathcal{A}^* \subset \mathcal{A}$ be the $\mathcal{G}$-invariant subset of irreducible connections which satisfy the integrability condition (4.10). Recall that a connection A is irreducible if the Kernel of the induced covariant derivative d_A in $\mathrm{End}(E,h)$ equals $i\mathbb{R}\mathrm{Id}$. For

$A \in \mu^{-1}(0) \cap \mathcal{A}^*$, h is a Hermite-Einstein metric on $\mathcal{E} = (E, \bar{\partial}_A)$. We define the *moduli space of irreducible Hermite-Yang-Mills connections* as the quotient

$$\mu^{-1}(0) \cap \mathcal{A}^*/\mathcal{G}.$$

This set has a natural Hausdorff topology, and can be endowed with a finite dimensional real analytic structure (which may be non-reduced) [77, Prop. 4.2.7]. Furthermore, by Proposition 4.8 the moduli space inherits a real-analytic symplectic structure away from its singularities [77, Cor. 5.3.9].

Building on the proof of Theorem 4.10, one can prove that for $\mu^{-1}(0) \cap \mathcal{A}^*$ one has $\bar{\partial}_A \in \mathcal{C}^s$ (see [77, Remark 2.3.3]) and therefore there is a natural map

$$\mu^{-1}(0) \cap \mathcal{A}^*/\mathcal{G} \to \mathcal{C}^s/\mathcal{G}^c.$$

This map induces a real analytic isomorphism by Theorem (4.5), and a Kähler structure on $\mathcal{C}^s/\mathcal{G}^c$ away from its singularities (see [77, Cor. 4.4.4] and [77, Cor. 5.3.9]).

5 The Strominger system

5.1 Definition and first examples

Let (X, Ω) be a Calabi-Yau manifold of dimension n, with underlying smooth manifold M and almost complex structure J. Let $\mathcal{E}$ be a holomorphic vector bundle over X, with underlying smooth complex vector bundle E. To define the Strominger system, we consider integrable Dolbeault operators $\bar{\partial}_T$, that is, satisfying $\bar{\partial}_T^2 = 0$, on the smooth complex vector bundle (TM, J).

Definition 5.1. The *Strominger system*, for a hermitian metric g on (X, Ω), a hermitian metric h on $\mathcal{E}$, and an integrable Dolbeault operator $\bar{\partial}_T$ on (TM, J), is

$$\begin{aligned} \Lambda_\omega F_h &= 0, \\ \Lambda_\omega R &= 0, \\ d^*\omega - d^c \log \|\Omega\|_\omega &= 0, \\ dd^c\omega - \alpha\,(\operatorname{tr} R \wedge R - \operatorname{tr} F_h \wedge F_h) &= 0. \end{aligned} \tag{5.1}$$

Here, α is non-vanishing real constant and $R = R_{g,\bar{\partial}_T}$ denotes the curvature of the Chern connection of g, regarded as a hermitian metric on the holomorphic vector bundle $\mathcal{T} = (TM, J, \bar{\partial}_T)$. The first two equations in the system correspond to the Hermite-Einstein condition for the curvatures F_h and R with vanishing constant λ, with respect to the Kähler form ω (see Definition 4.1). The third equation, involving the Kähler form ω and the holomorphic volume form Ω, is the dilatino equation (3.2). The new ingredient in the system is an equation for 4-forms, known as the *Bianchi identity*

$$dd^c\omega - \alpha\,(\operatorname{tr} R \wedge R - \operatorname{tr} F_h \wedge F_h) = 0, \tag{5.2}$$

which intertwines the exterior differential of the torsion $-d^c\omega$ of the Bismut connection of g, with the curvatures F_h and R.

Remark 5.2. The second equation in (5.1), that is, $\Lambda_\omega R = 0$, is often neglected in the literature. In this case, R is typically taken to be the Chern connection of g on the holomorphic tangent bundle TX (see e.g. [76]). Motivation for considering the Hermite-Einstein condition for R comes from physics, and it will be explained in Section 6.1.

Fixing the holomorphic structure $\bar\partial_T$ on the smooth complex vector bundle (TM, J) essentially determines the hermitian metric which solves the Hermite-Einstein equation on $(TM, J, \bar\partial_T)$ (see Section 4.2), and therefore lead us to an overdetermined system of equations. To put the bundles E and (TM, J) on equal footing, it is convenient to take a gauge-theoretical point of view, by fixing the hermitian metric h on E and substitute the unknowns h and $\bar\partial_T$ in Definition 5.1 by a unitary connection A on (E, h) and a unitary connection ∇ on (TM, J, g), with curvature R_∇ (cf. Section 4.2). This lead us to the following equivalent definition of the Strominger system.

Definition 5.3. The *Strominger system*, for a hermitian metric g on (X, Ω), a unitary connection A on (E, h) and a unitary connection ∇ on (TM, J, g), is

$$\begin{aligned} \Lambda_\omega F_A &= 0, \qquad F_A^{0,2} = 0\\ \Lambda_\omega R_\nabla &= 0, \qquad R_\nabla^{0,2} = 0\\ d(\|\Omega\|_\omega \omega^{n-1}) &= 0,\\ dd^c\omega - \alpha\,(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A) &= 0. \end{aligned} \tag{5.3}$$

From this alternative point of view, the Strominger system couples a pair of Hermite-Yang-Mills connections A and ∇ (see Definition 4.9) with a conformally balanced metric ω, by means of the Bianchi identity (5.2). For the equivalence between (5.1) and (5.3), we use Li-Yau characterization of the dilatino equation (3.2) in terms of the conformally balanced equation (3.3).

The system (5.3) makes more transparent three types of necessary conditions for the existence of solutions of the Strominger system. Firstly, for (X, Ω, E) to admit a solution of the Strominger system there are some evident cohomological obstructions on the Chern classes

$$\deg_\tau(E) = 0, \qquad c_1(X) = 0, \tag{5.4}$$

and also

$$ch_2(E) = ch_2(X) \in H^{2,2}_{BC}(X,\mathbb{R}), \tag{5.5}$$

where $ch_2(E)$ and $ch_2(X)$ denote the second Chern character of E and X. On a general complex manifold, the condition (5.5) depends on the complex structure of X, since the natural map

$$H^{2,2}_{BC}(X,\mathbb{R}) \to H^4(X,\mathbb{R})$$

may have a Kernel. Secondly, we have two further conditions on the complex structure on X, that is, the complex manifold must have trivial canonical bundle and it must be balanced. Recall that the balanced condition for X can be expressed as a positivity condition on the homology of X, which involves complex currents (see Proposition 2.9 and [79]). Finally, if (g,∇,A) is a solution with balanced class

$$\tau = [\|\Omega\|_\omega \omega^{n-1}] \in H^{n-1,n-1}_{BC}(X,\mathbb{R}),$$

Theorem 4.5 implies that the holomorphic bundle $\mathcal{E} = (E,\bar{\partial}_A)$ and the holomorphic bundle $\mathcal{T} = (TM,J,\bar{\partial}_\nabla)$ must be τ-polystable.

In the rest of this section we discuss some basic examples of solutions of the Strominger system. We postpone more complicated existence results to Section 5.2. Let us start with complex dimension 1.

Example 5.4. When $n = 1$, X is forced to be an elliptic curve and the Bianchi identity is an empty condition. Furthermore, the dilatino equation reduces to the Kähler Ricci-flat condition on X (see Section 3.1). Hence, by Theorem 4.5 solutions of the Strominger system with a prescribed Kähler class are given by degree-zero polystable holomorphic vector bundles over the elliptic curve X (see e.g. [97]).

The next example shows that the system can always be solved for Calabi-Yau surfaces, provided that (5.4) and (5.5) are satisfied. Note that a compact Calabi-Yau surface must be a $K3$ surface or a complex torus, and therefore is always kählerian. Hence, the condition (5.5) in this case is topological. We follow an argument of Strominger in [93] and do not assume that the connection ∇ is unitary (to the knowledge of the author, with the unitary assumption there is no general existence result for $n = 2$).

Example 5.5. Let (X,Ω) be a compact Calabi-Yau manifold of dimension 2. Let g be a Kähler Ricci-flat metric on X, with Kähler form ω. By Example 3.7, g solves the dilatino equation. Let ∇ be a unitary Hermite-Yang-Mills connection on the smooth hermitian bundle (TM,J,g) (e.g. we can take the Chern connection of g). Let (E,h) be a smooth complex hermitian vector bundle satisfying (5.4) and (5.5), with a Hermite-Yang-Mills connections A. We want to show that we can find a metric $\tilde{g}$ on the conformal class of g solving the Strominger system. Consider $\tilde{g} = e^f g$ for $f \in C^\infty(X)$. Using that $n = 2$ we have

$$\|\Omega\|_{\tilde{\omega}}\tilde{\omega}^{n-1} = \|\Omega\|_\omega \omega^{n-1},$$

and therefore $\tilde{\omega}$ solves the dilatino equation. Furthermore, ∇ and A are Hermite-Yang-Mills connections also for $\tilde{\omega}$ (note that ∇ is no longer $\tilde{\omega}$-unitary). Hence, to solve the Strominger system (5.3) with this ansatz, we just need to solve the Bianchi identity (5.2) for $\tilde{g}$. Now, using that g is Kähler, (5.2) is equivalent to

$$\Delta_\omega(e^f) = \Lambda^2_\omega \alpha(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A).$$

The obstruction to solve this equation is $ch_2(E) = ch_2(X)$ in $H^4(X, \mathbb{R})$, which holds by assumption. Note that we can assume our solution to be positive, since X is compact. As a concrete example, we can consider X to be a $K3$ surface and take $\mathcal{T} = TX$ and $\mathcal{E}$ a small holomorphic deformation of TX or $TX \oplus \mathcal{O}_X$ [67], and apply Theorem 4.5.

We discuss next the arguably most basic examples of solutions of the Strominger system in dimension $n \geq 3$.

Example 5.6. Let (X, Ω) be a compact kählerian Calabi-Yau manifold of dimension n. Let g be a Kähler Ricci-flat metric on X. By Example 3.7, g solves the dilatino equation and by Example 4.7, the Levi-Civita connection ∇^g is Hermite-Einstein. Set $\nabla = \nabla^g$ and denote by h_0 a constant hermitian metric on the trivial bundle $\oplus_{i=1}^{r-n}\mathcal{O}_X$ over X. Define

$$\mathcal{E} = TX \bigoplus (\oplus_{i=1}^{r-n}\mathcal{O}_X),$$

that we consider endowed with the hermitian metric $g \oplus h_0$, with Chern connection $A = \nabla \oplus d$. Then, it is immediate that $g \oplus h_0$ is Hermite-Einstein. Finally, the Bianchi identity (5.2) is satisfied, because $trR^2 = trF_A^2$ and $dd^c\omega = 0$, since g is Kähler.

When $n = r = 3$, these are called *standard embedding solutions* in the physics literature, based on the natural homomorphism of Lie groups

$$SU(3) \to E_8.$$

5.2 Existence results

The Strominger system is a fully non-linear coupled system of partial differential equations for the hermitian metric g, and the unitary connections ∇ and A. Note that the system is of mixed order, since the Hermite-Yang-Mills equations are of first order in ∇ and A, the dilatino equation is of order one in the Kähler form ω, while the Bianchi identity is of second order in ω. The most demanding and less understood condition of the Strominger system is, indeed, the Bianchi identity (5.2). In dimension $n \geqslant 3$, this condition is the ultimate responsible of the non-Kähler nature of this PDE problem, as the non-vanishing of the *Pontryagin term* $\operatorname{tr} R_\nabla^2 - \operatorname{tr} F_A^2$ prevents the hermitian form ω to be closed and hence allows the complex manifold X to be non-kählerian. To the present day, we have a very poor understanding of the Bianchi identity from an analytical point of view.

The first solutions of the system for $\mathcal{E}$ a stable holomorphic vector bundle with rank $r = 4, 5$ over an algebraic Calabi-Yau threefold were found by Li and Yau [76] (cf. Example 5.6). Solutions in non-kählerian threefolds were first obtained by Fu and Yau [45], on suitable torus fibrations over a $K3$ surface. Further solutions in non-kählerian homogeneous spaces, specially on nilmanifolds, have been found over the last years (see [34, 32, 57, 83] and references therein). For examples in non-compact threefolds we refer to [41, 30, 31, 33].

There are essentially three known methods to solve the Strominger system in a compact complex threefold: by perturbation in kählerian manifolds, by reduction in a non-kählerian fibration over a Kähler manifold, and the method of invariant solutions in homogeneous spaces. The aim of this section is to illustrate these three methods with concrete results and examples. By the end, we will comment on a conjecture by Yau, which is one of the main open problems in this topic.

We start with a general result on kählerian manifolds [7], that builds in the seminal work of Li and Yau in [76].

Theorem 5.7 ([7])**.** *Let (X, Ω) be a compact Calabi-Yau threefold endowed with a Kähler Ricci-flat metric ω_∞ with holonomy $SU(3)$. Let $\mathcal{E}$ be a holomorphic vector bundle over X satisfying* (5.4) *and* (5.5)*. If $\mathcal{E}$ is stable with respect to $[\omega_\infty^2]$, then there exists a 1-parameter family of solutions $(h_\delta, \omega_\delta, \bar\partial_\delta)$ of the Strominger system* (5.1) *such that $\frac{\omega_\delta}{\delta}$ converges to ω_∞ as $\delta \to \infty$.*

To explain the main idea, we note that the Strominger system is invariant under rescaling of the hermitian form ω, except for the Bianchi identity. Given a positive real constant δ, if we change $\omega \to \delta\omega$ and define $\epsilon := \alpha/\delta$ we obtain a new system, with all the equations unchanged except for the Bianchi identity, which reads

$$dd^c\omega - \epsilon(\operatorname{tr}(R_\nabla \wedge R_\nabla) - \operatorname{tr}(F_A \wedge F_A)) = 0.$$

In the *large volume limit* $\delta \to \infty$, a solution of the system is given by prescribing degree zero stable holomorphic vector bundles $\mathcal{E}$ and $\mathcal{T}$ over a Calabi-Yau threefold with hermitian metric ω_∞ satisfying

$$d(\|\Omega\|_{\omega_\infty}\omega_\infty^2) = 0, \qquad dd^c\omega_\infty = 0.$$

The combination of these two conditions implies that ω_∞ is actually Kähler Ricci-flat, and by the Hermite-Yang-Mills condition for ∇ we also have that $\mathcal{T} \cong TX$ (see [7, Lem. 4.1]). The final step is to perturb a given solution with $\epsilon = 0$ to a solution with small $\epsilon > 0$, that is, with large δ, provided that (5.5) is satisfied. This is done via the Implicit Function Theorem in Banach spaces. The perturbation leaves the holomorphic structure of $\mathcal{E}$ unchanged while the one on TX is shifted by a complex gauge transformation and so remains isomorphic to the initial one.

We give a concrete example where the hypothesis of Theorem 5.7 are satisfied. For further examples of stable bundles on algebraic Calabi-Yau threefolds satisfying (5.4) and (5.5) we refer to [7, 8, 69].

Example 5.8. For X a generic quintic in $\mathbb{CP}^4$ (see Example 3.9), any Kähler Ricci-flat metric has holonomy $SU(3)$. Then, by a result of Huybrechts [67], the bundle $TX \oplus \mathcal{O}_X$ admits stable holomorphic deformations $\mathcal{E}$, which therefore have the same Chern classes as TX. The application of Theorem 5.7 in this example recovers [76, Th. 5.1].

We recall next the reduction method of Fu and Yau [45], based on the non kählerian fibred threefolds constructed by Goldstein and Prokushkin [55]. This result does not impose the Hermite-Yang-Mills condition on ∇, that is taken to be the Chern connection of the hermitian metric on X. Let (S, Ω_S) be a $K3$ surface with a Kähler Ricci-flat metric g_S and Kähler form ω_S. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on S such that

$$[\omega_i/2\pi] \in H^2(S,\mathbb{Z}).$$

Let X be the total space of the fibred product of the $U(1)$ line bundles determined by $[\omega_1/2\pi]$ and $[\omega_2/2\pi]$. Given a function u on S, consider the hermitian form

$$\omega_u = p^*(e^u\omega_S) + \frac{i}{2}\theta \wedge \overline{\theta}, \tag{5.6}$$

where θ is a connection on X such that $iF_\theta = \omega_1 + \omega_2$, and the complex threeform

$$\Omega = \Omega_S \wedge \theta.$$

Then, using that ω_1 and ω_2 are anti-self-dual, it is easy to check that ω_u satisfies the dilatino equation (3.2) and $d\Omega = 0$. Let $\mathcal{E}_S$ be a degree zero $[\omega_S]$-stable holomorphic vector bundle over S. Define $\mathcal{E} = p^*\mathcal{E}_S$ and $h = p^*h_S$, where h_S is the Hermite-Einstein metric on $\mathcal{E}_S$. Then, h is a Hermite-Einstein metric for ω_u and hence with this ansatz the Strominger system reduces to the Bianchi identity. This identity is actually equivalent to the following complex Monge-Ampère equation on S

$$dd^c(e^u\omega - \alpha e^{-u}\rho) + \frac{1}{2}dd^cu \wedge dd^cu = \mu\omega_S^2/2,$$

where ρ is a smooth real $(1,1)$-form on S independent of u and

$$\mu\omega_S^2 = (|\omega_1|^2 + |\omega_2|^2)\omega_S^2 + \alpha(\operatorname{tr} F_h \wedge F_h - R_{\omega_S} \wedge R_{\omega_S}).$$

Here, R_{ω_S} denotes the curvature of the Chern connection of ω_S on S.

Theorem 5.9 ([45]). *The equation* (5.6) *admits a solution for* $\alpha > 0$, *provided that*

$$0 = \int_S \mu \omega_S^2 = \int_S (|\omega_1|^2 + |\omega_2|^2)\omega_S^2 - 8\pi^2\alpha(24 - c_2(\mathcal{E}_S)).$$

The case $\alpha < 0$ was proved in [44], and more recently in [85] with different methods.

We consider now the method of invariant solutions in homogeneous spaces. Following [32, 83], we describe an explicit solution of the form $X = \mathrm{SL}(2,\mathbb{C})/\Gamma$, for Γ a cocompact lattice in $\mathrm{SL}(2,\mathbb{C})$. The group $\mathrm{SL}(2,\mathbb{C})$ is unimodular and therefore it admits a biinvariant holomorphic volume form. Furthermore, any right invariant metric on $\mathrm{SL}(2,\mathbb{C})$ is balanced, and solves the dilatino equation (cf. Example 3.8). With the ansatz $F_A = 0$ for the connection A, the Strominger system (5.3) reduces to the conditions

$$\begin{aligned} \Lambda_\omega R_\nabla = 0, \qquad R_\nabla^{0,2} = 0 \\ dd^c\omega - \alpha\,(\mathrm{tr}\, R_\nabla \wedge R_\nabla) = 0. \end{aligned} \tag{5.7}$$

We want to check that the system is satisfied for $\nabla = \nabla^g - \frac{1}{2}g^{-1}d^c\omega$ the Bismut connection of ω. To see this, consider a right invariant basis $\{\sigma^1, \sigma^2, \sigma^3\}$ of $(1,0)$-forms satisfying

$$d\sigma^1 = \sigma^2 \wedge \sigma^3, \qquad d\sigma^2 = -\sigma^1 \wedge \sigma^3, \qquad d\sigma^3 = \sigma^1 \wedge \sigma^2.$$

Consider the biinvariant holomorphic volume form

$$\Omega = \sigma^1 \wedge \sigma^2 \wedge \sigma^3$$

and the right invariant hermitian metric

$$\omega_t = \frac{i}{2}t^2(\sigma^1 \wedge \overline{\sigma}^1 + \sigma^2 \wedge \overline{\sigma}^2 + \sigma^3 \wedge \overline{\sigma}^3)$$

for $t \in \mathbb{R}\backslash\{0\}$. Define a real basis of right invariant 1-forms by

$$e^1 + ie^2 = t\sigma^1, \qquad e^3 + ie^4 = t\sigma^2, \qquad e^5 + ie^6 = t\sigma^3.$$

Then, a direct calculation shows that (see [83, Th. 4.3] and [32, eq. (8)]

$$dd^c\omega_t = -\frac{4}{t^2}(e^{1234} + e^{1256} + e^{3456}),$$

and for ∇ the Bismut connection

$$\alpha\,\mathrm{tr}\, R_\nabla \wedge R_\nabla = -\alpha\frac{16}{t^4}(e^{1234} + e^{1256} + e^{3456}).$$

Therefore, for $\alpha > 0$ taking t such that $\alpha = t^2/4$ we obtain a solution of the Bianchi identity. Furthermore, by [83, Prop. 4.1] the Bismut connection is also a solution of the Hermite-Yang-Mills equations.

Although the three methods we have just explained provide a large class of solutions of the Strominger system in complex dimension 3, the existence problem is widely open. The following conjecture by Yau is one of the main open problems in this topic.

Conjecture 5.10 (Yau [105])**.** *Let (X, Ω) be a compact Calabi-Yau threefold endowed with a balanced class τ. Let $\mathcal{E}$ be a holomorphic vector bundle over X satisfying* (5.4) *and* (5.5)*. If $\mathcal{E}$ is stable with respect to τ, then $(X, \Omega, \mathcal{E})$ admits a solution of the Strominger system.*

Even for kählerian manifolds, Conjecture 5.10 is not completely understood. In this setup, Theorem 5.7 provides a solution of Conjecture 5.10 for balanced classes of the form $\tau = [\omega]^2$, where $[\omega]$ is a Kähler class on X. We note however that Fu and Xiao [43] have proved that for projective Calabi-Yau n-folds the cohomology classes

$$[\beta] \in H^{1,1}(X, \mathbb{R})$$

such that $[\beta]^n > 0$ – known as *big classes* – satisfy that $[\beta]^{n-1}$ is a balanced class. An interesting example of a big class which is not Kähler is provided by Example 3.9 on conifold transitions. With the notation stated there, if L is the pull-back of any ample divisor on X_0, then $c_1(L)$ is a big (and nef) class on X. By a result of Tosatti [95], the smooth Ricci-flat metrics on X with classes approaching $c_1(L)$ have a well-defined limit, given by the pull-back of the unique singular Ricci-flat metric on X_0. It is plausible that this result combined with Theorem 5.7 can be used to prove Conjecture 5.10 for algebraic Calabi-Yau threefolds. We should also note that the method of Theorem 5.7 does not have any control on the balanced class of the final solution. On general grounds, it is expected that the solution predicted by Conjecture 5.10 has balanced class τ.

For non-kählerian manifolds, Yau's Conjecture is widely open. To illustrate this, we state a basic question that should be addressed before dealing with the more general Conjecture 5.10.

Question 5.11. Let X be a compact complex manifold with balanced class $\tau \in H^{n-1,n-1}_{BC}(X, \mathbb{R})$. Let $\rho \in \Omega^{n-1,n-1}$ be a real dd^c-exact form on X. Is there a balanced metric g on X with balanced class τ solving the following equation?

$$dd^c\omega = \rho. \tag{5.8}$$

A promising approach to Conjecture 5.10 using geometric flows – which in particular treats a question closely related to Question 5.11 – has been recently proposed in [84].

Setting Question 5.11 in the affirmative in the case of Clemens-Friedman non-kählerian complex manifolds would provide important support for Yau's proposal of a metric approach to Reid's Fantasy (see Section 3.2). Equation (5.8) (as well as the Strominger system) pins down a particular solution of the dilatino equation (3.2) in a given balanced class, via a condition on the torsion $-d^c\omega$ of the Bismut connection. As we have pointed out earlier in this section, when $\rho = 0$ the combination of (5.8) with the dilatino equation (3.2) is equivalent to the metric being Calabi-Yau. The mechanism whereby a conifold transition creates a 4-form ρ which couples to the metric is still unknown (in physics, ρ can be interpreted as the Poincaré dual four-current of a holomorphic submanifold wrapped by a NS5-brane [96]).

6 Physical origins and string classes

6.1 The Strominger system in heterotic supergravity

The Strominger system arises in the low-energy limit of the heterotic string theory. This theory is described by a σ-model, a quantum field theory with fields given by smooth maps $C^\infty(S, N)$, from a smooth surface S – the worldsheet of the string – into a target manifold N. From the point of view of the worldsheet, the theory leads to a superconformal field theory. In the low-energy limit, the heterotic string can be described from the point of view of N, yielding a supergravity theory. We start with a discussion of classical heterotic supergravity, which allows a rigorous derivation of the Strominger system (and completely omits perturbation theory). We postpone a conceptual explanation of the Bianchi identity to the next section, where we adopt the worldsheet approach.

Heterotic supergravity is a ten-dimensional supergravity theory coupled with super Yang-Mills theory (see e.g. [27, p. 1101]). It is formulated on a 10-dimensional spin manifold N, i.e. oriented, with vanishing second Stiefel-Whitney class

$$w_2(N) = 0,$$

and with a choice of element in $H^1(N, \mathbb{Z}_2)$. The manifold is endowed with a principal bundle P_K, with compact structure group K, contained in $SO(32)$ or $E_8 \times E_8$. We will assume $K = SU(r)$.

The (bosonic) field content of the theory is given by a metric g_0 of signature $(1, 9)$ (in the string frame), a (dilaton) function $\phi \in C^\infty(N)$, a 3-form $H \in \Omega^3$ and a (gauge) connection A on P_K. We ignore the fermionic fields in our discussion.

The equations of motion can be written as

$$
\begin{aligned}
\operatorname{Ric}^{g_0} -2\nabla^{g_0}(d\phi) - \frac{1}{4}H\circ H + \alpha \operatorname{tr} F\circ F - \alpha \operatorname{tr} R\circ R = 0,\\
d^*(e^{2\phi}H) = 0,\\
d^*_A(e^{2\phi}F) + \frac{e^{2\phi}}{2} * (F\wedge *H) = 0,\\
S^{g_0} - 4\Delta\phi - 4|d\phi|^2 - \frac{1}{2}|H|^2 + \alpha(|R|^2 - |F|^2) = 0,
\end{aligned} \tag{6.1}
$$

where α as positive real constant – the *slope string parameter* –, F is the curvature of A, and R is the curvature of an auxiliary connection ∇_0 on TN. Here, $H\circ H$ is a symmetric 2-tensor constructed by contraction with the metric (and similarly for $\operatorname{tr} F\circ F - \operatorname{tr} R\circ R$)

$$(H\circ H)_{mn} = g_0^{ij}g_0^{kl}H_{ikm}H_{jln}.$$

The introduction of the connection ∇_0 – which is not considered as a physical field – is due to the cancellation of anomalies (this is the failure of a classical symmetry to be a symmetry of the quantum theory). The Green-Schwarz mechanism of anomaly cancellation [58] sets a particular local ansatz for the three-form

$$H = db - \alpha(CS(\nabla_0) - CS(A)), \tag{6.2}$$

in terms of a 2-form $b\in\Omega^2$ and the Chern-Simons 3-forms of A and ∇_0. We use the convention $dCS(A) = -\operatorname{tr} F_A\wedge F_A$. Up to the Chern-Simons term, H can be regarded as the *field strength* of the locally-defined *B-field* b. Although (6.2) fails to be globally well-defined on N, it imposes the global *Bianchi identity* constraint

$$dH = \alpha(\operatorname{tr} R\wedge R - \operatorname{tr} F\wedge F). \tag{6.3}$$

We postpone the conceptual explanation of this equation to the worldsheet approach.

We note that the equations (6.1) do not arise as critical points of any functional, e.g. due to the term $*(F\wedge *H)$ in the third equation. Rather, physicists consider the *pseudo-action*

$$\int_N e^{-2\phi}(S^{g_0} + 4|d\phi|^2 - \frac{1}{12}|H|^2 + \frac{\alpha'}{2}(|R|^2 - |F|^2))\operatorname{Vol}_{g_0},$$

where the norm squared of F and R is taken with respect to the Killing form $-\operatorname{tr}$. Remarkably, calculating the critical points of this functional and taking the local form (6.2) of H into account, yields the equations of motion (6.1).

In supergravity theories, supersymmetry distinguishes special solutions of the equations of motion which are fixed by the action of a (super) Lie algebra on the space of fields. Generators for this action are typically given in terms of spinors. In the case of heterotic supergravity, ($N = 1$) supersymmetry requires the existence of a non-vanishing Majorana-Weyl spinor ϵ with positive chirality (see [35, p. 9]), satisfying the *Killing spinor equations*

$$\begin{aligned} F \cdot \epsilon &= 0 \\ \nabla^{-} \epsilon &= 0, \\ (H + 2d\phi) \cdot \epsilon &= 0, \end{aligned} \tag{6.4}$$

Here ∇^{-} is the metric connection with skew torsion $-H$ obtained from the Levi-Civita connection ∇^{g_0}

$$\nabla^{-} = \nabla^{g_0} - \frac{1}{2} g_0^{-1} H. \tag{6.5}$$

The relation between the heterotic supergravity equations and the Strominger system arises via a mechanism called *compactification*, whereby the 10-dimensional theory is related to a theory in 4-dimensions. Strominger [93] and Hull [64] characterized the geometry of a very general class of compactifications of heterotic supergravity, inducing a 4-dimensional supergravity theory with $N = 1$ supersymmetry. The geometric conditions that they found in the so called *internal space* is what is known today as the Strominger system. Mathematically, Strominger-Hull compactifications amount to the following ansatz: the space-time manifold is a product

$$N = \mathbb{R}^4 \times M,$$

where M is a compact smooth oriented spin 6-dimensional manifold – the internal space –, with metric given by

$$g_0 = e^{2(f-\phi)}(g_{1,3} \times g)$$

for $g_{1,3}$ a flat Lorentz metric, g a riemannian metric on M and $f \in C^\infty(M)$ a smooth function on M. The fields H and A are pull-back from M, and ∇_0 is the product of the Levi-Civita connection of $g^{1,3}$ with a connection ∇ on TM compatible with g. The condition of $N = 1$ supersymmetry in 4-dimensions imposes that $f = \phi$ and also

$$\epsilon = \zeta \otimes \eta + \zeta^* \otimes \eta^*,$$

where ζ is a positive chirality spinor for $g_{1,3}$ and η is a positive chirality spinor for g (living in complex representations of the corresponding real Spin group), while ζ^* and η^* denote their respective conjugates. With this ansatz, the equations (6.1), (6.4) and (6.3) are equivalent, respectively, to equations for $(g, \phi, H, \nabla, A, \eta)$ on

M:

$$\begin{aligned} \operatorname{Ric}^g - 2\nabla^g(d\phi) - \frac{1}{4}H \circ H - \alpha F_A \circ F_A + \alpha R_\nabla \circ R_\nabla &= 0, \\ d^*(e^{2\phi}H) &= 0, \\ d_A^*(e^{2\phi}F_A) + \frac{e^{2\phi}}{2} * (F_A \wedge *H) &= 0, \\ S^g - 4\Delta\phi - 4|d\phi|^2 - \frac{1}{2}|H|^2 + \alpha(|R_\nabla|^2 - |F_A|^2) &= 0 \end{aligned} \tag{6.6}$$

$$\begin{aligned} \nabla^- \eta &= 0, \\ (d\phi + \frac{1}{2}H) \cdot \eta &= 0, \\ F_A \cdot \eta &= 0, \end{aligned} \tag{6.7}$$

$$dH - \alpha(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A) = 0. \tag{6.8}$$

The following characterization of (6.7) and (6.8) in terms of $SU(3)$-structures is due to Strominger and Hull.

Theorem 6.1 ([64, 93])**.** *A solution $(g, \phi, H, \nabla, A, \eta)$ of* (6.7) *and* (6.8) *is equivalent to a Calabi-Yau structure (X, Ω) on M, with hermitian metric g and connection A on P_K solving*

$$\begin{aligned} \Lambda_\omega F_A = 0, \qquad F_A^{0,2} &= 0, \\ d^*\omega - d^c \log \|\Omega\|_\omega &= 0, \\ dd^c\omega - \alpha\,(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A) &= 0. \end{aligned} \tag{6.9}$$

where

$$H = d^c\omega, \qquad d\phi = -\frac{1}{2} d \log \|\Omega\|_\omega.$$

For the proof, one can use that the stabilizer of η in Spin(6) is SU(3), and write the equations in terms of the corresponding SU(3)-structure. The key point is that, since ∇^- is unitary and has totally skew-torsion $-H$, by [53, Eq. (2.5.2)],

$$H = -N + (d^c\omega)^{2,1+1,2},$$

where N denotes the Nijenhuis tensor of the almost complex structure determined by η. We refer to [48, Th. 6.10] for a detailed proof of this result. We note that the same result holds on an arbitrary even-dimensional manifold M, provided that the spinor η is pure (see [74, Lem. 9.15] and [74, Rem. 9.12]).

Supersymmetric vacuum of heterotic supergravity compactified on M – with the Strominger-Hull ansatz – correspond to solutions of the system of equations formed by (6.6), (6.7) and (6.8). Therefore, finding a solution of (6.9) is a priori not enough to find a supersymmetric classical solution of the theory. This problem was understood by Fernandez, Ivanov, Ugarte and Villacampa [34, 68], who provided a characterization of the solutions of the Killing spinor equations (6.7) and the Bianchi identity (6.8) which also solve the equations of motion (6.6).

Theorem 6.2 ([34, 68]). *A solution of the Killing spinor equations* (6.7) *and the Bianchi identity* (6.8) *is a solution of the equations of motion* (6.6) *if and only if*

$$R_\nabla \cdot \epsilon = 0. \tag{6.10}$$

The *instanton condition* (6.10) is equivalent to ∇ being a Hermite-Yang-Mills connection, that is,

$$R_\nabla^{0,2} = 0, \quad R_\nabla \wedge \omega^2 = 0. \tag{6.11}$$

Combined with Theorem 6.1, Theorem 6.2 establishes the link between the equations in heterotic supergravity and the Strominger system (5.3). To state a precise result, which summarizes the previous discussion, we observe that it follows from the proof of Theorem 6.2 that the instanton condition $R_\nabla \cdot \epsilon = 0$ jointly with (6.7) and (6.8) implies the following *equation of motion for* ∇

$$d^*_\nabla(e^{2\phi}R_\nabla) + \frac{e^{2\phi}}{2} * (R_\nabla \wedge *H) = 0.$$

Theorem 6.3. *A solution* $(g, \phi, H, \nabla, A, \eta)$ *of the system*

$$\begin{aligned} \nabla^-\eta &= 0, \\ (d\phi + \frac{1}{2}H) \cdot \eta &= 0, \\ F_A \cdot \eta &= 0, \\ R_\nabla \cdot \eta &= 0, \\ dH - \alpha(\operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A) &= 0. \end{aligned} \tag{6.12}$$

is equivalent to a Calabi-Yau structure (X, Ω) *on* M*, with hermitian metric* g *and connection* A *on* P_K *solving the Strominger system* (5.3)*, where*

$$H = d^c\omega, \qquad d\phi = -\frac{1}{2}d\log\|\Omega\|_\omega.$$

Furthermore, any solution of (6.12) *solves the equations of motion*

$$\begin{aligned} \operatorname{Ric}^g - 2\nabla^g(d\phi) - \frac{1}{4}H \circ H - \alpha F_A \circ F_A + \alpha R_\nabla \circ R_\nabla = 0, \\ d^*(e^{2\phi}H) = 0, \\ d_A^*(e^{2\phi}F_A) + \frac{e^{2\phi}}{2} * (F_A \wedge *H) = 0, \\ d_\nabla^*(e^{2\phi}R_\nabla) + \frac{e^{2\phi}}{2} * (R_\nabla \wedge *H) = 0, \\ S^g - 4\Delta\phi - 4|d\phi|^2 - \frac{1}{2}|H|^2 + \alpha(|R_\nabla|^2 - |F_A|^2) = 0. \end{aligned} \tag{6.13}$$

The formal symmetry of the equations (6.12) in the connections ∇ and A (which flips a sign in the Bianchi identity), seems to be crucial for the understanding of the geometry and the moduli problem for the Strominger system, that we review in Section 7.

Remark 6.4. We note that in the formulation of the Strominger system in [93] the condition (6.11) was not included, probably relying on the general principle that supersymmetry implies the equations of motion of the theory (which is not valid for the heterotic string even in perturbation theory [25]). The analysis in [64, 65] takes this fact into account and proposes $\nabla = \nabla^+$ (given by changing $H \to -H$ in (6.5)) as the preferred connection to solve (6.8). From a mathematical perspective, this last statement has to be taken rather formally. It can be regarded as a perturbative version of Theorem 6.2, in the following sense: expanding g and H in a formal power series in the parameter α it follows that

$$R_{\nabla^+} \cdot \eta = O(\alpha),$$

provided that $\nabla^- \eta = 0$ is satisfied (see e.g. [25, App. C]).

6.2 The worldsheet approach and string classes

A more fundamental approach to heterotic string theory is provided by the (non-linear) σ-model. We start with a brief (and rather naive) description of this theory, in order to explain the Bianchi identity in heterotic supergravity and its relation with the notion of *string class*.

The heterotic non-linear σ-model is a two-dimensional quantum field theory with (bosonic) fields given by smooth maps $f \in C^\infty(S, N)$ from an oriented surface S into a target manifold N. To describe the classical action, we fix a metric γ on S – with volume Vol_γ and scalar curvature R_γ – and *background fields* (g^0, ϕ, b, A) on N. Here, (g^0, ϕ, A) are as in the previous section, and b is a *B-field*, that by now we treat as a (local) two-form on N (more invariantly, it will correspond to a trivialization of a bundle 2-gerbe with connection [99]). The action is

$$\frac{1}{4\pi\alpha}\int_S |df|^2 \mathrm{Vol}_\gamma + f^*b - \frac{\alpha}{2}\phi R_\gamma \mathrm{Vol}_\gamma + \dots \tag{6.14}$$

where $|df|^2$ denotes the norm square of df with respect to γ and g. The terms denoted $\dots$ correspond to the fermionic part of the action – depending on the connections A and $\nabla^- = \nabla^g - \frac{1}{2}g^{-1}db$, and a choice of spin structure on S – that we omit for simplicity (see e.g. [93]).

The constant α in (6.14) is $2\ell^2$, where ℓ is the Planck length scale (and hence positive). The background fields in the σ-model appear as coupling functions (generalizing the notion of coupling constant). The value of the dilaton ϕ at a point determines the string coupling constant, i.e. the strength with which strings interact with each other. The dilaton is a special field, as the term

$$\int_S \phi R_\gamma \mathrm{Vol}_\gamma$$

destroys the conformal invariance of the action (6.14) (classical Weyl invariance). Nonetheless, the inclusion of this term is crucial for the conformal invariance of the theory at the quantum level [19].

The quantum theory constructed from the action (6.14) is defined perturbatively, in an expansion in powers of α. Conformal invariance of the *effective action* corresponds to the vanishing of the *β-functions*, which in the critical dimension $\dim N = 10$ are given by

$$\begin{aligned}
\beta^G &= \mathrm{Ric}^{g_0} - 2\nabla^{g_0}(d\phi) - \frac{1}{4}H\circ H + \alpha \operatorname{tr} F\circ F - \alpha \operatorname{tr} R\circ R + O(\alpha^2),\\
\beta^B &= d^*(e^{2\phi}H) + O(\alpha^2),\\
\beta^A &= d^*_A(e^{2\phi}F) + \frac{e^{2\phi}}{2} * (F\wedge *H) + O(\alpha^2),\\
\beta^\phi &= S^{g_0} - 4\Delta\phi - 4|d\phi|^2 - \frac{1}{2}|H|^2 + \alpha(|R|^2 - |F|^2) + O(\alpha^2),
\end{aligned} \tag{6.15}$$

where H is a three-form on N and R is the curvature of an auxiliary connection ∇_0 on TN, locally related with b and A by the Green-Schwarz ansatz (6.2). The sudden appearance of the extra connection ∇_0 is explained by the way the fields are treated in perturbation theory, as formal expansions in the parameter α: even though the connection in (6.2) in the perturbation theory analysis is ∇^+ [66], the truncation to second order in α-expansion enables to remove its dependence from the rest of fields (cf. Remark 6.4).

We observe that the classical equations of motion of the heterotic supergravity in the target (6.1) are given by first-order conditions for conformal invariance (in α-expansion) of the quantum theory in the worldsheet of the string. Similarly, the killing spinor equations (6.4) are obtained as first order conditions in α-expansion in order to define a supersymmetric theory [65, 93]. When the β-functions vanish for a choice of background fields, the heterotic σ-model is expected to yield a two-dimensional superconformal field theory (with $(0,2)$-supersymmetry, when the killing spinor equations are satisfied). Unfortunately, to the present day a closed form of (6.15) to all orders in α-expansion is unknown. Despite of this fact, several rigorous attempts to construct this theory by indirect methods can be found in the literature (see e.g. [28, 56, 78]).

Aside from the disturbing perturbation theory, the (fermionic) terms omitted in the classical action (6.14) provide a conceptual explanation of the Bianchi identity (6.3), that we have ignored so far. The path integral quantization of the σ-model yields a (determinant) line bundle $\mathcal{L}$ over a space of Bose fields

$$B = \mathrm{Conf}(S) \times C^\infty(S,N)/\mathrm{Diff}_0(S),$$

where $\mathrm{Conf}(S)$ is the space of conformal structures on S and $\mathrm{Diff}_0(S)$ is the identity component of the diffeomorphism group of S. There is a canonical section s_0 of $\mathcal{L}$ – determined by the fermionic terms in the action – that should be integrated over B, and hence one tries to find a trivialization of $\mathcal{L}$, so as to express s_0 as a function on B. The obstruction to finding a trivialization is called the *anomaly*. This connection between anomalies and determinant line bundles was pioneered by Atiyah and Singer, in close relation to the Index Theorem. Here we follow closely a refined geometric version by Witten [101] and Freed [38].

The construction of the line bundle $\mathcal{L}$ in [38] assumes the compactification ansatz $N = \mathbb{R}^4 \times M^6$, discussed in the previous section. Further, M is endowed with an integrable almost complex structure J compatible with g, such that $c_1(TM,J) = 0$. Let (E,h) be a smooth hermitian vector bundle over M. We assume in this section that $c_1(E) = 0$ and that E has rank 16. The aim is to give an explanation of the *topological Bianchi identity*

$$dH = \alpha\left(\mathrm{tr}\, R_\nabla \wedge R_\nabla - \mathrm{tr}\, F_A \wedge F_A\right), \tag{6.16}$$

as an equation for an arbitrary three-form $H \in \Omega^3$ on M and a pair of special unitary connections ∇ on (TM,J) and A on (E,h). Observe that (6.16) implies a condition in the real first Pontryagin class $p_1(M)$ of M, namely

$$p_1(E) = p_1(M) \in H^4(M,\mathbb{R}). \tag{6.17}$$

Given the data (g,∇,A), Freed constructs in [38] a (Pfaffian) complex line bundle

$$\mathcal{L} \to B,$$

which is trivializable provided that

$$\frac{1}{2}p_1(E) = \frac{1}{2}p_1(M) \in H^4(M,\mathbb{Z}). \tag{6.18}$$

Furthermore, this line bundle is endowed with a natural connection $\mathbb{A}^0$ on $\mathcal{L}$ whose curvature $F_{\mathbb{A}^0}$ can be identified with

$$4\pi i F_{\mathbb{A}^0} \equiv \operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A.$$

Note here that a 4-form ρ on Y can be regarded as a 2-form ψ on $C^\infty(S,N)$, by

$$\psi(V_1,V_2) = \int_S f^* \iota_{V_1}\iota_{V2}\rho,$$

where $f \in C^\infty(S,N)$ and $V_1, V_2 \in T_f C^\infty(S,N)$, where the tangent space at f is identified with $C^\infty(S, f^*TN)$.

Assuming that (6.18) is satisfied, we try to parametrize flat connections with trivial holonomy on $\mathcal{L}$. This is an important question from a physical perspective, as different trivializations of $\mathcal{L}$ correspond to different partition functions of the heterotic string theory. The answer is closely related to the notion of *string class* [87], that we introduce next. Let P_g be the bundle of special unitary frames of the hermitian metric g on (TM,J), with structure group $\mathrm{SU}(3)$, and let P_h be the bundle of special unitary frames of the hermitian metric g, with structure group $\mathrm{SU}(r)$. Consider the principal bundle $p \colon P \to M$ given by

$$P = P_g \times_M P_h,$$

with structure group $G = \mathrm{SU}(3) \times \mathrm{SU}(r)$. Let σ denote the (left) Maurer-Cartan 1-form on G. We fix a biinvariant pairing on $\mathfrak{g}$

$$c = \alpha(\operatorname{tr}_{\mathfrak{su}(3)} - \operatorname{tr}_{\mathfrak{u}(r)}).$$

We assume that c is suitably normalized so that the $[\sigma_3] \in H^3(G,\mathbb{Z})$, where

$$\sigma_3 = \frac{1}{6} c(\sigma,[\sigma,\sigma]).$$

Definition 6.5 ([87])**.** A *string class* on P is a class $[\hat{H}] \in H^3(P,\mathbb{Z})$ such that the restriction of $[\hat{H}]$ to any fibre of P yields the class $[\sigma_3] \in H^3(G,\mathbb{Z})$.

String classes form a torsor over $H^3(M,\mathbb{Z})$, where the action is defined by pull-back and addition [87, Prop. 2.16]:

$$[\hat{H}] \to [\hat{H}] + p^*[H],$$

where $p \colon P \to M$ is the canonical projection on the principal bundle P and $[H] \in H^3(M,\mathbb{Z})$.

Flat connections with trivial holonomy on the line bundle $\mathcal{L}$ were interpreted by Bunke in [15] as (very roughly) enriched representatives of a string class in P. Here we give a simple-minded version of his construction, by choosing special

3-form representatives. Let $[\hat{H}] \in H^3(P,\mathbb{Z})$ be a string class. For our choice of connection $\theta = \nabla \times A$ on P we can take a G-invariant representative $\hat{H} \in \Omega^3(P)$ of the form (see Section 7.1)

$$\hat{H} = p^*H - CS(\theta) \tag{6.19}$$

for a choice of 3-form $H \in \Omega^3(M)$, determined up to addition of an exact three-form, where

$$CS(\theta) = -\frac{1}{6}c(\theta \wedge [\theta,\theta]) + c(F_\theta \wedge \theta).$$

Using that $d\hat{H} = 0$, we obtain that (H, ∇, A) solves the topological Bianchi identity (6.16), since $dCS(\theta) = c(F_\theta \wedge F_\theta)$. In conclusion, up to addition of an exact three-form, a string class determines a preferred solution of the topological Bianchi identity for a fixed connection θ.

We construct now a flat connection $\mathbb{A}^H$ on $\mathcal{L}$ using this fact. The connection $\mathbb{A}^H$ is defined by modification of $\mathbb{A}^0$ as follows

$$d_{\mathbb{A}^H} \log s = d_{\mathbb{A}^0} \log s - \frac{\alpha^{-1}}{4\pi i}\int_S f^*H,$$

where the left hand side is evaluated at the point $[(x,f)] \in B$ for $f \in C^\infty(S,N)$. The curvature of $\mathbb{A}^H$ can be identified with

$$4\pi i F_{\mathbb{A}^H} \equiv \operatorname{tr} R_\nabla \wedge R_\nabla - \operatorname{tr} F_A \wedge F_A - \alpha^{-1} dH,$$

and therefore $\mathbb{A}^H$ is flat. In [15, Th. 4.14], it is proved that $\mathbb{A}^H$ admits a parallel unit norm section s, therefore providing a trivialization of $\mathcal{L}$. Note here that if we chose a different three-form $H + db$ on M to represent $[\hat{H}]$, then this corresponds to a gauge transformation

$$s \to e^{\frac{\alpha^{-1}}{4\pi i}\int_S f^*b} s$$

of the section s.

String classes were introduced in [87] to parametrize *string structures* up to homotopy. String structures emerged from two-dimensional supersymmetric field theories in work by Killingback and Witten, and several definitions have been proposed so far. Given a spin bundle P over M, McLaughlin defines a string structure on P as a lift of the structure group of the looped bundle $LP = C^\infty(S^1,P)$ over the loop space

$$LM = C^\infty(S^1,M)$$

from $\mathrm{LSpin}(k) = C^\infty(S^1,\mathrm{Spin}(k))$ to its universal Kac-Moody central extension. Stolz and Teichner interpreted a string structure as a lift of the structure group of P from $\mathrm{Spin}(k)$ to a certain three-connected extension, the topological group $\mathrm{String}(k)$. For recent developments on this topic, in relation to the topological Bianchi identity (6.16), we refer the reader to [90, 91].

7 Generalized geometry and the moduli problem

7.1 The Strominger system and generalized geometry

In this section we review on recent developments on the geometry of the Strominger system, based on joint work of the author with Rubio and Tipler [48]. As we will see, the interplay of the Strominger system with the notion of string class (see Definition 6.5) leads naturally to an interesting relation with Hitchin's theory of generalized geometry [63], proposed in [47].

To start the discussion, we draw a parallel between the Strominger system and Maxwell equations in electromagnetism (cf. [96]). The Maxwell equations in a 4-manifold Y take the form

$$\begin{aligned} dF &= 0 \\ d * F &= j_e \end{aligned}$$

where F is a two-form – the electromagnetic field strength – and j_e is the three-form *electric current*. The cohomology class $[F] \in H^2(Y, \mathbb{R})$ is known as the *magnetic flux* of the solution. Whereas in classical electromagnetism the magnetic flux is allowed to take an arbitrary value in $H^2(Y, \mathbb{R})$, in the quantum theory Dirac's law of charge/flux quantization implies that magnetic fluxes are constrained to live in a full lattice inside $H^2(Y, \mathbb{R})$, namely

$$[F/2\pi] \in H^2(Y, \mathbb{Z}).$$

This changes the geometric nature of the problem. A geometric model which implements flux quantization takes the electromagnetic field to be the $i/2\pi$ times the curvature of a connection A on a $U(1)$ line bundle over Y. More generally, fixing the class $[F/2\pi] \in H^2(Y, \mathbb{R})$ amounts to fix the isomorphism class of a Lie algebroid over Y, and we can regard A as a 'global splitting' of the sequence defining the Lie algebroid.

Given a solution of the Strominger system (5.3), the three-form $d^c\omega$ and the connections ∇ and A determine a solution of the Bianchi identity (6.16). Relying on the discussion in Section (6.2), this data determines a *real string class* (the analogue of Definition 6.5 in real cohomology). The aim here is to understand how the global nature of the geometric objects involved in the Strominger system changes, upon fixing the real string class of the solutions.

We will consider a simplified setup, where the principal bundle considered in Definition 6.5 is not necessarily related to the tangent bundle of the manifold. Let M be a compact spin manifold M of dimension $2n$. Let P be a principal bundle, with structure group G. We assume that there exists a non-degenerate pairing c on the Lie algebra $\mathfrak{g}$ of G such that the corresponding first Pontryagin class of P

vanishes

$$p_1(P) = 0 \in H^4(M,\mathbb{R}). \tag{7.1}$$

Let $\mathcal{A}$ denote the space of connections θ on P. We denote by $\Omega^3_0 \subset \Omega^3_{\mathbb{C}}$ the space of complex 3-forms Ω such that

$$T^{0,1} := \{V \in TM \otimes \mathbb{C} \mid \iota_V \Omega = 0\}$$

determines an almost complex structure J_Ω on M, that we assume to be non-empty. Consider the parameter space

$$\mathcal{P} \subset \Omega^3_0 \times \mathcal{A} \times \Omega^2,$$

defined by

$$\mathcal{P} = \{(\Omega, \theta, \omega) \mid \omega \text{ is } J_\Omega - \text{compatible}\}.$$

The points in $\mathcal{P}$ are regarded as unknowns for the system of equations

$$\begin{aligned} d\Omega &= 0, & d(\|\Omega\|_\omega \omega^2) &= 0, \\ F_\theta^{0,2} &= 0, & F_\theta \wedge \omega^2 &= 0, \\ dd^c\omega - c(F_\theta \wedge F_\theta) &= 0, \end{aligned} \tag{7.2}$$

where F_θ denotes the curvature of θ, given explicitly by

$$F_\theta = d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(\operatorname{ad} P),$$

where θ is regarded as a G-invariant 1-form in P with values in $\mathfrak{g}$ and the bracket is the one on the Lie algebra. The induced covariant derivative on the bundle of Lie algebras $\operatorname{ad} P = P \times_G \mathfrak{g}$ is

$$\iota_V d_\theta r = [\theta^\perp V, r],$$

which satisfies $d_\theta \circ d_\theta = [F_\theta, \cdot]$.

To see the relation with the Strominger system, consider the particular case that P is the fibred product of the bundle of oriented frames of M and an $SU(r)$-bundle, with

$$c = \alpha(-\operatorname{tr} - c_{\mathfrak{gl}}). \tag{7.3}$$

Here, $c_{\mathfrak{gl}}$ is a non-degenerate invariant metric on $\mathfrak{gl}(2n,\mathbb{R})$, which extends the non-degenerate Killing form $-\operatorname{tr}$ on $\mathfrak{sl}(2n,\mathbb{R}) \subset \mathfrak{gl}(2n,\mathbb{R})$. Then, solutions (Ω, ω, θ) of the system (7.2) correspond to solutions of (5.3), provided that θ is a product connection $\nabla \times A$ and ∇ is compatible with the hermitian structure (Ω, ω). The compatibility between ∇ and (Ω, ω) leads to some difficulties in the construction, that we shall ignore here.

Going back to the general case, following Definition 6.5 we denote

$$H^3_{str}(P,\mathbb{R}) \subset H^3(P,\mathbb{R})$$

the set of *real string classes* in P. By condition (7.1) this set is non-empty, and it is actually a torsor over $H^3(M,\mathbb{R})$. We note that any solution $x = (\Omega,\omega,\theta) \in \mathcal{P}$ of (7.2) satisfies

$$dd^c\omega - c(F_\theta \wedge F_\theta) = 0,$$

and therefore x induces a string class

$$[\hat{H}_x] \in H^3_{str}(P,\mathbb{R}),$$

where

$$\hat{H}_x = p^*d^c\omega - CS(\theta).$$

To understand the geometric meaning of the set of solutions with fixed string class, we note that a choice $[\hat{H}] \in H^3_{str}(P,\mathbb{R})$ determines an isomorphism class of exact Courant algebroids over P (see e.g. [59]). More explicitly, for a choice of representative $\hat{H} \in [\hat{H}]$, the isomorphism class of exact Courant algebroids is represented by

$$\hat{E} = TP \oplus T^*P,$$

with (Dorfman) bracket

$$[\hat{X}+\hat{\xi},\hat{Y}+\hat{\eta}] = [\hat{X},\hat{Y}] + L_{\hat{X}}\hat{\eta} - \iota_{\hat{Y}}d\hat{\xi} + \iota_{\hat{Y}}\iota_{\hat{X}}\hat{H},$$

and pairing

$$\langle \hat{X}+\hat{\xi},\hat{X}+\hat{\xi}\rangle = \hat{\xi}(\hat{X}),$$

for vector fields $\hat{X},\hat{Y}$ and 1-forms $\hat{\xi},\hat{\eta}$ on P.

The exact Courant algebroid $\hat{E}$ comes equipped with additional structure, corresponding to the string class condition for $[\hat{H}]$. Firstly, we note that $[\hat{H}]$ is fixed by the G-action on P – as it always admits a G-invariant representative of the form (6.19) – and therefore $\hat{E}$ is G-equivariant. Secondly, $\hat{E}$ admits a *lifted G-action* [16], given by an algebra morphism $\rho\colon \mathfrak{g} \to \Omega^0(\hat{E})$ making commutative the diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\rho} & \Omega^0(\hat{E}) \\ & \searrow^{\psi} & \downarrow{\scriptstyle\pi} \\ & & \Omega^0(TP) \end{array} \tag{7.4}$$

and such that the infinitesimal $\mathfrak{g}$-action on $\Omega^0(\hat{E})$ induced by the Courant bracket integrates to a (right) G-action on $\hat{E}$ lifting the action on P. The previous data is determined up to isomorphism by the choice of real string class (see [12, Prop. 3.7]).

To be more explicit, writing $\rho(z) = Y_z + \xi_z$ for $z \in \mathfrak{g}$ we have

$$d\xi_z = i_{Y_z}\hat{H}$$

Then, for a choice of connection θ on P there exists a 2-form $\hat{b}$ on P such that

$$\rho(z) = e^{\hat{b}}(Y_z - c(z, \theta\cdot)),$$

and

$$\hat{H} = p^*H - CS(\theta) + d\hat{b},$$

where $e^{\hat{b}}(\hat{X} + \hat{\xi}) = \hat{X} + \iota_{\hat{X}}\hat{b} + \hat{\xi}$.

Applying the general theory in [16], the exact Courant algebroid $\hat{E}$ can be reduced, by means of the lifted action ρ, to a (transitive) Courant algebroid E over the base manifold M, whose isomorphism class only depends on the choice of string class $[\hat{H}]$. Any choice of connection θ on P determines an isomorphism

$$E \cong TP/G \oplus T^*$$

and a 3-form H on M, uniquely up to exact 3-forms on M, such that the symmetric pairing on E is given by

$$\langle \hat{X} + \xi, \hat{Y} + \eta \rangle = \frac{1}{2}(i_X\eta + i_Y\xi) + c(\theta\hat{X}, \theta\hat{Y}),$$

where $p\hat{X} = X$, and $p\hat{Y} = Y$, and the Dorfman Bracket is given by

$$\begin{aligned} [\hat{X} + \xi, \hat{Y} + \eta] = {} & [\hat{X}, \hat{Y}] + L_X\eta - \iota_Y d\xi + \iota_Y\iota_X H \\ & + 2c(d_\theta(\theta\hat{X}), \theta\hat{Y}) + 2c(F_\theta(X, \cdot), \theta\hat{Y}) - 2c(F_\theta(Y, \cdot), \theta\hat{X}). \end{aligned} \tag{7.5}$$

In [48] it is proved that solutions of the system (7.2) (and hence of the Strominger system (5.3)) with fixed string class $[\hat{H}]$, can be recasted in terms of natural geometry in the Courant algebroid E over M. This implies a drastic change in the symmetries of the problem: the system (7.2), with natural symmetries given by the automorphism group $\operatorname{Aut} P$ of P, is preserved by the automorphism group of the Courant algebroid E (see [48, Prop. 4.7]) once we fix the string class. In particular, this includes B-field transformations for any closed 2-form $b \in \Omega^2$ on M, given by

$$\hat{X} \to \hat{X} + \iota_X b.$$

To give the main idea, we have to go back to the physical origins of the Strominger system in supergravity, as explained in Section 6.1. By Theorem 6.3, the system (7.2) is equivalent to the Killing spinor equations

$$\begin{aligned} \nabla^-\eta &= 0, \\ (d\phi + \frac{1}{2}H) \cdot \eta &= 0, \\ F_\theta \cdot \eta &= 0, \\ dH - c(F_\theta \wedge F_\theta) &= 0, \end{aligned} \tag{7.6}$$

for a tuple $(g, \phi, H, \theta, \eta)$, given by a riemannian metric g on M, a function ϕ, a three-form H, a connection θ on P, and a non-vanishing pure spinor η with positive chirality.

Theorem 7.1 ([48]). *Solutions of the Killing spinor equations* (7.6) *with fixed string class* $[\hat{H}]$ *are preserved by the automorphism group of* E.

The proof is based on the fact that a tuple (g, H, θ) satisfying the Bianchi identity $dH = c(F_\theta \wedge F_\theta)$ is equivalent to a generalized metric on E, while the function ϕ determines a specific choice of torsion-free compatible connection (which in generalized geometry is not unique). Furthermore, in [23, 47] it was proved in that the equations of motion (6.13) correspond to the Ricci and scalar flat conditions for the metric connection determined by (g, ϕ, H, θ).

7.2 The moduli problem

In this section we review on the recent progress made in the study of the moduli space of solutions for the Strominger, following [48] (see also [6, 24]).

The moduli problem for the Strominger system (5.3) in dimension $n = 1$ reduces to the study of moduli space of pairs $(X, \mathcal{E})$, where X is an elliptic curve and $\mathcal{E}$ is a polystable vector bundle over X, with rank r and degree 0 (see Section 5.1). Due to results of Atiyah and Tu (see [97] and references therein), this moduli space corresponds to the fibred r-th symmetric product of the universal curve over the moduli space of elliptic curves

$$\mathbb{H}/\operatorname{SL}(2, \mathbb{Z}),$$

where $\mathbb{H} \subset \mathbb{C}$ denotes the upper half-plane.

For $n = 2$, Example 5.5 shows that the moduli problem corresponds essentially to the study of tuples $(X, [\omega], \mathcal{E}, \mathcal{T})$, where X is a complex surface with trivial canonical bundle (a $K3$ surface or a complex torus), $[\omega]$ is a Kähler class on X, $\mathcal{E}$ is a degree zero polystable holomorphic vector bundle over X satisfying (5.5), and $\mathcal{T}$ is a polystable holomorphic vector bundle with the same underlying smooth bundle as TX. Although the moduli problem for such tuples is not fully understood even in the algebraic case, it can be tackled with classical methods of algebraic geometry and Kähler geometry (see [5] and references therein).

As observed earlier in this work, the critical dimension for the study of the Strominger system is $n = 3$. This is the lowest dimension for which the Calabi-Yau manifold (X, Ω) may be non-kählerian, and therefore new phenomena is expected to occur. To see this explicitly, we review the construction of the local moduli for the Strominger system in [48, 49]. For simplicity, we will follow the setup introduced in the previous section, and deal with the system (7.2), for a six-dimensional compact spin manifold M. We start defining the symmetries that we will use to construct the infinitesimal moduli for (7.2). These are given by the group

$$\operatorname{Aut}_0 P \subset \operatorname{Aut} P$$

where $\operatorname{Aut} P$ is the group of automorphism of P, that is, the group of G-equivariant diffeomorphisms of P, and $\operatorname{Aut}_0 P$ denotes the connected component of the identity. Given $g \in \operatorname{Aut} P$ we denote by $\check{g} \in \operatorname{Diff}(M)$ the diffeomorphism in the base that it covers. Then, $\operatorname{Aut}_0 P$ acts on $\mathcal{P}$ by

$$g \cdot (\Omega, \omega, \theta) = (\check{g}_* \Omega, \check{g}_* \omega, g \cdot \theta),$$

preserving the subspace of solutions of (7.2). We define the moduli space of solutions of (7.2) as the following set

$$\mathcal{M} = \operatorname{Aut}_0 P \backslash \{x \in \mathcal{P} : x \text{ is a solution of } (7.2)\}$$

Theorem 7.2 ([48]). *The system* (7.2) *is elliptic.*

Relying on this result, the moduli space $\mathcal{M}$ is finite-dimensional, provided that it can be endowed with a natural differentiable structure. In order to do this, a finite dimensional vector space $H^1(S^*)$ parametrizing infinitesimal variations of a solution of (7.2) modulo the infinitesimal action of $\operatorname{Aut}_0 P$ is constructed in [48], using elliptic operator theory. Further, in [49] the Kuranishi method is applied to build a local slice to the $\operatorname{Aut}_0 P$-orbits in $\mathcal{P}$ through a point $x \in \mathcal{P}$ solving (7.2). By general theory, the local moduli space of solutions around x is defined by a (typically singular) analytic subset of the slice, quotiented by the action of the isotropy group of x.

The construction in Section 7.1 induces a well-defined map from the moduli space to the set of string classes

$$\vartheta \colon \mathcal{M} \to H^3_{str}(P, \mathbb{R}). \tag{7.7}$$

Relying on the parallel with Maxwell theory, we call this the *flux map*. We note that $H^3_{str}(P, \mathbb{R})$ is in bijection with $H^3(M, \mathbb{R})$, which corresponds to the space of infinitesimal variations of the Calabi-Yau structure Ω on M. Thus, potentially, restricting to the level sets of ϑ on should obtain a manifold of lower dimension (in relation to the physical problem of *moduli stabilitization*). By Theorem 7.1, each level set

$$\vartheta^{-1}([\hat{H}]) \subset \mathcal{M}$$

can be interpreted as a moduli space of solutions of the Killing spinor equations (7.6) on the transitive Courant algebroid $E_{[\hat{H}]}$. On general grounds, it is expected that the moduli space $\vartheta^{-1}([\hat{H}])$ is related to a Kähler manifold, generalizing the special Kähler geometry in the moduli problem for polarised kählerian Calabi-Yau manifolds. Note here that $\vartheta^{-1}([\hat{H}])$ contains a family of moduli spaces of τ-stable holomorphic vector bundles – with varying complex structure and balanced class on M –, each of them carrying a natural Kähler structure away from its singularities (see Section 4.2).

Based on the relation with string structures, it is natural to ask which enhanced geometry can be constructed in the moduli space $\vartheta^{-1}([\hat{H}])$ using an integral string class

$$[\hat{H}] \in H^3_{str}(P, \mathbb{Z}).$$

This integrality condition appears naturally in the theory of T-duality for transitive Courant algebroids, as defined by Baraglia and Hekmati [12], and it should be important for the definition of a Strominger-Yau-Zaslow version of mirror symmetry for the Strominger system [104].

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Geometry and large N limits in Laughlin states

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Abstract

In these notes I survey geometric aspects of the lowest Landau level wave functions, integer quantum Hall state and Laughlin states on compact Riemann surfaces. In particular, I review geometric adiabatic transport on the moduli spaces, derivation of the electromagnetic and gravitational anomalies, Chern-Simons theory and adiabatic phase, and the relation to holomorphic line bundles, Quillen metric, regularized spectral determinants, bosonisation formulas on Riemann surfaces and asymptotic expansion of the Bergman kernel.

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Contents

1 Introduction

Quantum Hall effect is observed in certain two-dimensional electron systems, such as GaAs heterostructures [100] and more recently in graphene [82], at large magnetic fields and low temperatures. In the most basic setup, the current I_x is forced through a 2d sample in direction x and the voltage V_y is measured across the sample in y direction, as shown on Fig. 1. The outcome of the measurement is that the Hall conductance $\sigma_H = I_x/V_y$ (Hall resistance shown on Fig. 1 is inverse Hall conductance, $R_{xy} = 1/\sigma_H$) as a function of magnetic field strength at a fixed chemical potential undergoes a series of plateaux. There it takes on fractional values $\sigma_H \in \mathbb{Q}$, with small denominators, as measured in units of e^2/h. This effect is referred to as the "quantization" of Hall conductance. Even more remarkable, taking into account impurities of the samples, is the fact that quantization happens to a very high degree of accuracy, of the order 10^{-8}.

Quantum Hall effect comes in two varieties: integer QHE, where $\sigma_H \in \mathbb{Z}_+$, and fractional QHE with σ_H rational and non-integer. On Fig. 1 the integer QHE plateaux are labelled as $1, 2, 3, 4$ and all other plateaux corresponds the fractional QHE. The physics of integer and fractional QHE is very different: the former corresponds to non-interacting fermions, while the latter is a strongly interacting system (we refer to the classical survey Ref. [42] for the introduction to the physics of QHE). However, there is a degree of similarity in that in both cases the mechanism behind the quantisation of Hall conductance alludes to Chern classes of certain vector bundles.

The standard approach to the theory of QHE is to assign a collective multiparticle electron wave function, or "state" $\Psi(z_1, ..., z_N)$ to each plateaux, and then test its various properties against the experiment or numerics. In the integer QHE

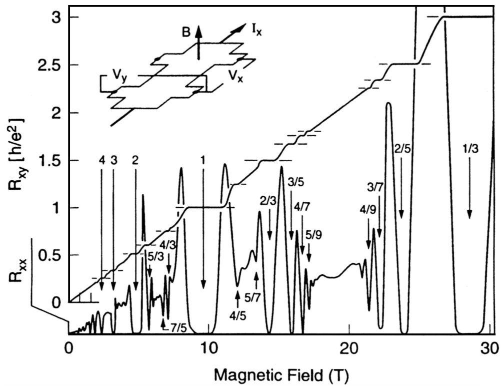

Figure 1. Hall resistance R_{xy} and longitudinal resistance R_{xx} vs. magnetic field B (borrowed with permission from Ref. [100])

one could argue that the physics is captured by the first several energy levels in the tower of Landau levels of an electron in a strong magnetic field. The lowest Landau level (LLL) becomes highly degenerate at strong magnetic fields and is gapped due to the scale set by the cyclotron energy $\omega_c = eB/mc$. To satisfy the Pauli principle, the exact collective wave function of N fermions on the fully filled LLL is completely antisymmetric and thus is given by the Slater determinant of one-particle wave functions, see Eq. (3.4). One-particle wave functions are holomorphic functions on $\mathbb{C}$ and so is the integer QH state, apart from the overall non-holomorphic gaussian factor.

In the fractional QHE the many-body Hamiltonian contains interaction terms and thus it is hard to find an exact ground state analytically. Usually one proceeds by making educated guesses for the trial states. The most successful choice is the celebrated Laughlin state [74], corresponding to the values of Hall conductance given by simple fractions $\sigma_H = 1/\beta$, $\beta \in Z_+$, see e.g. plateau labelled 1/3 on Fig. 1. This wave function is not made out of one-particle states, although one could look at it as corresponding to a partially filled LLL (notation $\nu = 1/\beta$ is also widely used, where ν is the "filling fraction" for the LLL). In particular, it is also a holomorphic function of coordinates times the gaussian factor, see Eq. (4.1).

Another well-known state is Pfaffian state [79], corresponding to $\sigma_H = 5/2$ plateau, which is not pictured on Fig. 1. Other states were proposed to describe various other plateaux, as an incomplete list of best known examples we refer to the hierarchy states of Refs. [49, 52], composite fermion states [57], Read-Rezayi states [90], but our main focus here is on Laughlin states. Interestingly, the Laughlin states are also widely used in the contexts other than the FQHE, in particular we shall mention $d + id$-wave superconductors [88] and chiral spin liquids [59].

The standard approach to the explanation of the quantization of Hall conductance, put forward in Ref. [97], is inherently geometric (the first explanation of the quantization is Laughlin's transport of charge argument, which also invokes non-trivial geometry of annulus [73]). There the integer QH effect was considered for a 2d electron gas in a periodic potential. Then the Hall conductance was essentially described as the first Chern number of the line bundle over the Brillouin zone, which is a torus in the momentum space, see Ref. [5] for this interpretation. We shall mention that this approach can be generalized to include impurities in the framework of Chern classes in non-commutative geometry [12].

In these notes we follow a closely related, but not equivalent line of thought, known as the *geometric adiabatic transport*, see e.g. Ref. [9] for introduction. One considers a model 2d electron system either with periodic boundary conditions or on a compact Riemann surface Σ [6, 7]. In the latter case the magnetic field is created by a configuration of k magnetic monopoles inside the surface and is described by the line bundle L^k of degree k over Σ. When the Riemann surface has nontrivial topology, i.e., for genus $\mathrm{g} > 0$, one can create magnetic field flux thought the holes of the surface using solenoids wrapped around the nontrivial one-cycles of Σ. On the surface this leads to the magnetic field acquiring flat gauge connection part, characterized by Aharonov-Bohm phases, e.g. in the case of torus, the phases are real numbers $\varphi_1, \varphi_2 \in [0,1]^2$. The space of solenoid phases is itself a 2g-dimensional torus $T^{2\mathrm{g}}$, which is known to mathematicians as the moduli space of flat connections, or the Jacobian variety $Jac(\Sigma)$. The non-zero time derivative of phases $\dot{\varphi}$ gives rise to the electric current along the surface.

The integer QH state is a section of the line bundle, called determinant line bundle, over $Jac(\Sigma)$. With the help of the Kubo's formulas Hall conductance can be expressed as the first Chern class of this line bundle, see e.g. [6, 7] for details. In particular, when the integer QH wave function is transported adiabatically along a smooth closed contour $\mathcal{C} \in Jac(\Sigma)$, it acquires an adiabatic Berry phase [14]. For contractible contours it is equal to the integral of the Chern curvature of the canonical Berry connection, known in this context as adiabatic connection and curvature, over the area enclosed by the process. This argument can be extended to the Hall conductance in the fractional case [95, 81], where the novel feature is topological degeneracy of the fractional QH states on the Riemann surfaces of genus $\mathrm{g} > 0$.

Studying QH states on a Riemann surface can also help uncover hidden properties of the QH states, such as e.g. the Hall viscosity. Apart from the moduli space of flat connections, complex structure moduli of Riemann surfaces $\mathcal{M}_\mathrm{g}$ provide another parameter space as a new arena for the geometric adiabatic transport. Avron-Seiler-Zograf [8] (see also [76]) considered the integer QH state as a section of the line bundle over the moduli space of complex structures of the torus

$\mathcal{M}_1$. The latter is the fundamental domain of the action of $PSL(2,\mathbb{Z})$ on the complex upper half plane, pictured on Fig. 4. They computed the curvature of the adiabatic connection on this line bundle over $\mathcal{M}_1$, which is proportional to the Poincaré metric on the upper half plane. The coefficient of proportionality was interpreted as the non-dissipative (anomalous, Hall) viscosity of the quantum Hall "electronic liquid", we refer to [54] for the recent review.

Geometric adiabatic transport was subsequently generalized to integer QH state on higher genus Riemann surfaces [77]. The Hall viscosity for various fractional QH states was derived Refs. [98, 99, 89, 91, 39], see especially Ref. [89] for the comprehensive study. In general, the coefficients entering the adiabatic curvature are called *adiabatic transport coefficients*. In this sense the Hall conductance σ_H is a transport coefficient for the adiabatic transport on the moduli space of flat line bundles. To sum up, mathematically one would like to construct various QHE states a Riemann surface and describe the vector bundles arising on the moduli space of Riemann surface, for a large number of particles.

QH states can be also constructed on surfaces with curved metric and inhomogeneous magnetic field. This allows to determine the effect of gravitational anomaly in QHE starting directly from the wave functions. In Ref. [65] the integer QH state was defined on a compact Riemann surface of any genus with an arbitrary metric $g_{z\bar{z}}$ and for the constant magnetic field with the integer flux k through the surface. The main task is to compute the asymptotics for large number of particles N of the normalization factor of the integer QH state on curved backgrounds, which also has the meaning of the generating functional for the density-density correlation functions. The tool for the derivation of the asymptotics is the Bergman kernel expansion for large powers of the holomorphic line bundle L^k. Since the number of particles N is of order k, this asymptotics is equivalent to large N limit in the number of electrons. The gravitational anomaly appears as the order $\mathcal{O}(1)$ term in the $1/k$ expansion of the generating functional [65]. This calculation was generalized to the case of inhomogeneous magnetic field in [66].

The advantage of the Bergman kernel method is that it is mathematically rigorous, and that its large k expansion is very well understood, see [112, 25, 78, 102]. However, this method does not appear to be generalizable to the fractional QH case. There exists several physics methods to compute the asymptotics of the generating functional in this case. In Refs. [21, 22, 71] the generating functional and the gravitational anomaly for the Laughlin states was derived using the Ward identity method, developed in the important Ref. [110]. In Ref. [37] an alternative derivation was given, based on the vertex operator construction of Laughlin states [79] adapted to curved backgrounds. This derivation, which we review here relies on path-integral arguments and does not directly refers to the standard "plasma screening" argument for large N scaling limit in QH states [74], see also [48, 18].

Asymptotic expansion of the generating functional for QH states consists of two parts: the anomalous part, where non-local functionals of the metric and magnetic field enter, and exact part, which includes an infinite series of local invariants of scalar curvature and its derivatives. The anomalous part consists of three terms corresponding to electromagnetic, mixed and gravitational anomalies, with three independent coefficients, see e.g. Eq. (5.13). The coefficient in front of the electromagnetic anomaly is the Hall conductance $\sigma_H = 1/\beta$ and the coefficient ς_H in front of the mixed gravitational-electromagnetic anomaly is related to Hall viscosity (for the Laughlin states on torus the Hall viscosity is $\eta_H = N_\phi/4$). Gravitational anomaly gives rise to a new adiabatic transport coefficient which was dubbed c_H, for "Hall central charge" in [67], and "apparent central charge" in Ref. [20]. For Laughlin states the Hall central charge is $c_H = 1 - 3q^2$, where q is background charge, see Eq. (4.22). We shall stress that the theory is not conformally invariant, since there is a scale associated with the magnetic field $l_B^2 = 1/B$, and c_H is one of infinitely many coefficients appearing in the asymptotic expansion (in l_B) of the generating functional. For a closely related point of view on the gravitational anomaly in QH states we refer to Ref. [20].

In parallel to these developments the gravitational anomaly in QHE was derived from the $2+1$d picture where 2 stand for the space and 1 for the time dimensions. Description of the $2+1$d long distance effective action in QHE in terms of Chern-Simons action goes back to Refs. [40, 41, 106]. There the non-relativistic Chern-Simons theory was constructed using $2+1$d gauge A and spin ω connections and has schematic form $\int AdA + Ad\omega$, where the first term represents the electromagnetic anomaly and the second term corresponds to the mixed anomaly. Recently the gravitational anomaly contribution $\int \omega d\omega$ to this effective was computed starting from $2+1$d non-relativistic fermions in the integer QHE [1], see [46] for the fractional case, and also Refs. [94, 44, 45, 19, 24]. These results are in complete agreement with the 2d generating functional Eq. (5.13), although the exact meaning of the matching between the terms in 2d and $2+1$d actions is not immediately clear. It was understood in Ref. [66] that the geometric part of the adiabatic phase for the integer QH state can be expressed in Chern-Simons form equivalent to that of Ref. [1], thus connecting 2d and $2+1$d approaches to QHE effective action in the integer QHE case. The derivation of the Chern-Simons action as adiabatic phase is based on Quillen theory [87] and on the Bismut-Gillet-Soulé formula [17] for the curvature of the Quillen metric for the holomorphic section of the determinant line bundle, i.e. integer QH state (Bismut-Gillet-Soulé formula was also invoked in QHE context in Ref. [96]).

In these notes we review these developments and make an attempt to put them into a broader mathematical physics context. In particular, in the QHE context we cover such topics as holomorphic line bundles and $\bar\partial$-operator, asymptotic expansion of the Bergman kernel, regularized spectral determinants and the Quillen metric, bosonisation formulas on Riemann surfaces, Bismut-Gillet-Soulé anomaly formula and Chern-Simons action. In this regard the lectures are aimed at both theoretical physicists working on QHE and mathematicians interested in learning about the subject.

We begin in Section 2 with the one-particle states on Riemann surfaces, where we introduce the holomorphic line bundle, $\bar{\partial}$-operator and describe the one-particle wave functions on the lowest Landau level as holomorphic sections of the magnetic monopole line bundle. Our main working example is the torus, where we construct the wave functions and study geometric adiabatic transport on the moduli space in detail. In section 3 we introduce the integer QHE wave function and the generating functional on Riemann surfaces with arbitrary magnetic field and Riemannian metric. We then use Bergman kernel expansion to compute the generating functional to all orders in $1/k$. We relate the non-local terms in the expansion to the gauge, mixed and gravitational anomalies and the local part of the expansion to the regularized determinant of the spectral laplacian for the line bundle.

In Section 4 we define the Laughlin states, review their construction on the round sphere and flat torus. We then review in detail the vertex operator construction of the Laughlin states and explain how it reduces to the bosonisation formulas on higher genus Riemann surfaces in the integer QHE case. We work out vertex operator construction on the torus, paying particular attention to the role of the spin structures and modular transformations. In Section 5 we proceed to the definition of the generating functional for the Laughlin states and review its asymptotic expansion for large number of particles. Then we study geometric adiabatic transport and derive adiabatic curvature for the integer QH and for the Laughlin states. Finally we review the relation between the geometric part of the adiabatic phase and Chern-Simons action.

These lectures reflect the point of view on the geometry of Laughlin states taken in Refs. [65, 34, 37, 67, 66]. In addition to the work already mentioned here, a number of exciting recent developments in geometry of QHE appeared in recent years, which regrettably are not surveyed here. These include emergent geometry approach to Laughlin states [51, 58], Newton-Cartan approach to the geometry of QHE [94], recently experimentally realized QH states on singular surfaces [92] and emergent conformal symmetry [72], genons [10, 47], Dehn twist process on the torus [62, 111], quantum Hall effect on Kähler manifolds [60, 34, 13, 61, 55], to mention just a few. We also plan to provide a more detailed account of higher genus Laughlin states in a separate publication [68]. Apart form that, we do not discuss a vast topic of quasi-hole excitations above the Laughlin states and their non-abelian statistics, although this is something which can be treated by methods reviewed here. We hope to address some of the aforementioned topics elsewhere.

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2 Lowest Landau level on Riemann surface

2.1 Background

We consider a compact connected Riemann surface Σ of genus g, a positive Hermitian holomorphic line bundle (L,h) of degree $\deg L=1$ and the kth tensor power (L^k,h^k), where $h(z,\bar z)$ is a Hermitian metric. Given some complex structure J there exist local complex coordinates $z,\bar z$ where the Riemannian metric on Σ is diagonal $ds^2=2g_{z\bar z}|dz|^2$, and the area of Σ is normalized as $\int_\Sigma\sqrt{g}d^2z=2\pi$. The curvature two-form of the Hermitian metric $h^k(z,\bar z)$,

$$F:=F_{z\bar z}idz\wedge d\bar z=-(\partial_z\partial_{\bar z}\log h^k)idz\wedge d\bar z, \tag{2.1}$$

where $i=\sqrt{-1}$, defines the magnetic field strength two-form on Σ. We will mainly work with the scalar magnetic field density B, defined as follows

$$B=g^{z\bar z}F_{z\bar z}. \tag{2.2}$$

The total flux N_ϕ of the magnetic field through the surface is an integer

$$N_\phi=\frac{1}{2\pi}\int_\Sigma F=\frac{1}{2\pi}\int_\Sigma B\sqrt{g}d^2z, \tag{2.3}$$

and it is equal to the degree $\deg L^k=k$ of the line bundle L^k,

$$N_\phi=k.$$

In this section and in Sec. 3 we will use k for the degree/flux of magnetic field, and reserve the notation N_ϕ for Sec. 4.

The scalar curvature R of the metric is given by

$$R=2g^{z\bar z}R_{z\bar z}=-2g^{z\bar z}\partial_z\partial_{\bar z}\log\sqrt{g}=-\Delta_g\log\sqrt{g},$$

where $\sqrt{g} = 2g_{z\bar{z}}$ and the scalar Laplacian is $\Delta_g = 2g^{z\bar{z}}\partial_z\partial_{\bar{z}}$. The Euler characteristic of Σ is the integral of the scalar curvature over the surface

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R\sqrt{g}d^2z = 2 - 2\mathrm{g}.$$

We will also introduce the gauge connection for the magnetic field $F = dA$ and spin connection for the curvature written in components as follows

$$F_{z\bar{z}}idz \wedge d\bar{z} = (\partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z)dz \wedge d\bar{z},$$
$$R_{z\bar{z}}\,idz \wedge d\bar{z} = (\partial_z \omega_{\bar{z}} - \partial_{\bar{z}} \omega_z)dz \wedge d\bar{z}.$$

Sometimes it will be convenient to use the symmetric gauge, where

$$A_z = \frac{1}{2}i\partial_z \log h^k, \quad A_{\bar{z}} = -\frac{1}{2}i\partial_{\bar{z}} \log h^k,$$
$$\omega_z = \frac{1}{2}i\partial_z \log g_{z\bar{z}}, \quad \omega_{\bar{z}} = -\frac{1}{2}i\partial_{\bar{z}} \log g_{z\bar{z}}. \tag{2.4}$$

Here A_z and $A_{\bar{z}}$ (id. ω_z and $\omega_{\bar{z}}$) are complex conjugate and the components of the connections A_x, A_y in real coordinates defined as $A_z dz + A_{\bar{z}} d\bar{z} = A_x dx + A_y dy$ are real-valued.

We will consider a more general choice of the line bundle which is the tensor product $\mathrm{L} = L^k \otimes K^s \otimes L_\varphi$, where K is the canonical line bundle and L_φ is the flat line bundle, which has degree zero. This choice is motivated by different physical meaning of the components of L. As we already mentioned, the line bundle L^k corresponds to the magnetic field, created by a distribution of monopole charges inside the compact surface, with the total flux k through Σ, Eq. (2.3). If $s > 0$, the sections of K^s correspond to tensors of weight $(s, 0)$, i.e. invariant objects of the form $t_{zz\ldots z}(dz)^s$ with s holomorphic indices z (for $s < 0$ one considers holomorphic vector fields of weight $-s$), see e.g. [28, §II.E]. The Hermitian norm-squared $||t(z,\bar{z})||^2$ of a section $t(z,\bar{z})$ of $L^k \otimes K^s$ reads

$$||t(z,\bar{z})||^2 = |t(z,\bar{z})|^2 h^k(z,\bar{z}) g_{z\bar{z}}^{-s}(z,\bar{z}).$$

Physically this means that parameter s is the gravitational (or conformal) spin of the wave function. The curvature of the Hermitian metric on K^s is thus $isR_{z\bar{z}}dz \wedge d\bar{z}$ and $\deg K^s = \frac{1}{2\pi}\int_\Sigma isR_{z\bar{z}}dz \wedge d\bar{z} = -s\chi(\Sigma)$. Since $\chi(\Sigma)$ is even this allows for half-integer values of the spin s.

Finally, the flat line bundle L_φ takes into account the flat connection part of the gauge field of total flux zero (line bundle of degree zero), which nonetheless can have non-trivial monodromy around the 1-cycles of the Riemann surface. Physically this corresponds to the magnetic field, created by the solenoid coils wrapped around the 1-cycles of the surface and φ labels the solenoid phases.

Namely, let $(A_a, B_b) \in H_1(\Sigma, \mathbb{Z}),\ a, b = 1, ..., \mathrm{g}$ be a canonical basis of one-cycles in Σ and $\alpha_a, \beta_b \in H^1(\Sigma, \mathbb{Z})$ be the dual basis of harmonic one-forms

$$\int_{A_a} \alpha_b = \delta_{ab}, \quad \int_{A_a} \beta_b = 0$$
$$\int_{B_a} \alpha_b = 0, \quad \int_{B_a} \beta_b = \delta_{ab}.$$

The space of phases $(\varphi_{1a}, \varphi_{2b}) \in [0,1]^{2\mathrm{g}}$ span 2g dimensional torus $T^{2\mathrm{g}}_{[\varphi]}$, also known as the Jacobian variety $Jac(\Sigma)$. The flat connections can be explicitly parameterized as follows

$$A^{\varphi} = 2\pi \sum_{a=1}^{\mathrm{g}} (\varphi_{1a}\alpha_a - \varphi_{2a}\beta_a). \tag{2.5}$$

This gauge connection has the monodromy $e^{2\pi i\varphi_{1a}}$ around the cycle A_a and $e^{-2\pi i\varphi_{2b}}$ around B_b and so do the wave functions, which we will define in a moment.

It will be useful to write the flat connection (2.5) in terms of the basis of holomorphic differentials ω_a, normalized as

$$\int_{A_a} \omega_b = \delta_{ab}, \quad \int_{B_a} \omega_b = \Omega_{ab},$$

where Ω is the period matrix of Riemann surface, which is a $\mathrm{g} \times \mathrm{g}$ complex symmetric matrix with $\mathrm{Im}\,\Omega > 0$, see e.g. [80, Ch. 2,§2] and [2]. Then the harmonic one-forms are related to the holomorphic differentials by the following linear transformation

$$\alpha = -\bar\Omega(\Omega - \bar\Omega)^{-1}\omega + \Omega(\Omega - \bar\Omega)^{-1}\bar\omega,$$
$$\beta = (\Omega - \bar\Omega)^{-1}\omega - (\Omega - \bar\Omega)^{-1}\bar\omega,$$

where summation over matrix and vector indices is understood. We can now write the connection (2.5) as

$$A^{\varphi} = 2\pi\varphi(\Omega - \bar\Omega)^{-1}\bar\omega - 2\pi\bar\varphi(\Omega - \bar\Omega)^{-1}\omega, \tag{2.6}$$

where

$$\varphi_a = \varphi_{2a} + \Omega_{ab}\varphi_{1b} \tag{2.7}$$

is the complex coordinate on $Jac(\Sigma)$.

2.2 Lowest Landau level, $\bar\partial$-equation and holomorphic sections

The Hamiltonian for a particle with gravitational spin s on surface (Σ, g) in the magnetic field B can be written in complex coordinates as

$$H = \frac{1}{m} D_z D_{\bar z} + \frac{2-g_s}{4} e\hbar B - \frac{s}{4} R + cR, \tag{2.8}$$

where g_s is the Landé g-factor and in our conventions mass, charge and Planck's constant will be set to one. The derivative operator here is $D_{\bar z} = g_{z\bar z}^{-\frac{1}{2}}(i\partial_{\bar z} - s\omega_{\bar z} + A_{\bar z})$. The additional term cR, where c is a numerical constant, is sometimes added to the Hamiltonian depending on the quantization scheme, see e.g. [63] for review. For the large flux k of the magnetic field, the lowest energy level (lowest Landau level, LLL) for this hamiltonian is highly degenerate, provided the sum of the last three terms in (2.8) is constant

$$E_0 = \frac{2-g_s}{4} B - \frac{s}{4} R + cR = \frac{2-g_s}{4} k + \left(c - \frac{s}{4}\right) \frac{\chi(\Sigma)}{2}, \tag{2.9}$$

and this constant is equal to the ground state energy E_0. Note that the constant magnetic field $B = \mathrm{const}$ corresponds to $F_{z\bar z} = k g_{z\bar z}$, where $g_{z\bar z}$ is not necessarily a constant scalar curvature metric. There exist various choices of constants when Eq. (2.9) holds for inhomogeneous magnetic field and/or non-constant scalar curvature metrics. Since we are interested in the case when LLL is highly degenerate, we will ignore the curvature and magnetic field terms in (2.8) in what follows. In this case the LLL wave functions solve the first order PDE:

$$D_{\bar z}\Psi = 0. \tag{2.10}$$

As we will see in a moment, equation (2.10) is a local version of a globally defined (on Σ) $\bar\partial$-equation,

$$\bar\partial_{\mathrm{L}} s(z) = 0, \tag{2.11}$$

for a line bundle $\mathrm{L} = L^k \otimes K^s \otimes L_\varphi$. Here the $\bar\partial$-operator acts from $\mathcal{C}^\infty$ sections of L to (0,1) forms with coefficients in $\mathcal{C}^\infty$ sections,

$$\bar\partial_{\mathrm{L}} : \mathcal{C}^\infty(\Sigma, \mathrm{L}) \to \Omega^{0,1}(\Sigma, \mathrm{L}).$$

The global solutions to Eq. (2.11) are called the holomorphic sections of L, and these will be our LLL wave functions. The vector space of holomorphic sections of L is usually denoted as $H^0(\Sigma, \mathrm{L})$.

By the Riemann-Roch theorem, the dimension of the space of holomorphic sections $\dim H^0(\Sigma, \mathrm{L})$ satisfies the relation

$$\dim H^0(\Sigma, \mathrm{L}) - \dim H^1(\Sigma, \mathrm{L}) = \deg(\mathrm{L}) + 1 - \mathrm{g},$$

see e.g. [43, p. 245-6]. Here $H^1(\Sigma, \mathrm{L})$ is the first Dolbeaux cohomology group, which by the Kodaira vanishing theorem [43, p. 154] vanishes $H^1(\Sigma, \mathrm{L}) = 0$ for k large enough, and the latter is exactly the large magnetic flux condition relevant for applications to QHE. The precise technical condition for vanishing is $\deg L^k \otimes K^s > \deg K$, i.e., $k + 2(\mathrm{g}-1)(s-1) > 0$ (using $\deg L_\varphi = 0$ and $\deg \mathrm{L} = \deg L^k + \deg K^s = k + 2s(\mathrm{g}-1)$), see [43, p. 215]. Hence, we have for the total number of LLL wave functions

$$N \equiv N_{k,s} = \dim H^0(\Sigma, \mathrm{L}) = k + (1-\mathrm{g})(1-2s)\,. \tag{2.12}$$

This formula again reminds us that we can allow for half-integer spins s, in which case the wave function is a spinor on Σ.

In quantum mechanics the wave functions are always normalized with respect to the L^2 norm. The L^2 inner product of sections reads

$$\langle s_1, s_2\rangle_{L^2} = \frac{1}{2\pi}\int_\Sigma \bar{s}_1 s_2\, h^k g_{z\bar{z}}^{-s}\sqrt{g}d^2z. \tag{2.13}$$

One could also formally write the wave function in some local coordinate system as

$$\Psi_l(z,\bar{z}) = s_l(z)h^{k/2}(z,\bar{z})g^{-s/2}(z,\bar{z}),$$

although other inequivalent choices for Ψ will be considered in what follows. Then the inner product looks like the standard quantum-mechanical one

$$\langle\Psi_1|\Psi_2\rangle_{L^2} = \frac{1}{2\pi}\int_\Sigma \Psi_1^*\Psi_2\sqrt{g}d^2z.$$

Now, in the symmetric gauge (2.4), the $\bar{\partial}$-equation for the locally defined wave functions $\Psi_l(z,\bar{z})$ reduces to the local form Eq. (2.10) that we started with, $D_{\bar{z}}\Psi = 0$. The operator $\bar{D}$ is a twisted version of the global operator $\bar{\partial}_{\mathrm{L}}$ in (2.11), namely

$$D_{\bar{z}} = h^{\frac{k}{2}} g_{z\bar{z}}^{-\frac{s+1}{2}} \circ i\bar{\partial}_L\big(h^{-\frac{k}{2}} g_{z\bar{z}}^{\frac{s}{2}}\circ\big) = g_{z\bar{z}}^{-\frac{1}{2}}(i\partial_{\bar{z}} - s\omega_{\bar{z}} + A_{\bar{z}}), \tag{2.14}$$

where spin s couples to the spin-connection.

Note that in the discussion above we did not impose any conditions on B and R, and we can consider holomorphic sections on Σ with an arbitrary metric and with inhomogeneous magnetic field. Suppose g_0 is constant scalar curvature metric and $B_0 = g_0^{z\bar{z}}F_{0z\bar{z}}$ is constant magnetic field. Integrating magnetic field over Σ we conclude that the constant equals the total flux $B_0 = k$. A natural way to parameterize an arbitrary curved Riemannian metric g and inhomogeneous magnetic field B is via the Kähler potential $\phi(z,\bar{z})$ and magnetic potential $\psi(z,\bar{z})$,

$$g_{z\bar{z}} = g_{0z\bar{z}} + \partial_z\partial_{\bar{z}}\phi, \tag{2.15}$$

$$F_{z\bar z} = F_{0z\bar z} + k\partial_z\partial_{\bar z}\psi, \quad h^k = h_0^k e^{-k\psi}. \tag{2.16}$$

Here ϕ is a scalar function, satisfying $\partial_z\partial_{\bar z}\phi > -g_{0z\bar z}$, so that the metric $g_{z\bar z}$ is everywhere positive on Σ. In other words, the (1,1) forms F and F_0 (2.1) (cf. $g_{z\bar z} i dz\wedge d\bar z$ and $g_{0z\bar z} i dz\wedge d\bar z$) are in the same Kähler class. We use lower-case ψ for the magnetic potential and and upper-case Ψ for the wave functions throughout the paper, which shall not be confused.

The area of the surface computed with the metrics g_0 and g is the same, which is a natural parameterization in view of incompressibility of QH electronic liquid. It follows from the formulas above and from the definition of magnetic field Eq. (2.2), that constant magnetic field implies magnetic and Kähler potentials being equal up to an irrelevant constant and vice versa,

$$B = k \iff \phi = \psi + \text{const}. \tag{2.17}$$

We will now consider examples where we explicitly construct some reference orthonormal bases of the LLL states, for constant scalar curvature metric and constant magnetic field. We start here with the sphere, and in the next subsections review LLL on the torus.

Lowest Landau level on sphere. In the complex coordinate z induced from $\mathbb{C}$ by stereographic projection, the round metric on the sphere reads

$$g_{0z\bar z} = \frac{1}{(1+|z|^2)^2}. \tag{2.18}$$

It has constant scalar curvature $R(g_0) = 4$ (the subscript 0 will mostly be reserved for the metric of constant scalar curvature on the surface). For the constant magnetic field $F_{0z\bar z} = kg_{0z\bar z}$, and thus one can choose the Hermitian metric (2.1) as

$$h_0^k(z,\bar z) = \frac{1}{(1+|z|^2)^k}. \tag{2.19}$$

There are no flat connections with nontrivial monodromy since there are no non-contractible cycles ($H^1(S^2,\mathbb{Z})$ is trivial) and basis of holomorphic sections of $L^k\otimes K^s$ can be constructed from the polynomial-valued $(s,0)$-forms $(dz)^s$, $z(dz)^s$, $z^2(dz)^s$,... of the maximal degree $N-1 = k-2s$ constrained by the condition of finiteness of the L^2-norm Eq. (2.13). Denoting the orthogonal basis as $s_l(z) = c_l z^{l-1}(dz)^s$, we can derive the normalization coefficients c_l from orthogonality condition on the basis

$$\langle s_l, s_m\rangle_{L^2} = \delta_{lm} c_l^2 \frac{1}{2\pi}\int_{\mathbb{C}} \frac{|z|^{2l-2}}{(1+|z|^2)^{k-2s+2}} 2d^2z = \frac{c_l^2}{(k-2s+1)C_{k-2s}^{l-1}}\delta_{lm},$$

where C_{k-2s}^{l-1} is the binomial coefficient. The integral above converges for $j \leqslant k-2s$. Thus the reference orthonormal basis of sections of $L^k\otimes K^s$ on sphere reads

$$s_l(z) = \sqrt{k-2s+1}\sqrt{C_{k-2s}^{l-1}}\, z^{l-1}(dz)^s, \quad l = 1,...,k-2s+1. \tag{2.20}$$

The basis is non-empty when $k \geqslant 2s$. For spin zero particles there are $k+1$ states on LLL, one extra normalizable state compared to the Landau problem in the planar domain.

2.3 Moduli spaces and adiabatic curvature

Torus is the first nontrivial example where moduli parameters appear. The flat torus can be represented as a quotient $T^2 = \mathbb{C}/\Lambda$ of the complex plane by a lattice $\Lambda = m + n\tau,\ m, n \in \mathbb{Z}$ and $\tau \in \mathbb{H}$ ($\mathbb{H}$ is the complex upper-half-plane) is the complex structure modulus, see picture on Fig. 2 (left).

In addition to τ there are moduli parameters associated with the moduli space of flat connections $Jac(T^2) = T_{[\varphi]}$ (moduli of the flat line bundle L_φ). These are two "solenoid phases" $(\varphi_1, \varphi_2) \in [0,1]^2 = T_{[\varphi]}$. The corresponding flat connections, defined in Eq. (2.5), read in this case

$$A^\varphi = 2\pi(\varphi_1\alpha_1 - \varphi_2\beta_1), \quad \alpha_1 = \frac{\tau d\bar z - \bar\tau dz}{\tau - \bar\tau}, \quad \beta_1 = \frac{dz - d\bar z}{\tau - \bar\tau}. \tag{2.21}$$

The normalization is such that $\int_{A_1} A^\varphi = 2\pi\varphi_1$, $\int_{B_1} A^\varphi = -2\pi\varphi_2$ for the 1-cycles $A_1 = [0,1]$, $B_1 = [0,\tau]$. We can give the Jacobian torus $T_{[\varphi]}$ a natural complex structure, defining the complex coordinate on $T_{[\varphi]}$ according to Eq. (2.7),

$$\varphi = \varphi_2 + \varphi_1\tau. \tag{2.22}$$

In this complex coordinate $T_{[\varphi]}$ looks exactly like the coordinate space torus, see the picture on the right on Fig. 2. Of course, one should remember that these are two different spaces – one is the physical space where particles live and the other one is the parameter space.

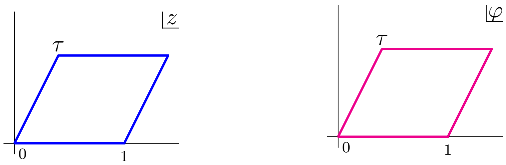

Figure 2. Coordinate space torus T^2 (left) and Jacobian torus $T_{[\varphi]} = Jac(T^2)$ (right).

Our wave functions on T^2 will also depend on the moduli space coordinates, i.e., on two complex parameters (τ, φ), while the energy of the ground state is independent of (τ, φ). This situation falls into the broader context of Berry phases [14] and, when the parameter space is the moduli space, this is also known as *geometric adiabatic transport.* Before constructing the wave functions on the torus explicitly, we briefly review adiabatic (Berry) connection and curvature on moduli spaces. Suppose the wave functions depend on a point in the moduli space, which is a complex space Y of complex dimension $\dim Y$. In our case the parameter space will be

$$Y = \mathcal{M}_{\mathrm{g}} \times Jac(\Sigma), \tag{2.23}$$

where $\mathcal{M}_{\mathrm{g}}$ is the moduli space of complex structures of genus-g Riemann surface and $Jac(\Sigma)$ is the moduli space of flat connections (2.5). To be more precise, Eq. (2.23) is not exactly a direct product, but a fibration, since $Jac(\Sigma)$ as a complex manifold varies over $\mathcal{M}_{\mathrm{g}}$ due to the choice of complex structure (2.22). This will not be important for us here, but should be kept in mind. We consider some local coordinates $(y^\mu, \bar{y}^{\bar{\mu}}), \mu = 1, .., \dim Y$, and we will usually suppress the index μ for simplicity. All wave functions we will encounter here have the following schematic form

$$\Psi_l(x|y, \bar{y}) = \frac{F_l(x|y)}{\sqrt{Z(y, \bar{y})}}, \tag{2.24}$$

where $F_l(x|y)$, $l = 1, .., n$ are locally holomorphic functions of y. The wave functions can be one-particle wave functions or multi-particle states, so x schematically denotes coordinates of one or several particles on Σ. The normalization factor Z is fixed by the condition that the wave functions are canonically normalized $\langle \Psi_l | \Psi_m \rangle = \delta_{lm}$, and crucially Z is independent of the index r. The basis of wave functions on Σ varies over Y as a frame of certain Hermitian vector bundle E of rank n over Y. Moreover, in quantum mechanics there is a canonical Hermitian connection $\mathcal{A}$ on E, called Berry, or adiabatic connection [14], given by

$$(\mathcal{A}_y)_{lm} = i \langle \Psi_l | \partial_y \Psi_m \rangle_{L^2}, \tag{2.25}$$

$$\mathcal{R}_{lm} = d\mathcal{A}_{lm} + (\mathcal{A} \wedge \mathcal{A})_{lm}. \tag{2.26}$$

This is a Chern connection [93] since its curvature is a $(1,1)$-form on Y. Specifically for the wave functions of the form (2.24) the formulas above simplify and the connection and curvature can be expressed via the normalization factor as

$$(\mathcal{A}_y)_{lm} = i\partial_y \langle \Psi_l | \Psi_m \rangle_{L^2} - i \langle \partial_y \Psi_l | \Psi_m \rangle_{L^2} = \delta_{lm} \frac{i}{2} \partial_y \log Z, \tag{2.27}$$

$$\mathcal{R}_{lm} = -\delta_{lm} (\partial_y \partial_{\bar{y}} \log Z) \, i dy \wedge d\bar{y}. \tag{2.28}$$

Adiabatic connection and curvature in Eq. (2.28) are scalar matrices, i.e., $\mathcal{A}$ is the projectively flat connection, and normalization factor Z plays the role of the Hermitian metric on E. The vector bundle E appears to be very close to being a direct sum of line bundles for each degenerate state. However this will only be the case for the integer QH state (which is non degenerate), but not for the LLL states on the torus and not for the Laughlin states, because these states have non-abelian monodromy around 1-cycles of Y, as we will see later.

2.4 Lowest Landau level on the torus

For the discussion of various aspects of quantum mechanics of an electron in the magnetic field on the torus we refer to [35, 9, 76].

The metric on the flat torus reads

$$g_{0z\bar z} = \frac{2\pi i}{\tau - \bar\tau}, \tag{2.29}$$

and its area is normalized to be 2π. The Hermitian metric on L^k, corresponding to the constant magnetic field can be chosen as

$$\tilde h_0^k(z,\bar z) = \exp\left(\frac{\pi i k}{\tau-\bar\tau}(z-\bar z)^2\right), \tag{2.30}$$

so that $F_0 = kg_0$. The canonical bundle K is a trivial holomorphic line bundle on the torus, so we set gravitational spin $s=0$ in this case.

The transformation rules for the holomorphic parts of the wave functions under the lattice shifts can be derived from the condition that $\bar{\tilde s}(z)\tilde s(z)\tilde h_0^k(z,\bar z)$ transforms as a scalar. Then the lattice shifts act on $s(z)$ via the automorphy factor

$$\tilde s(z+t_1+t_2\tau) = (-1)^{2t_2\delta+2t_1\varepsilon}e^{-\pi ikt_2^2\tau-2\pi ikt_2z+2\pi i(t_1\varphi_1-t_2\varphi_2)}\tilde s(z), \quad t_1,t_2\in\mathbb{Z}. \tag{2.31}$$

Here in addition to the solenoid phases φ_1,φ_2 we introduced the spin structure parameters $\varepsilon,\delta\in\{0,\frac12\}$. These label periodic or anti-periodic boundary conditions for the wave function around the 1-cycles of the torus. The four spin structures are divided into two classes: $(0,0)$, $(0,\frac12)$, $(\frac12,0)$ are called even and $(\frac12,\frac12)$ is odd. At first glance introducing spin structures appears to be a redundancy since ε,δ correspond to points in the space of phases φ_1,φ_2, but as we will see later spin structures and phases transform in a different fashion under modular transformations, so it is instructive to keep track of the spin structure.

The transformation rules (2.31) are holomorphic in τ, but not in φ. In order to stay in accordance with Eq. (2.24) we slightly change the Hermitian metric (2.30) and consequently the automorphy factors as follows

$$h_0^k(z,\bar z) = \exp\left(\frac{\pi ik}{\tau-\bar\tau}(z-\bar z)^2 + \frac{2\pi i}{\tau-\bar\tau}(z-\bar z)(\varphi-\bar\varphi)\right), \tag{2.32}$$

$$s(z+t_1+t_2\tau) = (-1)^{2t_2\delta+2t_1\varepsilon}e^{-\pi ikt_2^2\tau-2\pi ikt_2z-2\pi it_2\varphi}s(z), \tag{2.33}$$

so that the transformations are holomorphic in φ. These are the transformation properties satisfied by theta functions with characteristics, see Eq. (6.1) in the Appendix and Ref. [80] for their definition and properties. A particularly convenient choice of the basis of theta functions is

$$s_l^{\varepsilon,\delta}(z) = \vartheta\begin{bmatrix}\frac{\varepsilon+l}{k}\\ \delta\end{bmatrix}(kz+\varphi,k\tau), \quad l=1,...,k. \tag{2.34}$$

Here we explicitly indicated the choice of spin-structure ε, δ. We stress that l labels the degenerate LLL states while ε, δ are treated as external quantum numbers, labelling the choice of boundary conditions. Thus we have constructed k LLL states, and in accordance with the Riemann-Roch theorem (2.12), $\dim H^0(T^2, L^k) = k$.

Taking into account that the complex coordinate depends on the complex structure as $z = x_1 + \tau x_2$, in the basis Eq. (2.34) we can write the wave functions on the torus as

$$\Psi_l^{\varepsilon,\delta}(x|\tau,\varphi) = \frac{1}{\sqrt{Z(\tau,\bar\tau,\varphi,\bar\varphi)}} \cdot e^{2\pi i\varphi x_2 + \pi i k x_2^2 \tau}\, \vartheta \begin{bmatrix} \frac{\varepsilon+l}{k} \\ \delta \end{bmatrix} (kz+\varphi, k\tau), \tag{2.35}$$

which is consistent with the generic abstract form given in Eq. (2.24) since φ now enters holomorphically in the numerator. Computing the L^2 inner product (2.13) for the basis (2.34)

$$\langle s_l, s_m\rangle_{L^2} = \sqrt{\frac{i}{k(\tau-\bar\tau)}} \cdot e^{-\frac{\pi i}{k}\frac{(\varphi-\bar\varphi)^2}{\tau-\bar\tau}} \delta_{lm} \tag{2.36}$$

allows us to determine the normalization Z-factor

$$Z(\tau,\bar\tau,\varphi,\bar\varphi) = \sqrt{\frac{i}{k(\tau-\bar\tau)}} \cdot e^{-\frac{\pi i}{k}\frac{(\varphi-\bar\varphi)^2}{\tau-\bar\tau}}, \tag{2.37}$$

cf. Eq. (2.24).

2.5 Modular group and geometric adiabatic transport

In addition to the adiabatic connection and curvature (2.28), which will be determined momentarily from (2.37), the vector bundle over Y of ground states (2.34) on Σ is characterized by the monodromies around non-trivial one-cycles in the moduli space. The flat connection moduli φ belong to the torus $T_{[\varphi]} = \mathbb{C}/\Lambda,\ \Lambda = t_1 + t_2\tau,\ t_1, t_2 \in \mathbb{Z}$ and the group $\mathbb{Z}+\mathbb{Z}$ of lattice shifts $\varphi \to \varphi + t_1 + t_2\tau$ acts on the holomorphic wave functions (2.34), Hermitian metric (2.32) and the Z-factor (2.37). We have

$$\begin{aligned} s_l^{\varepsilon,\delta}(z|\varphi+t_1+t_2\tau,\tau) &= e^{-\frac{\pi i}{k}t_2^2\tau - \frac{2\pi i}{k}t_2(kz+\varphi)} \cdot \sum_{m=1}^{k} U_{lm} s_m^{\varepsilon,\delta}(z|\varphi,\tau), \\ &\quad \text{where} \quad U_{lm} = e^{\frac{2\pi i}{k}(t_1 l + t_1\varepsilon - t_2\delta)}\delta_{l,m-t_2}, \\ h_0^k(z,\bar z|\varphi+t_1+t_2\tau,\tau) &= e^{2\pi i t_2(z-\bar z)} h_0^k(z,\bar z|\varphi,\tau), \\ Z(\varphi+t_1+t_2\tau, \bar\varphi+t_1+t_2\bar\tau,\tau,\bar\tau) &= e^{-\frac{\pi i}{k}t_2^2(\tau-\bar\tau) - \frac{2\pi i}{k}t_2(\varphi-\bar\varphi)} \cdot Z(\varphi,\bar\varphi,\tau,\bar\tau). \end{aligned}$$

We see that group group of lattice transformations on $T_{[\varphi]}$ acts on the basis of states in a unitary representation, given by the unitary matrix U, and preserves the spin structure. Note that while shifts by t_1 act diagonally on the basis, the action of shift in t_2 is non-diagonal and amounts to relabelling the basis $l \to l+t_2$. We will see shortly that the adiabatic curvature is given by scalar matrix (2.43), yet due to the non-diagonal action of φ_1 shifts the vector bundle of LLL states is not a direct sum of one-dimensional bundles for each wave function.

Next we discuss the monodromies on the complex structure moduli space. The symmetry group of the lattice Λ is the modular group $PSL(2,\mathbb{Z})$, which preserves the torus. It consists of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in \mathbb{Z}, \quad ad-bc=1,$$

defined up to an overall sign $a,b,c,d \to -a,-b,-c,-d$. Its action (modular transformation) on τ, has to be accompanied by the action on $z = x+y\tau$ and $\varphi = \varphi_2 + \varphi_2\tau$, since the latter depend on τ. The full action is given by

$$(\tau,\varphi,z) \to \left(\frac{a\tau+b}{c\tau+d}, \frac{\varphi}{c\tau+d}, \frac{z}{c\tau+d}\right),$$

while ε and δ do not transform (that is why their role is slightly different from solenoid phases φ, as was mentioned after Eq. (2.31)). The group $PSL(2,\mathbb{Z})$ is generated by two elements $T:\tau \to \tau+1$ and $S:\tau \to -1/\tau$, subject to relations $S^2=1$ and $(ST)^3=1$.

The action of the modular group on the wave functions (2.34) can be derived using Eqns. (6.5, 6.6) in the Appendix. We obtain

$$\text{(2.38)} \qquad \begin{aligned} &T \circ s_l^{\varepsilon,\delta}(z|\varphi,\tau) = \sum_{m=1}^{N} U^T_{lm} s_m^{\varepsilon,\delta+\varepsilon-\lambda}(z|\varphi,\tau), \quad U^T_{lm} = e^{\frac{\pi i}{k}(l+\varepsilon)(l-\varepsilon+2\lambda)}\delta_{lm},\\ &T \circ h_0^k(z,\bar z) = h_0^k(z,\bar z),\\ &T \circ Z(\varphi,\bar\varphi,\tau,\bar\tau) = Z(\varphi,\bar\varphi,\tau,\bar\tau),\end{aligned}$$

$$\text{(2.39)} \qquad \begin{aligned} &S \circ s_l^{\varepsilon,\delta}(z|\varphi,\tau) = \sqrt{-i\tau}\cdot e^{\frac{\pi i}{k}\frac{(kz+\varphi)^2}{\tau}} \sum_{m=1}^{k} U^S_{lm} s_m^{\delta,\varepsilon}(z|\varphi,\tau),\\ &\text{where} \quad U^S_{lm} = \frac{1}{\sqrt{k}} e^{-\frac{2\pi i}{k}(\varepsilon\delta + l(m+2\varepsilon))},\\ &S \circ h_0^k(z,\bar z) = e^{-\pi i k\frac{z^2}{\tau} + \pi i k \frac{\bar z^2}{\bar\tau} - 2\pi i \frac{z\varphi}{\tau} + 2\pi i \frac{\bar z\bar\varphi}{\bar\tau}} h_0^k(z,\bar z),\\ &S \circ Z(\varphi,\bar\varphi,\tau,\bar\tau) = \sqrt{\tau\bar\tau}\cdot e^{\frac{\pi i}{k}\frac{\varphi^2}{\tau} - \frac{\pi i}{k}\frac{\bar\varphi^2}{\bar\tau}} \cdot Z(\varphi,\bar\varphi,\tau,\bar\tau).\end{aligned}$$

Here we introduced the parity indicator constant for the degeneracy of LLL states

$$\text{(2.40)} \qquad \lambda = \frac{k}{2} - \left[\frac{k}{2}\right] = \begin{cases} 0, & \text{for } k \in \text{even} \\ \frac{1}{2}, & \text{for } k \in \text{odd.} \end{cases}$$

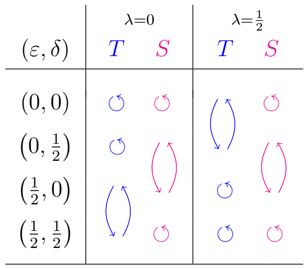

Figure 3. Modular group action on the states changes their spin-structure.

We see that the action of the modular group is given essentially by the unitary matrices U^S and U^T, and all other factors appearing in transformation formulas cancel out between $s(z), h_0$ and Z.

Note that the modular transformations in general mix between different spin structures, according to the table Fig. 3. While S transformation preserves the parity of the spin structure (it maps states with odd spin structure to odd, and even to even), the T transformation in general does not. According to Fig. 3 there exist several possibilities. For even number of particles $\lambda = 0$ the $(0,0)$ spin structure is preserved by the full modular group, and for odd number of particles $\lambda = \frac{1}{2}$ the odd $(\frac{1}{2}, \frac{1}{2})$ spin structure is preserved by full modular group. For the choices $\lambda = 0$, $(\frac{1}{2}, \frac{1}{2})$ and $\lambda = \frac{1}{2}$, $(0,0)$, the invariant subgroup of the modular group is Γ_θ, generated by (T^2, S), which appeared before in Refs. [76] and [67, 23] for Laughlin states, where the situation is completely equivalent, see §4.5. Finally, if we take into account $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ spin structures, then the minimal subgroup is the normal subgroup $\Gamma(2)$ generated by (T^2, ST^2S), see e.g. Ref. [70]. This subgroup is also called the modular group Λ, see Ref. [104] and its fundamental domain is pictured in Fig. 4 (right).

Let us focus on the states invariant under the full modular group: $(0,0)$ at $\lambda = 0$ and $(\frac{1}{2}, \frac{1}{2})$ at $\lambda = \frac{1}{2}$. In order to check that the action of $PSL(2,\mathbb{Z})$ is unitary for these two choices of spin structure, we need to check the action of S^2 and $(ST)^3$ on the basis. In both cases we obtain

$$(U^S U^T)^3 = e^{2\pi i\theta} C, \qquad (U^S)^2 = C, \tag{2.41}$$

where the constant $\theta = \frac{1}{8}$. Matrix C satisfies $C^2 = 1$, and for $\lambda = 0$ it has a particularly simple form $C_{lm} = \delta_{l,k-m}$. Thus the basis of states transform by the projective unitary representation of $SL(2,\mathbb{Z})$. For more details on projective unitary representations and modular tensor categories we refer to [64].

The fundamental domain for the action of $PSL(2,\mathbb{Z})$ on the upper-half plane is the complex structure moduli space $\mathcal{M}_1 = \mathbb{H}/PSL(2,\mathbb{Z})$, see Fig. 4 (left). It

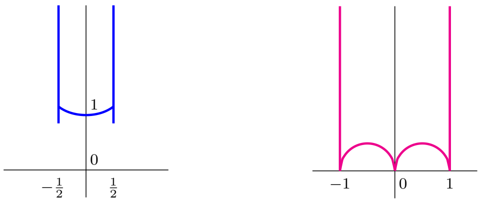

Figure 4. Fundamental domains: $\mathcal{M}_1$ of $PSL(2,\mathbb{Z})$ (left) and $R_{\Gamma(2)}$ of $\Gamma(2)$ (right).

has a cusp (at $\tau = i\infty$) and two orbifold points (at $\tau = i$ and $\tau = e^{\pi i/3} \sim e^{2\pi i/3}$). The fundamental domain $R_{\Gamma(2)}$ of $\Gamma(2)$ has three cusps at $i\infty, 0$ and $1 \sim -1$.

Now we recall basic setup of the geometric adiabatic transport on the moduli space [7, 8]. We choose a smooth closed contour $\mathcal{C}[t]$, $[0 \leqslant t \leqslant 1]$ in the moduli space $Y = T_{[\varphi]} \times \mathcal{M}_1$, or $Y = T_{[\varphi]} \times R_{\Gamma(2)}$. We consider first the transport on $T_{[\varphi]}$ with the phases $\dot{\varphi}_a(t)$ varying along the contour. This creates the electric field and thus the current I_b along the cycle b on the torus, according to

$$I_b = i\dot{\varphi}_a\sigma_{ab}, \tag{2.42}$$

where σ_{ab} is the conductance matrix. This is the coefficient matrix of the adiabatic curvature 2-form, traced over all LLL states $\mathrm{Tr}\,\mathcal{R}$ (2.28). Here we take the trace because we would like to consider k fermionic particles on the LLL, i.e., completely filled lowest Landau level (this point will become obvious in §5.3, where we make the same calculation for the integer QH state). On the torus using (2.37) we immediately obtain

$$\begin{aligned} \mathcal{R}_{lm} &= \frac{2\pi}{k}\delta_{lm}\frac{d\varphi \wedge d\bar{\varphi}}{\tau - \bar{\tau}} = \frac{2\pi}{k}\delta_{lm}\, d\varphi_1 \wedge d\varphi_2, \\ \mathrm{Tr}\,\mathcal{R} &= 2\pi\sigma_{12}\, d\varphi_1 \wedge d\varphi_2 = 2\pi\sigma_{12}\, d\varphi_1 \wedge d\varphi_2. \end{aligned} \tag{2.43}$$

Recall that the latter is actually the first Chern class of the ground state vector bundle E, defined as $c_1(E) = \frac{1}{2\pi}\mathrm{Tr}\,\mathcal{R}$. Therefore the quantization of the Hall conductance

$$\sigma_H = \sigma_{12} = 1$$

in the integer QHE follows from the integrality of $c_1(E)$. Indeed, we have

$$\sigma_H = \int_{T_{[\varphi]}} c_1(E|_{T_{[\varphi]}}) = 1, \tag{2.44}$$

where notation $E|_{T_{[\varphi]}}$ means that we restrict the vector bundle to the Jacobian torus $T_{[\varphi]}$.

Using Eq. (2.37) we can derive the adiabatic curvature for the full moduli space. We obtain,

$$\frac{1}{2\pi}\mathrm{Tr}\,\mathcal{R} = -\frac{i}{\tau-\bar\tau}\,id\varphi\wedge d\bar\varphi + \frac{k}{4\pi(\tau-\bar\tau)^2}id\tau\wedge d\bar\tau, \tag{2.45}$$

and the mixed terms $d\varphi\wedge d\bar\tau$ vanish. The second term in the adiabatic curvature was related in Ref. [8] to the non-dissipative component of the viscosity tensor, and gives rise to a new adiabatic transport coefficient $\eta_H = \frac{k}{4}$, called anomalous or Hall viscosity. By analogy with the Hall conductance calculation above (2.44), we can identify the coefficient η_H computing the integral of the first Chern class $c_1(E)$ restricting to the moduli space $\mathcal{M}_1$,

$$\int_{\mathcal{M}_1} c_1(E|_{\mathcal{M}_1}) = \frac{k}{8\pi}\int_{\mathcal{M}_1}\frac{d\tau_1 d\tau_2}{\tau_2^2} = \frac{k}{24} = \frac{\eta_H}{6},$$

since the volume of $\mathcal{M}_1$ in Poincaré metric equals $\pi/3$. This Chern number is a fraction, since $\mathcal{M}_1$ is an orbifold, but this is still a topological invariant of the vector bundle of LLL states. Finally, we note that one can get rid of 1/6 by replacing the moduli space $\mathcal{M}_1$ by $R_{\Gamma(2)}$, which seems appropriate in view of action of the modular group. Since $\Gamma(2)$ is a congruence subgroup of index 6, the volume of $R_{\Gamma(2)}$ in Poincaré metric is six times bigger than the volume of $\mathcal{M}_1$ and thus $\int_{R_{\Gamma(2)}} c_1(E|_{R_{\Gamma(2)}}) = \eta_H$.

3 Integer quantum Hall state

3.1 Free fermions on a Riemann surface

We consider Hermitian line bundle (L^k, h_0^k) on a compact Riemann surface (Σ, g_0). The basis of the states on the LLL is given by the holomorphic sections $\{s_l\}$, $l = 1,..,N = \dim H^0(\Sigma,\mathrm{L})$ of $\mathrm{L} = L^k\otimes K^s\otimes L_\varphi$. In the integer QHE we have a system of N free fermions, occupying the lowest Landau level. Such a system is described by a multi-particle wave function, which is a completely antisymmetric combination of the one-particle ground states. Thus the (holomorphic part of) integer quantum Hall wave function on Σ^N (more precisely on Σ^N/S_N since the particles are identical) is given by the Slater determinant

$$\mathcal{S}(z_1,...,z_N) = \frac{1}{\sqrt{N!}}\det[s_l(z_m)]|_{l,m=1}^N. \tag{3.1}$$

Similar to the one-particle case, the mod squared of the actual wave function is given by point-wise Hermitian norm of (3.1),

$$|\Psi(z_1,...,z_N)|^2 = ||\mathcal{S}(z_1,...,z_N)||^2 = \frac{1}{N!}|\det[s_l(z_m)]|^2\prod_{l=1}^N h_0^k(z_l,\bar z_l)g_{0z\bar z}^{-s}.$$

Now, if the basis $\{s_l\}$ is chosen to be orthonormal with respect to L^2 norm,

$$\frac{1}{2\pi}\int_\Sigma \bar{s}_l(\bar{z})s_m(z)h_0^k(z,\bar{z})g_{0z\bar{z}}^{-s}\sqrt{g}d^2z=\delta_{lm}, \tag{3.2}$$

then the integer quantum Hall state is automatically normalized (with respect to the L^2 norm on Σ^N)

$$\frac{1}{(2\pi)^N}\int_{\Sigma^N}|\Psi(z_1,...,z_N)|^2\prod_{j=1}^N\sqrt{g}d^2z_j= \tag{3.3}$$

$$=\frac{1}{(2\pi)^N N!}\int_{\Sigma^N}|\det[s_l(z_m)]|^2\prod_{l=1}^N h_0^k(z_l,\bar{z}_l)g_{0z\bar{z}}^{-s}\sqrt{g_0}d^2z_l=\det\langle s_l,s_m\rangle_{L^2}=1,$$

where in the last line we used a straightforward combinatorial identity, expressing the multiple integral as the determinant of one-particle L^2 norms (2.13).

Plane. On the complex plane $\mathbb{C}$ and for the constant perpendicular magnetic field B the number of states on the LLL is infinite, so we take the first N states $s_l=z^{l-1}, l=1,...,N$, imposing a cut-off on the angular momentum $r^m e^{2\pi i m\phi}$, $m<N$. Since the metric is flat, we can set the gravitational spin s to zero. Using the Vandermonde determinant formula $\det z_l^{m-1}=\prod_{l<m}(z_l-z_m)$, the integer QH state can be written as

$$\Psi(z_1,...,z_N)=\frac{\mathcal{N}}{\sqrt{N!}}\det[s_l(z_m)]\prod_{l=1}^N h_0^{B/2}(z_l,\bar{z}_l)=\frac{\mathcal{N}}{\sqrt{N!}}\prod_{l<m}^N(z_l-z_m)\cdot e^{-\frac{1}{4}B\sum_l|z_l|^2}, \tag{3.4}$$

up to an overall normalization constant $\mathcal{N}$.

Sphere. On the sphere with uniform constant magnetic field we can write the wave function (3.1) explicitly, using the basis (2.20) and the Vandermonde identity

$$\frac{1}{\sqrt{N!}}\det[s_l(z_m)]=\frac{1}{\sqrt{N!}}\prod_{l=1}^N c_l\cdot\prod_{l<m}^N(z_l-z_m)\cdot(dz_1)^s\otimes(dz_2)^s\otimes\cdots\otimes(dz_N)^s,$$

where number of particles $N=k-2s+1$ and normalization constants c_l are given in Eq. (2.20).

Torus. On the torus the analog of the Vandermonde formula above is known as the "bosonisation formula", see e.g. [36, Eq. 5.33]. For the constant magnetic field and flat metric the orthonormal basis of sections was constructed in (2.34) and the number of particles is $N=k$. The following identity holds,

$$\begin{aligned}\mathcal{S}(z_1,..,z_k)&=\det[s_l^{\varepsilon,\delta}(z_m)]\\&=e^{\pi i\epsilon}\eta(\tau)^{k-1}\vartheta\left[\begin{array}{c}\varepsilon-\lambda+\frac{1}{2}\\ \delta-\lambda+\frac{1}{2}\end{array}\right](z_{\rm cm}+\varphi,\tau)\prod_{l<m}^k\frac{\vartheta_1(z_l-z_m,\tau)}{\eta(\tau)},\end{aligned} \tag{3.5}$$

where ϵ is a constant depending only on ε, δ and k, λ is the parity indicator for k Eq. (2.40), and $\eta(\tau)$ is the Dedekind eta function. Also we introduced the following notation

$$z_{\rm cm} = \sum_{l=1}^{k} z_l, \tag{3.6}$$

for the so called center-of-mass coordinate. For the proof of this identity we first note that both sides are sections of L^k for each coordinate z_m. This can be checked by comparing automorphy factors under lattice shifts, which on the lhs can be read off immediately from Eq. (2.33), and on the rhs can be worked out from Eqns. (6.2, 6.3) in the Appendix. Then we check that the zeroes on both sides coincide. For each z_m there are manifestly $k-1$ zeroes at $z_m = z_1, .., z_{m-1}, z_{m+1}, ..z_k$ on both sides: on the lhs because determinant vanishes due to coincident rows, and on the rhs because $\theta_1(0) = 0$. Since lhs is a section of $L^k \otimes L_\varphi$ for z_m, there is an extra hidden zero, which is located exactly where the center-of-mass piece on the rhs vanishes. The indirect argument for the extra zero on the lhs relies upon the general correspondence between the line bundles and divisors [43, 36]. After that the overall z-independent factor can be fixed by computing the $PSL(2,\mathbb{Z})$ action on both sides with the help of Eqns. (6.5-6.10) and concluding that it must be a modular form with a prescribed behavior at infinity, i.e., a certain power of the Dedekind eta function.

The action of the T and S transformation preserves the spin structure, when $\varepsilon = \delta = \lambda = 0$ or $\varepsilon = \delta = \lambda = \frac{1}{2}$, according to the table Fig. 3. Since the integer QH state is not degenerate, in both cases the action is given by the phase factors,

$$T \circ \mathcal{S} = e^{\frac{\pi i}{12}(k^2+2)}\mathcal{S}, \quad S \circ \mathcal{S} = e^{-\frac{\pi i}{4}(k^2-k+2)}\mathcal{S},$$

which follows immediately from (2.38, 2.39).

3.2 Generating functional

In quantum mechanics the main objects are wave functions, which have to be L^2 normalized. Usually one can explicitly find normalization constants for one-particle LLL states and for the IQHE state on the surface with constant scalar curvature metric, like e.g. in the examples of plane, sphere and torus above. However our goal is to define IQHE state for an arbitrary metric g and inhomogeneous magnetic field B. Thus we have two sets of geometric data: background metric g_0 and Hermitian metric h_0^k on L^k defining background magnetic field B_0 (e.g. constant scalar curvature metric and constant magnetic field) and an arbitrary metric $g = g_0 + \partial\bar\partial\phi$ and Hermitian metric $h^k = h_0^k e^{-k\psi}$, related to the background data exactly as in (2.15) and (2.16), so that g_0 and g and F_0 and F are in the same Kähler class. The surface Σ and the line bundle L^k is the same in both cases, only the corresponding metrics are changed, so the number of LLL states does not change.

Now, in order to construct the IQHE state for arbitrary g and B we start with a basis of one-particle states, normalized with respect to the background metrics, as in (3.2). Then we write the norm-squared of the IQHE state as

$$|\Psi[g,B](z_1,...,z_N)|^2 = \frac{1}{Z_k}\frac{1}{N!}|\det[s_l(z_m)]|^2 \prod_{l=1}^N h^k(z_l,\bar z_l) g_{z\bar z}^{-s}\sqrt{g}d^2z_l, \tag{3.7}$$

where the point-wise Hermitian norm of Slater determinant is taken with respect to the curved metric (2.15) and the potential (2.16). In order to make this state L^2 normalized $\langle\Psi,\Psi\rangle_{L^2}=1$, we included the normalization factor Z_k, given by the following integral

$$Z_k = \frac{1}{(2\pi)^N N!}\int_{\Sigma^N} |\det[s_l(z_m)]|^2 \prod_{l=1}^N h^k(z_l,\bar z_l) g_{z\bar z}^{-s}\sqrt{g}d^2z_l, \tag{3.8}$$

which we will call the partition function. This is a functional of $Z_k = Z_k[h_0, g_0, \psi, \phi]$, or equivalently $Z_k = Z_k[B_0, g_0, B, g]$. However we note that the normalized IQHE state $|\Psi[g,B](z_1,...,z_N)|^2$ depends only on the metrics g and inhomogeneous B and not on g_0 and B_0. Indeed, under the change of the background metric $g_0\to g_0'$ the basis of sections transforms linearly $s\to s' = As$, where $A\in GL(N,\mathbb{C})$. It follows immediately that dependence on A cancels out between the numerator and denominator in (3.7), hence the normalized IQHE state is independent of the choice of background metrics and of the choice of the basis of sections.

The logarithm of partition function $\log Z_k$ is called the generating functional and is the main object of our interest, since it contains a wealth of information. For example, it generates the density-density connected correlation functions, produced by variations wrt ψ,

$$\frac{\delta}{\delta\psi(w_1,\bar w_1)}\cdots\frac{\delta}{\delta\psi(w_m,\bar w_m)}\log Z_k = (-k)^m\big\langle\rho(w_1,\bar w_1)...\rho(w_m,\bar w_m)\big\rangle_{\rm conn}, \tag{3.9}$$

where the density of states operator reads $\rho(z,\bar z) = \sum_{l=1}^N \delta(z,z_l)$.

The remarkable property of the generating functional is the existence of the asymptotic expansion for large magnetic field, i.e., large k expansion. This expansion can be derived as follows [65, 66]. First, we write (3.8) in the determinant form

$$Z_k = \det\int_\Sigma \bar s_l(\bar z)s_m(z)h^k(z,\bar z)g_{z\bar z}^{-s}\sqrt{g}d^2z, \tag{3.10}$$

by the same combinatorial identity used previously in Eq. (3.3). Next, we use the following variational method (suggested in Ref. [33] in a different context).

Denoting the matrix inside det in (3.10) as G_{lm}, we can write for the variational derivative of $\log Z_k$ with respect to $\delta\phi$ and $\delta\psi$,

$$\begin{aligned}\delta \log Z_k &= \delta \,\mathrm{Tr}\, \log G_{lm} \\ &= -\frac{1}{2\pi}\sum_{l,m}(G^{-1})_{ml}\int_\Sigma\left(\frac{s-1}{2}(\Delta_g\delta\phi)+k\delta\psi\right)\bar s_l s_m h^k g_{z\bar z}^{-s}\sqrt{g}d^2z \\ &= -\frac{1}{2\pi}\int_\Sigma\left(\frac{s-1}{2}(\Delta_g B_k(z,\bar z))\,\delta\phi+kB_k(z,\bar z)\,\delta\psi\right)\sqrt{g}d^2z.\end{aligned} \tag{3.11}$$

The function $B_k(z,\bar z)$ here

$$B_k(z,\bar z)=\sum_{l,m}(G^{-1})_{ml}\bar s_l(\bar z)s_m(z)h^k(z,\bar z)g_{z\bar z}^{-s}(z,\bar z), \tag{3.12}$$

is called the Bergman kernel, which is a well known object in the theory of holomorphic line bundles. Its physical meaning becomes apparent when with the help of (3.9),(3.11) we obtain

$$\langle\rho(z,\bar z)\rangle=\frac{1}{2\pi}B_k(z,\bar z).$$

Hence the Bergman kernel is the average density of particles. The asymptotic expansion of $\log Z_k$ will follow from the asymptotic expansion of the Bergman kernel. Thus we now review the Bergman kernel expansion and then resume from Eq. (3.11) the derivation of the asymptotic expansion of $\log Z_k$ in §3.4.

3.3 Bergman kernel and density of states

Bergman kernel on the diagonal for the line bundle $L^k\otimes K^s$ has a straightforward interpretation as the density of states function on the LLL. Indeed, we write the Hermitian matrix in Eq. (3.12) as $G^{-1}=AA^+$ for $A\in GL(N,\mathbb{C})$, the basis $s'_l=A_{ml}s_m$ (up to $U(N)$ rotation) becomes orthonormal with respect to h^k, g, and we can write the Bergman kernel as the sum over the orthonormal LLL ground states

$$B_k(z,\bar z)=\sum_{l=1}^N \bar s'_l(\bar z)s'_l(z)h^k(z,\bar z)g_{z\bar z}^{-s}(z,\bar z).$$

They key fact about the Bergman kernel is the existence of the complete asymptotic expansion for large degree of the line bundle k on Kähler manifold of any dimension, which was proven in [112, 25]. In complex dimension, i.e. for the Riemann surfaces, the first few terms in the expansion can be read off from [66, Eq. (44)],

$$B_k(z,\bar z)=B+\frac{1-2s}{4}R+\frac{1}{4}\Delta_g\log B+\frac{2-3s}{24}\Delta_g(B^{-1}R) \tag{3.13}$$

$$+\frac{1}{24}\Delta_g(B^{-1}\Delta_g \log B) + \mathcal{O}(1/k^2)\,.$$

The expansion involves only magnetic field and curvature invariants and their covariant derivatives. It goes in the inverse powers of the magnetic field B. For this reason we formally should require $B > 0$ everywhere on Σ and also that B is of order k. Also the potentials ϕ, ψ here are $\mathcal{C}^\infty(\Sigma)$ functions on Σ. In particular, the expansion in the form of Eq. (3.13) in general breaks down near singularities of the curvature and the magnetic field.

The asymptotic expansion Eq. (3.13) can be derived by quantum mechanical methods [34]. The density of states on the LLL can be represented as the path integral

$$B_k(z,\bar z) = \lim_{T\to\infty}\int_{x(0)=z}^{x(T)=z} e^{-\frac{1}{\hbar}\int_0^T (g_{a\bar b}\dot x^a \dot{\bar x}^{\bar b} + A_a \dot x^a + A_{\bar b}\dot{\bar x}^{\bar b})dt}\,\mathcal{D}x(t)$$

for a particle in the magnetic field $F = dA$ on Σ (more generally, on a Kähler manifold of complex dimension $n \geqslant 1$), here at spin $s = 0$. The $T \to \infty$ limit projects the density of states to the lowest Landau level. The large k expansion can be derived via perturbation theory techniques [34], and the Planck constant $\hbar$ enters as the order-counting parameter in (3.13).

There exists a closed formula for the coefficients of the Bergman kernel to all orders in k in any complex dimension n, see Ref. [102]. Let us illustrate this formula for the case when $s = 0$ and magnetic field is constant $B = k$. We consider a local normal coordinate system around a point z_0, where

$$g_{i\bar j}(z_0) = \delta_{i\bar j}, \quad g_{i\bar j_1\ldots\bar j_m} = g_{i_1\bar j i_2\ldots i_m} = 0.$$

The mth term a_m in the expansion of the Bergman kernel

$$B_k(z,\bar z) = a_0(z)k^n + a_1(z)k^{n-1} + \ldots a_m(z)k^{n-m} + \ldots$$

involves exactly $2m$ derivatives of the metric $g_{i\bar j}$ in the local coordinate system. For example the order 4 term will involve the structures as e.g. $g^{i_1\bar j_2}g^{i_2\bar j_1}g^{k_1\bar l_2}g^{k_2\bar l_1}g_{i_1\bar j_1 k_1\bar l_1}$ $g_{i_2\bar j_2 k_2\bar l_2}$. In general one can associate a directed graph G to the structures of this kind, where the positions of $g_{i_1\bar j_1 k_1\bar l_1}$ and $g_{i_2\bar j_2 k_2\bar l_2}$ are represented by vertices and contractions with respect to $g^{i_1\bar j_2}$, etc., are represented by directed arrows between the vertices. At each vertex the number of incoming and outgoing vertices is at least 2. The local coefficient a_m is then given by the sum

$$a_m(z) = \sum_{G\in G(m)} z(G)\cdot G$$

over the set of all such not necessarily connected graphs $G(m)$ at level m. The remarkable fact is that coefficients $z(G)$ are given by easily computable formulas.

For strongly connected graphs (when there exists a directed path from each vertex in G to every other vertex)

$$z(G) = -\frac{\det(A-I)}{Aut(G)},$$

where A is the adjacency matrix of the graph. For connected but not strongly connected graphs $z(G) = 0$ and for disconnected sum of p subgraphs G_j the coefficient is given by

$$z(G) = \prod_{j=1}^{p} z(G_j)/|\mathrm{Sym}(G_1, ..., G_p)|,$$

where $\mathrm{Sym}(G_1, ..., G_p)$ is the permutation group of these subgraphs. The hard part of the calculation (at high orders of m) is to transfer the expressions of the type $g^{i_1\bar{j}_2}g^{i_2\bar{j}_1}g^{k_1\bar{l}_2}g^{k_2\bar{l}_1}g_{i_1\bar{j}_1k_1\bar{l}_1}g_{i_2\bar{j}_2k_2\bar{l}_2}$ in the normal coordinate system back to the invariant form involving scalar curvature, Ricci and Riemann tensors and their derivatives, see Ref. [103] for the state of the art.

3.4 Anomalies and geometric functionals

Going back to the variational formula (3.11) and plugging the expansion Eq. (3.13) we can now integrate it, imposing the boundary condition $\log Z_k[\phi = 0, \psi = 0] = 0$. The calculation was performed for the constant magnetic field in Ref. [65] and generalized to inhomogeneous magnetic fields in Ref. [66, Thm. 1]. We refer to these papers for more details and here we only state the result. The asymptotic expansion has the following general form

$$\log Z_k = \log \frac{Z_H}{Z_{H0}} + \mathcal{F} - \mathcal{F}_0, \tag{3.14}$$

where $\log Z_H$ is the "anomalous part" of the expansion and $\mathcal{F}$ is the "exact part". The former consists of only three terms

$$\begin{aligned} \log Z_H - \log Z_{H0} = &- k^2 S_2(g_0, B_0, \phi) + k\frac{1-2s}{2} S_1(g_0, B_0, \phi, \psi) \\ &- \left(\frac{1}{12} - \frac{(1-2s)^2}{4}\right) S_L(g_0, \phi), \end{aligned} \tag{3.15}$$

where the following functionals appear

$$S_2(g_0, B_0, \psi) = \frac{1}{2\pi}\int_\Sigma \left(\frac{1}{4}\psi\Delta_0\psi + \frac{1}{k}B_0\psi\right)\sqrt{g_0}d^2z, \tag{3.16}$$

$$S_1(g_0, B_0, \phi, \psi) = \frac{1}{2\pi}\int_\Sigma \Big(-\frac{1}{2}\psi R_0 \tag{3.17}$$

$$+\Big(\frac{1}{k}B_0+\frac{1}{2}\Delta_0\psi\Big)\log\big(1+\frac{1}{2}\Delta_0\phi\big)\Big)\sqrt{g_0}d^2z,$$

$$S_L(g_0,\phi)=\frac{1}{2\pi}\int_\Sigma\Big(-\frac{1}{4}\log\big(1+\frac{1}{2}\Delta_0\phi\big)\,\Delta_0\log\big(1+\frac{1}{2}\Delta_0\phi\big)+\frac{1}{2}R_0\log\big(1+\frac{1}{2}\Delta_0\phi\big)\Big)\sqrt{g_0}d^2z. \tag{3.18}$$

These are geometric functionals, which do not have a local expression in terms of the metric and magnetic field, and thus physically they correspond to anomaly terms, as we explain below. The last functional is the Liouville action S_L, and the first two are certain energy functionals well-known in Kähler geometry, see below.

The exact part $\mathcal{F}=\mathcal{F}[g,B]$ and $\mathcal{F}_0=\mathcal{F}[g_0,B_0]$ consists of infinitely many terms, which are local integrals of the magnetic field and curvature and their derivatives. Terms up to the order $\mathcal{O}(1)$ in $1/k$ read

$$\mathcal{F}[g,B]=-\frac{1}{2\pi}\int_\Sigma\Big[\frac{1}{2}B\log\frac{B}{2\pi}+\frac{2-3s}{12}R\log\frac{B}{2\pi}+\frac{1}{24}(\log B)\Delta_g(\log B)\Big]\sqrt{g}d^2z+\mathcal{O}(1/k). \tag{3.19}$$

Written in the form (3.16-3.18) the meaning of the anomalous action is not completely manifest. In order to make it transparent we now rewrite it in two equivalent forms. First, as the following double integral

$$\log Z_H=-\frac{1}{2\pi}\int_{\Sigma\times\Sigma}\Big(B+\frac{1-2s}{4}R\Big)\,|_z\Delta_g^{-1}(z,y)\Big(B+\frac{1-2s}{4}R\Big)\,|_y\sqrt{g}d^2z\sqrt{g}d^2y+\frac{1}{96\pi}\int_{\Sigma\times\Sigma}R(z)\Delta_g^{-1}(z,y)R(y)\sqrt{g}d^2z\sqrt{g}d^2y, \tag{3.20}$$

where the operator Δ_g^{-1} is formally defined as the inverse Laplacian $\Delta_g^z\Delta_g^{-1}(z,y)=\delta(z,y)$. The second term here is the gravitational anomaly in the form of the Polyakov effective action $\int_{\Sigma\times\Sigma}R\Delta^{-1}R$, see Ref. [86].

Yet another form uses the symmetric gauge (2.4), where by integration by parts we can rewrite $\log Z_H$ as

$$\log Z_H=\frac{2}{\pi}\int_\Sigma\Big[A_zA_{\bar z}+\frac{1-2s}{2}(A_z\omega_{\bar z}+\omega_zA_{\bar z})-\Big(\frac{1}{12}-\frac{(1-2s)^2}{4}\Big)\omega_z\omega_{\bar z}\Big]d^2z. \tag{3.21}$$

Here we see, that the first term corresponds to 2d $U(1)$ gauge anomaly, the last term is the gravitational anomaly and the middle term is the mixed anomaly, known as Wen-Zee term [106], see also [26]. This form of the generating functional is a 2d avatar of the Chern-Simons action which will appear later §5.5.

Two interesting special cases of Eq. (3.14) correspond to the two natural choices of magnetic field B on Σ: B =const and $B = \text{const} \cdot R$.

Conformal regime. For any Σ of genus $\mathrm{g} \neq 1$ we can choose magnetic field to be proportional to the scalar curvature $B = \frac{k}{4(1-\mathrm{g})}R$. For non-constant R, in order to keep B positive we should require $R > 0$ everywhere on Σ for the $\Sigma = S^2$ and $R < 0$ everywhere on Σ for higher genus surfaces $\mathrm{g} > 1$. In this case the anomalous part combines into one term

$$\log Z_H = \frac{1}{96\pi}(1-3Q^2)\int_{\Sigma\times\Sigma} R(z)\Delta_g^{-1}(z,y)R(y)\sqrt{g}d^2z\sqrt{g}d^2y.$$

This formally corresponds to gravitational anomaly in a CFT with central charge $c = 1-3Q^2$ with a very large background charge $Q = \frac{k}{1-\mathrm{g}} + 1 - 2s$.

Kähler regime. This is the case of constant magnetic field $B = k$ and arbitrary metric g. As was pointed out in Eq. (2.17) this means the Kähler and magnetic potentials are equal $\phi = \psi$, possibly up to an irrelevant constant. This case was considered in Ref. [65]. In this case the first two terms in the expansion of the anomalous part

$$\begin{aligned}(3.22)\qquad &\log Z_H - \log Z_{H0} = \\ &- kNS_{AY}(g_0,\phi) + k\frac{1-2s}{2}S_M(g_0,\phi) - \left(\frac{1}{12} - \frac{(1-2s)^2}{4}\right)S_L(g_0,\phi),\end{aligned}$$

reduce to the Aubin-Yau and Mabuchi functionals, ubiquitous in Kähler geometry see e.g. [84] for review. These are defined by their variational formulas

$$\begin{aligned}(3.23)\qquad \delta S_{AY}(g_0,\phi) &= \frac{1}{2\pi}\int \delta\phi\sqrt{g}d^2z, \\ \delta S_M(g_0,\phi) &= \frac{1}{4\pi}\int \delta\phi\,(2\chi(\Sigma)-R)\sqrt{g}d^2z,\end{aligned}$$

and explicit formulas can be given

$$\begin{aligned}(3.24)\qquad S_{AY}(g_0,\phi) &= S_2(g_0,kg_0,\phi), \\ S_M(g_0,\phi) &= \chi(\Sigma)S_2(g_0,kg_0,\phi,\phi) + S_1(g_0,kg_0,\phi,\phi),\end{aligned}$$

in terms of functionals defined in Eq. (3.16) and (3.17).

3.5 Regularized spectral determinant

Breaking up the generating functional into the anomalous and exact parts (3.14) has an interesting interpretation in terms of regularized determinants of spectral laplacian and Quillen metric, which we now recall following [66].

We consider the $\bar\partial_{\rm L}$ operator (2.11) and its adjoint $\bar\partial_{\rm L}^*:\ \Omega^{0,1}(\Sigma,{\rm L})\to\mathcal{C}^\infty(\Sigma,{\rm L})$ under the L^2 inner product (2.13). Thus we can define the laplacian acting on sections of the line bundle L (Dolbeault laplacian):

$$\Delta_{\rm L}=\bar\partial_{\rm L}^*\bar\partial_{\rm L}:\ \mathcal{C}^\infty(\Sigma,{\rm L})\to\mathcal{C}^\infty(\Sigma,{\rm L}). \tag{3.25}$$

This operator, which is just the kinetic term in the one-particle hamiltonian (2.8), is sometimes called the "magnetic laplacian". The regularized spectral determinant of this laplacian can be defined in the usual way. We consider the non-zero eigenvalues λ of $\Delta_{\rm L}$ taken with multiplicities and define the zeta-function $\zeta(u)=\sum_\lambda\lambda^{-u}$. Then $\det'\Delta_{\rm L}=\exp(-\zeta'(0))$.

The relation to our setup is as follows. The holomorphic part of the integer QH state (3.1) is a section $\mathcal{S}$ of the determinant line bundle $\mathcal{L}=\det H^0(\Sigma,L^k\otimes K^s)$ over the parameter space $Y=\mathcal{M}_{\rm g}\times Jac(\Sigma)$, Eq. (2.23). Quillen defined a Hermitian metric on sections $\mathcal{S}$ of $\mathcal{L}$ as follows

$$||\mathcal{S}||^2=\frac{Z_k}{\det'\Delta_{\rm L}}. \tag{3.26}$$

Using the determinantal formula Eq. (3.10) the previous formula can also be written as

$$||\mathcal{S}||^2=\frac{\det\langle s_l,s_m\rangle_{L^2}}{\det'\Delta_{\rm L}}.$$

Note that the Quillen metric (3.26) is defined with respect to a choice of h^k and g. Now we ask how $||\mathcal{S}||^2$ varies under the variations (2.15) and (2.16) of these metrics. The following exact formula holds

$$\log\frac{||\mathcal{S}||^2}{||\mathcal{S}||_0^2}=\log\frac{Z_H}{Z_{H0}}, \tag{3.27}$$

where $||\mathcal{S}||_0^2$ is defined for (h_0^k,g_0). In other words, the anomalous part (3.15) of the transformation formula for the partition function (3.14) is entirely due to the ratio of Z_k and determinant of laplacian that enters the Quillen metric (3.26). As an immediate consequence we see that the exact part $\mathcal{F}$ (3.19) corresponds to the regularized determinant of laplacian

$$\mathcal{F}-\mathcal{F}_0=\log\frac{\det'\Delta_{\rm L}}{\det'\Delta_{\rm L0}}, \tag{3.28}$$

where $\Delta_{\rm L}$ is defined wrt (h^k,g) and $\Delta_{\rm L0}$ wrt (h_0^k,g_0). The proof of (3.27) and (3.28) is a standard heat kernel calculation, see [66, Thm. 2] for details. On the compact Riemann surfaces the regularized determinants are known explicitly. On the round sphere with the metric (2.18) and magnetic potential (2.19) we have an exact formula,

$$\log\det{}'\Delta_{L^k}=2\sum_{j=1}^k(k-j)\log(j+1)-(k+1)\log(k+1)! \tag{3.29}$$

$$- 4\zeta'(-1) + \frac{(k+1)^2}{2} = -\frac{k}{2}\log\frac{k}{2\pi} - \frac{2}{3}\log k + \mathcal{O}(1),$$

see [66, §4] and references therein. This is in perfect agreement with Eq. (3.19) at $s = 0$. On the flat torus with constant magnetic field we have

$$\log \det{}'\Delta_{L^k} = -\frac{k}{2}\log\frac{k}{2\pi}, \tag{3.30}$$

valid for any $k > 0$, see [15]. This is also consistent with Eq. (3.19). In §5.3 we will consider the case of higher-genus surfaces, and the Quillen metric will turn out to be a useful tool for the study of the geometric adiabatic transport on the moduli space.

4 Laughlin states on Riemann surfaces

4.1 Definition of the Laughlin state

We consider N particles, labelled by their positions $z_1, ..., z_N$, confined to the plane in the perpendicular constant magnetic field B. The Laughlin state in this setup was introduced in Ref. [74],

$$\Psi(z_1, ..., z_N) \sim \prod_{l<m}(z_l - z_m)^\beta \cdot e^{-\frac{1}{4}B\sum_{l=1}^N |z_l|^2}, \quad \beta \in \mathbb{Z}_+, \tag{4.1}$$

up to a normalization factor. These states are associated with the Quantum Hall plateaux with the values of the Hall conductance $\sigma_H = 1/R_{xy} = 1/\beta$. The graph Fig. 1 includes only one such state, labelled 1/3. In this section we will define and construct the Laughlin state (4.1) on Riemann surfaces, and in the next section we come back to the relation between the Hall conductance and β via geometric adiabatic transport.

At $\beta = 1$ the Laughlin state (4.1) reduces to the integer QH state (3.4), which corresponds to free fermions. For $\beta > 1$ the Laughlin state takes into account Coulomb interactions between the electrons. However, it is not an exact ground state of the full interacting Hamiltonian. Nevertheless, there exists a model Hamiltonian with the short-range interaction potential, see e.g. Refs. [49] and [42, 107], for which the state (4.1) is an exact ground state with zero energy,

$$H = \sum_{l=1}^N D_l\bar{D}_l + \sum_{l,m} V_\beta(z_l - z_m), \quad V_\beta(z) = \sum_{p=1}^{\beta-1}(-1)^p v_p \partial_{\bar{z}}^p \delta^2(z)\partial_z^p, \tag{4.2}$$

where v_p are arbitrary positive constants and kinetic term is the sum over one particle operators (2.8).

In order to define the Laughlin state on a compact Riemann surface, we first note that (4.2) is already written in a covariant fashion, applicable to Riemann surfaces with arbitrary metric and magnetic field. The most important features of (4.1) is that it consists of the holomorphic function of coordinates $F(z_1, ..., z_N)$ and overall gaussian factor, depending on magnetic field. In order to minimize the kinetic energy in (4.2) written on a compact Riemann surface, for each coordinate z_m the function $F(..., z_m, ...)$ should transform as a holomorphic section of the line bundle L^{N_ϕ} of degree N_ϕ on Σ, where N_ϕ is the total flux of the magnetic field, as follows from Eq. (2.10) and discussion in §2.2. Also due to compactness the number of particles and total flux are related,

$$N_\phi \approx \beta N. \tag{4.3}$$

Indeed, on a compact surface N_ϕ is integer and the allowed number of LLL states $(\dim H^0(\Sigma, L^{N_\phi}))$ is of order N_ϕ (2.12). Therefore the number of points in Eq. (4.1) equals to $1/\beta$ times the number of allowed LLL states, while in the integer QH state (3.4) the number N of particles was exactly equal to the number of LLL states. Thus one could say that only a $1/\beta$ fraction of all available LLL states is activated in the Laughlin state ("fractional" Quantum Hall effect). This interpretation is not a precise statement, but only an analogy, since in general the Laughlin state cannot be constructed from one-particle LLL states.

The magnetic field flux on a compact surface is quantized $N_\phi \in \mathbb{Z}$ (2.3), and without loss of generality we can write $N_\phi = \beta k + p,\ k \in \mathbb{Z}_+,\ p = 0, ..., \beta - 1$, where p is the remainder of division of N_ϕ by β. The Laughlin state will be defined specifically at $p = 0$. The wave functions corresponding to $p > 0$ describe quasi-hole excitations over the Laughlin state [74], which are extremely important in physics of QHE, but we here we will focus only on Laughlin states $p = 0$.

Now we have to choose the number of particles N (of order $\sim N_\phi/\beta$), thus fixing the exact relation between k and N. At this point in the discussion we can also turn on the power of canonical line bundle K^{j}, which we take to be quantized as $\mathrm{j} = \beta s,\ s \in \frac{1}{2}\mathbb{Z}$. To make the formulas consistent with the integer QH case we will choose the number of particles exactly as before (2.12) at $\beta = 1$ (and $q = 0$),

$$N = \dim H^0(\Sigma, L^k \otimes K^s) = k + (1 - \mathrm{g})(1 - 2s), \quad s = \frac{\mathrm{j}}{\beta}. \tag{4.4}$$

Thus on the Riemann surface the relation between the flux and number of particles is

$$N_\phi = \beta N - \mathsf{S}, \quad \text{where} \quad \mathsf{S} = (1 - \mathrm{g})(\beta - 2\mathrm{j}), \tag{4.5}$$

generalizing the planar relation (4.3). Here S is usually called the shift in the QHE literature [106].

Given this data we can now define a Laughlin state on a compact Riemann surface Σ.

Definition 4.1. *Consider the holomorphic line bundle* $\mathrm{L} = L^{N_\phi} \otimes K^{\mathrm{j}} \otimes L_\varphi$ *on a compact Riemann surface* Σ*. Let* $N_\phi = \beta k$ *and* $\mathrm{j} = \beta s$ *as in Eq.* (4.4)*. Take* N *points* $z_1, ..., z_N$ *on* Σ*, where* N *is given by Eq.* (4.4) *and* N_ϕ *and* N *are related as in Eq.* (4.5)*. Then the (holomorphic part of the) Laughlin state on* Σ^N *is* $F(z_1, ..., z_N)$*, satisfying the following conditions*

1. $F(..., z_m, ...)$ *with all coordinates, except* z_m*, fixed, transforms as a holomorphic section of* $L^{N_\phi} \otimes K^{\mathrm{j}} \otimes L_\varphi$.
2. *When all* z_l*'s are near the diagonal in* Σ^N *at a generic point on* Σ*,* $F(z_1, ..., z_N)$ *satisfies the vanishing condition*

$$F(z_1, ..., z_N) \sim \prod_{l<m}^{N} (z_l - z_m)^\beta. \tag{4.6}$$

As we have already mentioned, the first condition ensures the vanishing of the kinetic term, while the second condition guarantees minimization of the short-range pseudopotential in (4.2). The Laughlin states on Σ are degenerate for $\mathrm{g} > 0$. This fact is usually referred to as the topological degeneracy.

So far Def. 4.1 defines only the holomorphic part of the wave function. Thus we need to define the Hermitian norm, which is induced from the choice of the Hermitian metric on the line bundle. As in Eq. (2.1), let $h^{N_\phi}(z, \bar z)$ be an Hermitian metric on L^{N_ϕ} so that the magnetic field is given by

$$F_{z\bar z} = -(\partial_z \partial_{\bar z} \log h^{N_\phi}), \quad B = g^{z\bar z} F_{z\bar z}, \quad N_\phi = \frac{1}{2\pi} \int_\Sigma B\sqrt{g} d^2 z.$$

The natural Hermitian metric for the holomorphic part of the Laughlin state is the point-wise product of $h^{N_\phi}(z, \bar z)$ on Σ^N,

$$||F(z_1, ..., z_N)||^2 = |F(z_1, ..., z_N)|^2 \prod_{l=1}^{N} h^{N_\phi}(z_l, \bar z_l) g_{z\bar z}^{-\mathrm{j}}(z_l, \bar z_l). \tag{4.7}$$

This is a scalar function on Σ^N and we can thus compute the L^2 norm and write the normalized Laughlin state Ψ as follows

$$|\Psi(z_1, ..., z_N)|^2 = \frac{1}{\mathcal{N}} ||F(z_1, ..., z_N)||^2,$$

$$\langle \Psi | \Psi \rangle_{L^2} = \frac{1}{\mathcal{N}} \frac{1}{(2\pi)^N} \int_{\Sigma^N} ||F(z_1, ..., z_N)||^2 \prod_{l=1}^{N} \sqrt{g} d^2 z_l = 1. \tag{4.8}$$

The normalization factor $\mathcal{N}$ is a functional of the geometric data: the metric g, magnetic field B, complex structure moduli J of Σ and line bundle moduli $\varphi \in Jac(\Sigma)$, $\mathcal{N} = \mathcal{N}[g, B, J, \varphi]$.

4.2 Examples

Sphere. The spherical Laughlin state was constructed in Ref. [49]. On the sphere the Laughlin state is unique. This is easy to see since, $\prod_{l<m}(z_l - z_m)^\beta$ is the only combination, which meets both conditions in Def. 4.1. As we have already emphasized, the definition of the Laughlin state does not make any reference to the lowest Landau level wave functions. However, specifically for the sphere we can express the Laughlin state as a power of the Slater determinant, for the basis of LLL states Eq. (2.20), as

$$\begin{aligned} &F(z_1, ..., z_N) = (\det s_l(z_m))^\beta, \\ (4.9)\qquad &|\Psi(z_1, ..., z_N)|^2 = \frac{1}{\mathcal{N}_0} \cdot |\det s_l(z_m)|^{2\beta} \prod_{l=1}^{N} h_0^{N_\phi}(z_l, \bar z_l) g_0^{-\mathrm{j}}(z_l, \bar z_l) \\ &= \frac{1}{\mathcal{N}_0} \prod_{l=1}^{N} c_l^{2\beta} \cdot \prod_{l<m}^{N} |z_l - z_m|^{2\beta} \prod_{l=1}^{N} \frac{1}{(1+|z_l|^2)^{N_\phi - 2\mathrm{j}}}, \end{aligned}$$

where the number of particles is $N = k + 1 - 2s$, c_l is given in Eq. (2.20) and constant $\mathcal{N}_0$ is such that the L^2 norm is one: $\langle \Psi|\Psi\rangle_{L^2} = 1$.

Torus. Laughlin states on the torus were constructed in Ref. [50]. We also refer to [75, 62, 89, 107] for other excellent accounts. On the torus the canonical line bundle is trivial and for this reason we set $\mathrm{j} = 0$. The first condition in Def. 4.1 implies that under the lattice shifts the wave function transforms with the same factors of automorphy as in Eq. (2.33) for each coordinate z_m,

$$\begin{aligned} (4.10)\qquad &F(z_1, ..., z_m + t_1 + t_2\tau, ..., z_N) \\ &= (-1)^{2t_2\delta + 2t_1\varepsilon} e^{-2\pi i N_\phi t_2 z_m - \pi i N_\phi t_2^2 \tau - 2\pi i t_2 \varphi} \cdot F(z_1, ..., z_m, ..., z_N). \end{aligned}$$

Here $\varepsilon, \delta \in \{0, \frac{1}{2}\}$ label the choice of spin structures, which are independent of m since the particles are identical. To fulfil the second condition Eq. (4.6), without any loss of generality we can assume the ansatz

$$F(z_1, ..., z_N) \sim f(z_1, ..., z_N) \prod_{l<m}^{N} (\vartheta_1(z_l - z_m, \tau))^\beta,$$

since $\vartheta_1(z)$ has only one simple zero at $z = 0$. From the lattice shift transformation formula Eq. (6.3) for the product of theta functions in the previous equation it follows that in order to be consistent with Eq. (4.10), the function f should transform as

$$\begin{aligned} (4.11)\qquad &f(z_1, ..., z_m + t_1 + t_2\tau, z_N) \\ &= (-1)^{t_2(2\delta - N_\phi + \beta) + t_1(2\varepsilon - N_\phi + \beta)} e^{-i\pi\beta\tau t_2^2 - 2\pi i t_2 \beta z_{\mathrm{cm}} - 2\pi i t_2 \varphi} \cdot f(z_1, ..., z_N). \end{aligned}$$

Comparing with Eq. (2.33) this condition essentially means that $f(z_1,...,z_N) = f(z_{\rm cm})$. Moreover, with respect to the center-of-mass coordinate $z_{\rm cm}$ the function f transforms as a section of the line bundle L^β. Since $\dim H^0(\Sigma, L^\beta) = \beta$ the degeneracy of the center-of-mass factor equals β. Thus we can write down the basis of solutions $f_r, r = 1,..,\beta$ to (4.11) explicitly using e.g. the basis of sections in Eq. (2.34). We consider first odd values of β and, by analogy with (2.40), we shall introduce the parity indicator parameter for the number of particles:

$$\lambda = \frac{N}{2} - \left[\frac{N}{2}\right] = \begin{cases} 0, & \text{for } N \in \text{even} \\ \frac{1}{2}, & \text{for } N \in \text{odd}. \end{cases} \tag{4.12}$$

Then the following basis solves the condition (4.11)

$$\beta \in \text{odd}: \quad f_r(z_{\rm cm}) = \vartheta \begin{bmatrix} \frac{r+\varepsilon}{\beta} - \lambda + \frac{1}{2} \\ \delta - \beta\lambda + \frac{\beta}{2} \end{bmatrix} (\beta z_{\rm cm} + \varphi, \beta\tau), \quad r = 1, ..., \beta.$$

Then the basis of Laughlin states reads

$$F_r^{\varepsilon,\delta}(z_1,...,z_N) = \vartheta \begin{bmatrix} \frac{r+\varepsilon}{\beta} - \lambda + \frac{1}{2} \\ \delta - \beta\lambda + \frac{\beta}{2} \end{bmatrix} (\beta z_{\rm cm} + \varphi, \beta\tau) \prod_{l<m}^N (\vartheta_1(z_l - z_m, \tau))^\beta, \quad \text{where } \beta \in \text{odd}. \tag{4.13}$$

Here index $r = 1, ..., \beta$ labels the topological degeneracy and ε, δ label the spin structure constants, i.e., the choice of boundary conditions Eq. (4.10). As a consistency check, note that for $\beta = 1$ we recover the integer QH state on the torus Eq. (3.5), up to a normalization constant, to which we will come back further down the road.

For $\beta \in$ even (and thus $N_\phi \in$ even) the state is bosonic, i.e., completely symmetric under exchange of the coordinates, and spin structures are redundant. In this case we set

$$\beta \in \text{even}: \quad f_r(z_{\rm cm}) = \vartheta \begin{bmatrix} \frac{r}{\beta} \\ 0 \end{bmatrix} (\beta z_{\rm cm} + \varphi, \beta\tau), \quad r = 1, ..., \beta.$$

We will mostly focus here on odd values of β. In order to see that the states Eq. (4.13) indeed form an orthogonal basis we consider flat torus and constant magnetic field, and rewrite the point-wise Hermitian norm on F_r (4.7) as follows

$$\begin{aligned} ||F_r||^2 &= |F_r(z_1,...,z_N)|^2 \prod_{l=1}^N h_0^{N_\phi}(z_l, \bar z_l) \\ &= |f_r(z_{\rm cm})|^2 h_0^\beta(z_{\rm cm}, \bar z_{\rm cm}) \cdot \prod_{l<m}^N |\vartheta_1(z_l - z_m, \tau)|^{2\beta} \cdot e^{\frac{\pi i \beta}{\tau - \bar\tau}(z_l - z_m - (\bar z_l - \bar z_m))^2}, \end{aligned} \tag{4.14}$$

where we used the Hermitian metric h_0 corresponding to the constant magnetic field (2.32). The L^2 inner product for the flat metric g_0 (2.29) then reads

$$\langle F_r, F_{r'}\rangle = \frac{1}{(2\pi)^{N_\phi}}\int_{\Sigma^N} \bar f_r(\bar z_{\rm cm}) f_{r'}(z_{\rm cm}) h_0^\beta(z_{\rm cm},\bar z_{\rm cm}) \cdot \prod_{l<m}^N |\vartheta_1(z_l-z_m,\tau)|^{2\beta}\cdot e^{\frac{\pi i\beta}{\tau-\bar\tau}(z_l-z_m-(\bar z_l-\bar z_m))^2}\prod_{l=1}^N\sqrt{g_0}d^2z_l.$$

Note that the dependence on the center-of-mass coordinate in the integrand decouples from the relative distances $z_l - z_m$ between the points. Thus, we can pass to the integration over $z_{\rm cm}$ and $z_1 - z_m$, $m = 2, ..., N$, and note that the latter is independent of $z_{\rm cm}$. Now, as we already noticed, $f_r(z_{\rm cm})$ is an orthonormal basis of holomorphic sections of the line bundle L^β over the center-of-mass, as can be seen from (2.34), formally replacing $k \to \beta$ and $z \to z_{\rm cm}$. Since this basis is orthogonal (2.36), we conclude that the overlap matrix of L^2 norms

$$\langle F_r, F_{r'}\rangle = \mathcal N_0(\tau,\bar\tau,\varphi,\bar\varphi)\delta_{rr'}, \tag{4.15}$$

is a scalar matrix, where the constant $\mathcal N_0$ is independent of the index r and is a function of only τ and φ. This is the normalization factor in Eq. (4.8) specified to the flat torus with the constant magnetic field $\mathcal N_0 = \mathcal N[g_0, B_0, \tau, \varphi]$.

Higher genus. The number of degenerate Laughlin states on a higher genus surface is $n_{\beta,{\rm g}} = \beta^{\rm g}$ [105]. For more details on higher genus states we refer to [27, 56] and [68].

4.3 Vertex operator construction

Vertex operator construction of Laughlin states was originally proposed in Ref. [79]. Here we review the construction of Laughlin states on a Riemann surface (Σ, g) of genus g following Refs. [37, 67], and consider the example on sphere and torus in full detail. We start with the gaussian free field $\sigma(z,\bar z)$, compactified on a circle $\sigma \sim \sigma + 2\pi R_c$, with radius $R_c = \sqrt{\beta}$ ("compactified boson"). The action functional has the form

$$S(g,B,\sigma) = \frac{1}{2\pi}\int_\Sigma\big(\partial_z\sigma\partial_{\bar z}\sigma + \frac{iq}{4}\sigma R\sqrt{g}\big)d^2z + \frac{i}{2\pi\sqrt{\beta}}\int_\Sigma A\wedge d\sigma, \tag{4.16}$$

where we will specify the constant q later in Eq. (4.22). The second term here is the coupling of the field to the Riemannian metric on Σ and the last term is the coupling to the magnetic field. The latter is written in this form to take into account a possible contribution from nontrivial flat connection part A^φ, see Eq. (2.5), of the gauge connection on surfaces of genus g > 0. We can write this contribution explicitly, writing the gauge connection as $A + A^\varphi$ and using product rule for derivative

(4.17)
$$S(g,B,\sigma)=\frac{1}{2\pi}\int_\Sigma\big(\partial_z\sigma\partial_{\bar z}\sigma+\frac{iq}{4}\sigma R\sqrt{g}+\frac{i}{\sqrt{\beta}}\sigma B\sqrt{g}\big)d^2z+\frac{i}{2\pi\sqrt{\beta}}\int_\Sigma A^\varphi\wedge d\sigma.$$

For inhomogeneous magnetic field and curved metric this action was proposed in QHE context in Refs. [69, 37, 67]. We emphasize that, while at the zero magnetic field this theory is a conformal field theory with background charge q and central charge $c=1-3q^2$, the coupling to the magnetic field breaks conformal invariance, since it introduces the magnetic length scale $l_B^2\sim 1/B$ to the theory.

We consider now the (unnormalized) correlation function of a N vertex operators at points $z_1,...,z_N$.

$$\mathcal{V}\big(g,B,\{z_l\}\big)=\int e^{i\sqrt{\beta}\sigma(z_1)}\dots e^{i\sqrt{\beta}\sigma(z_N)}e^{-S(g,B,\sigma)}\mathcal{D}_g\sigma. \tag{4.18}$$

Its integral over the coordinates will be denoted as

(4.19)
$$e^{\mathcal{F}_\beta(g,B)}=\frac{1}{(2\pi)^N}\int\int_\Sigma e^{i\sqrt{\beta}\sigma(z_1)}\sqrt{g}d^2z_1\cdots\int_\Sigma e^{i\sqrt{\beta}\sigma(z_N)}\sqrt{g}d^2z_N\;e^{-S(g,B,\sigma)}\mathcal{D}_g\sigma.$$

The key observation is that the correlation function (4.18) gives the sum over all normalized degenerate Laughlin states $\Psi_r^{\varepsilon,\delta}$, $r=1,...,n_{\beta,\mathrm{g}}$ and over all $2^{2\mathrm{g}}$ choices of spin structures $\vec\varepsilon,\vec\delta\in\{0,\frac12\}^{\mathrm{g}}$ on the Riemann surface Σ,

$$\frac{1}{2^{\mathrm{g}}\cdot n_{\beta,\mathrm{g}}}\sum_{\vec\varepsilon,\vec\delta}c_{\varepsilon,\delta}\sum_{r=1}^{n_{\beta,\mathrm{g}}}|\Psi_r^{\varepsilon,\delta}(z_1,...,z_N)|^2=e^{-\mathcal{F}_\beta(g,B)}\mathcal{V}\big(g,B,\{z_l\}\big), \tag{4.20}$$

where the constant $c_{\varepsilon,\delta}=\pm1$ depending on the parity of spin structure[1]

$$c_{\varepsilon,\delta}=e^{4\pi i\vec\varepsilon\cdot\vec\delta}=\begin{cases}+1, & \text{for } (\vec\varepsilon,\vec\delta)\in\text{even}\\ -1, & \text{for } (\vec\varepsilon,\vec\delta)\in\text{odd}.\end{cases}$$

The number of even spin structures $2^{\mathrm{g}-1}(2^{\mathrm{g}}+1)$ minus the number of odd spin structures $2^{\mathrm{g}-1}(2^{\mathrm{g}}-1)$ equals 2^{g}, which explains the overall normalization factor in (4.20). The formula above also holds on the sphere where $n_{\beta,\mathrm{g}}=1$ and spin structures are absent.

We now follow the standard prescription for computing bosonic path integrals, see e.g. [101, 28]. The field σ has a constant zero mode σ_0, defined as

$$\sigma(z,\bar z)=\sigma_0+\tilde\sigma(z,\bar z),\qquad \int\tilde\sigma\sqrt{g}d^2z=0.$$

[1]This holds for odd number of particles, for even number of particles the constant is slightly different, see Eq. (4.35)

Then integration over the zero mode yields the relation between the number of particles N and the coefficients in the action

$$N = \frac{1}{\beta} N_\phi + \frac{q}{2\sqrt{\beta}} \chi(\Sigma). \tag{4.21}$$

Hence to be in accordance with (4.5) we fix the parameter q as

$$q = \sqrt{\beta} - \frac{2\mathrm{j}}{\sqrt{\beta}}. \tag{4.22}$$

Since the number of particles is always an integer, it follows that $\frac{1}{\beta}(N_\phi - \mathrm{j}\chi(\Sigma))$ should be an integer. Therefore we assume $\frac{1}{\beta} N_\phi \in \mathbb{Z}$ and $\mathrm{j} \in \frac{\beta}{2}\mathbb{Z}$.

We consider now the path integral (4.18) on the sphere, where we can drop the flat connection part and compactification of the field does not play any role since all 1-cycles are contractible and there are no non-trivial winding configurations of the field. Then the integral over $\tilde{\sigma}$ can be computed according to standard rules for gaussian integrals. We introduce the standard Green function for scalar laplacian

$$\Delta_g G^g(z, y) = -2\pi\delta(z, y) + 1, \tag{4.23}$$

$$\int_M G^g(z, y)\sqrt{g} d^2 y = 0, \tag{4.24}$$

and the regularized Green function at coincident point,

$$G^g_{\mathrm{reg}}(z) = \lim_{z \to y} \big(G^g(z, y) + \log d_g(z, y) \big), \tag{4.25}$$

where $d_g(z, y)$ is the geodesic distance between the points in the metric g.

Next, we can make a linear shift of $\tilde{\sigma}$ without changing the measure of integration

$$\tilde{\sigma}(z) \to \tilde{\sigma}(z) + \int_M G^g(z, z') j(z') \sqrt{g} d^2 z',$$

$$j(z) = 2i\sqrt{\beta} \sum_{j=1}^{N} \delta(z, z_j) - \frac{iq}{4\pi} R(z) - \frac{i}{\sqrt{\beta}} B(z).$$

Then the integral (4.18) becomes purely gaussian and can be written as

$$\mathcal{V}\big(g, B, \{z_j\}\big) = \left[\frac{\det' \Delta_g}{2\pi} \right]^{-1/2} \tag{4.26}$$

$$\cdot \exp\left(-\frac{1}{4\pi^2} \int_{\Sigma \times \Sigma} \left(\frac{q}{4} R + \frac{1}{\sqrt{\beta}} B \right) \Big|_z G^g(z, z') \left(\frac{q}{4} R + \frac{1}{\sqrt{\beta}} B \right) \Big|_{z'} \sqrt{g} d^2 z \, \sqrt{g} d^2 z' \right)$$

$$\cdot \exp\left(\frac{\sqrt{\beta}}{\pi}\sum_{l=1}^{N}\int_\Sigma G^g(z_l,z)\left(\frac{q}{4}R+\frac{1}{\sqrt{\beta}}B\right)\big|_z\sqrt{g}d^2z\right.$$
$$\left.-\beta\sum_{l\neq m}^{N}G^g(z_l,z_m)-\beta\sum_{l=1}^{N}G^g_{\rm reg}(z_l)\right).$$

Here the regularized Green function $G^g_{\rm reg}(z_l)$ replaces $G^g(z_l,z_l)$ on the diagonal, where the latter is infinite.

Let us consider now the round sphere $R_0=4$ and constant magnetic field $B_0=N_\phi$. The round metric is given by Eq. (2.18) and the corresponding Green function reads

$$G^{g_0}(z,z')=-\log\frac{|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}-\frac{1}{2}.$$

The regularized Green function is just a constant $G^{g_0}_{\rm reg}(z)=-\frac{1}{2}$. Due to the property Eq. (4.24) the integrals over Σ in Eq. (4.26) vanish and we arrive at

$$\mathcal{V}\big(g_0,B_0,\{z_l\}\big)=e^{2\zeta'(-1)-\frac{1}{4}-\frac{1}{2}\beta N^2}\prod_{l<m}^{N}|z_l-z_m|^{2\beta}\prod_{l=1}^{N}\frac{1}{(1+|z_l|^2)^{N_\phi-2\mathrm{j}}},$$

where we used the value of the regularized determinant of the laplacian on the round sphere $\det'\Delta_0/2\pi=e^{\frac{1}{2}-4\zeta'(-1)}$ [83]. Comparing this equation to (4.9), we conclude that the normalized Laughlin state can be expressed as

$$|\Psi(z_1,...,z_N)|^2=e^{-\mathcal{F}_\beta(g_0,B_0)}\mathcal{V}\big(g_0,B_0,\{z_l\}\big)$$

and it has norm one by definition (4.19). We also see that the normalization constant in Eq. (4.9) is controlled by $e^{\mathcal{F}_\beta(g_0,B_0)}$ up to numerical factors.

Finally, let us comment on the $\beta=1$ case of the formula (4.20). At $\beta=1$ this construction reduces to the bosonisation formula on Riemann surfaces [3, 101]. Bosonisation formula is the statement that the correlation function (4.18) equals to the correlator of N insertions of $b\bar{b}$-operators in the theory of free fermions b,c with spins $\mathrm{j},1-\mathrm{j}$, see Eq. [3, Eq. (3.1)′]. The construction of Refs. [3, 101] applies to the case of the canonical line bundle and no magnetic field, but it can be straightforwardly generalized to the case of line bundle $\mathrm{L}=L^k\otimes K^s$ (recall that at $\beta=1$, $\mathrm{j}=s$, $N_\phi=k$). The main statement is that

$$\mathcal{V}_{\beta=1}\big(g,B,\{z_j\}\big)=\langle b(z_1)\bar{b}(z_1)...b(z_N)\bar{b}(z_N)\rangle=\frac{\det'\Delta_{\rm L}}{Z_k}||\det s_l(z_m)||^2,$$

cf. [3, Eq. (4.15)], where on the right hand side we recognize the Hermitian norm of the integer QH state (3.1) and the Quillen metric (3.26). Then from (4.20) we it follows that $\mathcal{F}_\beta$ at $\beta = 1$ reduces to the logarithm of the spectral determinant of laplacian (3.25) for the line bundle L,

$$\mathcal{F}_{\beta=1} = \log \det{}' \Delta_{\mathrm{L}}. \tag{4.27}$$

In this sense, the formula (4.20) for the Laughlin states can be thought of as a β-deformation of bosonisation formulas on Riemann surfaces.

4.4 Laughlin states on the torus from free fields

The computation of the correlation function Eq. (4.18) on the torus is a version of the standard computation in CFT [30, Ch. 10], slightly modified to include the magnetic field. We go over this calculation here in order to account for non-homogeneous magnetic field and curved metric. For CFT-type calculation for the Laughlin states on the flat torus and in constant magnetic field, we also refer to [27, 23] and excellent recent accounts [53, 39, 32], where also QH hierarchy states are constructed.

There are two nontrivial 1-cycles on the torus, hence there exist classical configurations $\sigma_{mm'}$ of the compactified boson, labelled by the integers $m, m' \in \mathbb{Z}$,

$$\sigma = \sigma_0 + \sigma_{mm'}(z) + \tilde{\sigma}(z), \tag{4.28}$$

$$\sigma_{mm'}(z + a\tau + b) = \sigma_{mm'}(z) + 2\pi R_c(mb + m'a), \tag{4.29}$$

winding m, m' times around each of the cycles, with $\tilde{\sigma}(z)$ being a single-valued scalar function. The last equation can be solved by

$$\sigma_{mm'}(z) = 2\pi R_c \left(\frac{m' - m\bar{\tau}}{\tau - \bar{\tau}}(z - z_0) - \frac{m' - m\tau}{\tau - \bar{\tau}}(\bar{z} - \bar{z}_0) \right), \tag{4.30}$$

where z_0 is so far an arbitrary point on the torus. After the zero-mode integration, which fixes the number of particles Eq. (4.21), the integral decomposes into the product of the classical part Z_{cl} due to $\sigma_{mm'}$ and quantum part Z_{qu} due to integration over $\tilde{\sigma}$

$$\mathcal{V} = Z_{\mathrm{cl}} Z_{\mathrm{qu}}. \tag{4.31}$$

Here Z_{cl} is the sum over the sectors with different m, m'.

In order to define Z_{cl} we need to compute the action Eq. (4.17) on the field configuration (4.30). There is a certain subtlety arising from multi-valuedness of the compactified boson, which manifests itself in ambiguity in the choice of base-point z_0 in Eq. (4.30). This has no effect on the first and the last terms in the action (4.17), but the second and third term need to be defined more carefully. Proper definition should ensure modular invariance of the correlation functions. For our purposes it suffices to choose the base-point as $z_0 = (\tau + 1)/2$, for which the second and third terms in the action (4.17) vanish. However, we shall note that various other prescriptions are possible for the terms of this type, on the torus [31] and on higher-genus surfaces [101], that also preserve the modular invariance.

Then the value of the action Eq. (4.17) on the configuration (4.30) is easily computed

$$S(g_0, N_\phi, \sigma_{mm'}) = \frac{\pi i R_c^2}{(\tau-\bar\tau)}|m'-m\tau|^2 - \frac{2\pi i R_c}{\sqrt\beta}(m\varphi_2 + m'\varphi_1),$$

where the first term comes form the kinetic term and the second term is the contribution of the flat connections Eq. (2.21). Taking into account the contribution from the vertex operators $e^{i\sqrt\beta\sigma_{mm'}(z_l)}$, $Z_{\rm cl}$ reads

$$Z_{\rm cl} = \frac{R_c}{\sqrt2}\sum_{m,m'\in\mathbb Z}\exp\Bigg(-\frac{\pi i R_c^2}{(\tau-\bar\tau)}|m'-m\tau|^2 + \frac{2\pi i R_c}{\sqrt\beta}(m\varphi_2+m'\varphi_1) \\ +2\pi i R_c\sqrt\beta\left(\frac{m'-m\bar\tau}{\tau-\bar\tau}z_{\rm cm} - \frac{m'-m\tau}{\tau-\bar\tau}\bar z_{\rm cm}\right)\Bigg).$$

Now we apply Poisson summation formula Eq. (6.4) to the sum over m'

$$Z_{\rm cl} = \sqrt{{\rm Im}\,\tau}\cdot e^{\frac{\pi i\beta}{\tau-\bar\tau}\left(z^\varphi_{\rm cm}-\bar z^\varphi_{\rm cm}\right)^2}\sum_{m,n\in\mathbb Z}\exp\Bigg(i\pi\tau\left(\frac{n}{R_c}-\frac{mR_c}{2}\right)^2 - i\pi\bar\tau\left(\frac{n}{R_c}+\frac{mR_c}{2}\right)^2 \\ -2\pi i\sqrt\beta\tau\left(\frac{n}{R_c}-\frac{mR_c}{2}\right)z^\varphi_{\rm cm} + 2\pi i\sqrt\beta\bar\tau\left(\frac{n}{R_c}+\frac{mR_c}{2}\right)\bar z^\varphi_{\rm cm}\Bigg), \tag{4.32}$$

where we introduced the short-hand notation

$$z^\varphi_{\rm cm} = z_{\rm cm} + \frac{\varphi}{\beta},\quad \bar z^\varphi_{\rm cm} = \bar z_{\rm cm} + \frac{\bar\varphi}{\beta}.$$

Now we set $R_c=\sqrt\beta$ and change the summation indices $m,n\to p,r,\varepsilon$ according to: $n=\beta p+r,\ p\in\mathbb Z,\ r=1,..,\beta;\ m=2(q+\varepsilon),\ q\in\mathbb Z,\ \varepsilon=\{0,\frac12\}$. Then the sum over m,n in Eq. (4.32) reads

$$\sum_{m,n\in\mathbb Z}... = \sum_{\varepsilon=0,\frac12}\sum_{r=1}^{\beta}\sum_{p,q\in\mathbb Z}\exp\Bigg(i\pi\beta\tau\left(p-q+\frac r\beta-\varepsilon\right)^2 + 2\pi i\left(p-q+\frac r\beta-\varepsilon\right)\beta z^\varphi_{\rm cm} \\ +i\pi\beta\bar\tau\left(p+q+\frac r\beta+\varepsilon\right)^2 - 2\pi i\left(p+q+\frac r\beta+\varepsilon\right)\beta\bar z^\varphi_{\rm cm}\Bigg).$$

Next we redefine the summation variables as follows, $n=p-q$, $m=p+q$, $n,m\in\mathbb Z$, which implies the constraint on $n+m$ being even,

$$\sum_{m,n\in\mathbb Z}... = \sum_{\varepsilon=0,\frac12}\sum_{r=1}^{\beta}\sum_{m,n\in\mathbb Z}\exp\Bigg(i\pi\beta\tau\left(n+\frac r\beta-\varepsilon\right)^2 + 2\pi i\left(n+\frac r\beta-\varepsilon\right)\beta z^\varphi_{\rm cm}$$

$$+i\pi\beta\bar{\tau}\left(m+\frac{r}{\beta}+\varepsilon\right)^2-2\pi i\left(m+\frac{r}{\beta}+\varepsilon\right)\beta\bar{z}^{\varphi}_{\rm cm}\Bigg)\left(\frac{1+e^{\pi i(n+m)}}{2}\right),$$

where the last term enforces this constraint. Writing

$$(1+e^{\pi i(n+m)})=\sum_{\delta=\{0,\frac12\}}e^{2\pi i\delta(n+m)},$$

we can finally recast $Z_{\rm cl}$ in the form of the sum of absolute values squared of theta functions

(4.33)

$$Z_{\rm cl}=\frac12\sqrt{{\rm Im}\,\tau}\cdot e^{\frac{\pi i\beta}{\tau-\bar\tau}\left(z_{\rm cm}-\bar z_{\rm cm}+\frac{\varphi-\bar\varphi}{\beta}\right)^2}\sum_{\varepsilon,\delta=\{0,\frac12\}}\sum_{r=1}^{\beta}e^{4\pi i\varepsilon\delta}\left|\vartheta\begin{bmatrix}\frac{r}{\beta}+\varepsilon\\ \delta\end{bmatrix}\left(\beta z_{\rm cm}+\varphi,\beta\tau\right)\right|^2.$$

The sum over ε,δ is nothing but the sum over four different spin structures, by analogy with Eq. (2.34). However, we note that for β even the sum over ε,δ collapses into one term:

$$\beta\in\text{even}:\quad Z_{\rm cl}=\sqrt{{\rm Im}\,\tau}\cdot e^{\frac{\pi i\beta}{\tau-\bar\tau}\left(z_{\rm cm}-\bar z_{\rm cm}+\frac{\varphi-\bar\varphi}{\beta}\right)^2}\sum_{r=1}^{\beta}\left|\vartheta\begin{bmatrix}\frac{r}{\beta}\\ 0\end{bmatrix}\left(\beta z_{\rm cm}+\varphi,\beta\tau\right)\right|^2.$$

From now on we will only consider odd values of β. In order to be consistent with Eq. (4.13) we rewrite the expression Eq. (4.33) in the equivalent form, using the parity indicator λ (4.12) for the number of particles,

$$\begin{aligned}Z_{\rm cl}=&\frac12\sqrt{{\rm Im}\,\tau}\cdot e^{\frac{\pi i\beta}{\tau-\bar\tau}\left(z_{\rm cm}-\bar z_{\rm cm}+\frac{\varphi-\bar\varphi}{\beta}\right)^2}\\ &\cdot\sum_{\varepsilon,\delta=\{0,\frac12\}}\sum_{r=1}^{\beta}e^{4\pi i(\varepsilon-\lambda+\frac12)(\delta-\lambda+\frac12)}\left|\vartheta\begin{bmatrix}\frac{r+\varepsilon}{\beta}-\lambda+\frac12\\ \delta-\beta\lambda+\frac{\beta}{2}\end{bmatrix}\left(\beta z_{\rm cm}+\varphi,\beta\tau\right)\right|^2.\end{aligned}$$

Next we compute the quantum part $Z_{\rm qu}$ in Eq. (4.31). This is given by Eq. (4.26), where the Green function (4.23) and regularized Green function (4.25) on the flat torus read

$$\begin{aligned}G^{g_0}(z,z')&=\frac12\frac{\pi i}{\tau-\bar\tau}\big(z-z-(\bar z-\bar z')\big)^2+\log\left|\frac{\theta_1(z-z',\tau)}{\eta(\tau)}\right|,\\ G^{g_0}_{\rm reg}(z)&=-\log\big(\sqrt{2\pi{\rm Im}\,\tau}\,|\eta(\tau)|^2\big).\end{aligned}$$

Plugging this to (4.26), and observing that the integrals over Σ vanish due to (4.24), we obtain

$$Z_{\rm qu}=\left[\frac{\det'\Delta_{g_0}}{2\pi}\right]^{-1/2}\cdot\exp\left(-\beta\sum_{l\neq m}^{N}G^{g}(z_l,z_m)-\beta\sum_{l=1}^{N}G^{g}_{\rm reg}(z_l)\right)\qquad(4.34)$$

$$= \sqrt{2\pi}\big(2\pi \mathrm{Im}\,\tau|\eta(\tau)|^4\big)^{\frac{N_\phi-1}{2}} \prod_{l<m}^{N} \left|\frac{\theta_1(z_l-z_m,\tau)}{\eta(\tau)}\right|^{2\beta} \cdot e^{\frac{\pi i N_\phi}{\tau-\bar\tau}\sum_l (z_l-\bar z_l)^2 - \frac{\pi i\beta}{\tau-\bar\tau}(z_{\rm cm}-\bar z_{\rm cm})^2},$$

where we used the formula for the regularized determinant of laplacian on the torus

$$\det{}'\Delta_{g_0} = 2\pi|\eta(\tau)|^4\,\mathrm{Im}\,\tau,$$

see e.g. Ref. [30]. Putting together $Z_{\rm cl}$ and $Z_{\rm qu}$ we arrive at

$$\begin{aligned}\mathcal{V} =&\frac{1}{2}\big(2\pi \mathrm{Im}\,\tau|\eta(\tau)|^4\big)^{\frac{N_\phi}{2}} \cdot e^{\frac{\pi i}{\beta(\tau-\bar\tau)}(\varphi-\bar\varphi)^2}\\ &\cdot \frac{1}{|\eta(\tau)|^2}\sum_{\varepsilon,\delta=\{0,\frac12\}}\sum_{r=1}^{\beta} e^{4\pi i(\varepsilon-\lambda+\frac12)(\delta-\lambda+\frac12)}\left|\vartheta\begin{bmatrix}\frac{r+\varepsilon}{\beta}-\lambda+\frac12\\ \delta-\beta\lambda+\frac{\beta}{2}\end{bmatrix}\big(\beta z_{\rm cm}+\varphi,\beta\tau\big)\right|^2\\ &\cdot\prod_{l<m}^{N}\left|\frac{\theta_1(z_l-z_m,\tau)}{\eta(\tau)}\right|^{2\beta}\cdot\prod_{l=1}^{N} e^{\frac{\pi i N_\phi}{\tau-\bar\tau}(z_l-\bar z_l)^2+\frac{2\pi i}{\tau-\bar\tau}(z_l-\bar z_l)(\varphi-\bar\varphi)}.\end{aligned}$$

This is the final result for the correlation function of vertex operators Eqns. (4.18, 4.31) on the torus.

4.5 Holomorphic structure and modular group action

Comparing with the (holomorphic parts) of the Laughlin state given in Eq. (4.13), and taking into account Eq. (4.14) we see that the sum above is in the form of Eq. (4.20),

(4.35)
$$\begin{aligned}\frac{1}{2\beta}\sum_{\varepsilon,\delta=\{0,\frac12\}} c_{\varepsilon,\delta}\sum_{r=1}^{\beta}|\Psi_r^{\varepsilon,\delta}|^2 =&\frac{e^{-\mathcal{F}_\beta(g_0,B_0)}}{Z_\beta(\tau,\bar\tau,\varphi,\bar\varphi)}\\ &\cdot\sum_{\varepsilon,\delta=\{0,\frac12\}}\sum_{r=1}^{\beta} c_{\varepsilon,\delta}|F_r^{\varepsilon,\delta}|^2\prod_{l=1}^{N} h_0^{N_\phi}(z_l,\bar z_l),\end{aligned}$$

with $c_{\varepsilon,\delta} = e^{4\pi i(\varepsilon-\lambda+\frac12)(\delta-\lambda+\frac12)}$. Here we introduced the the Z-factor and redefined the holomorphic part of the Laughlin state to include the η functions exactly as they appear from the path integral calculation

(4.36)
$$F_r^{\varepsilon,\delta}(\{z_l\}) =\eta(\tau)^{N_\phi-1}\vartheta\begin{bmatrix}\frac{r+\varepsilon}{\beta}-\lambda+\frac12\\ \delta-\beta\lambda+\frac{\beta}{2}\end{bmatrix}\big(\beta z_{\rm cm}+\varphi,\beta\tau\big)\prod_{l<m}^{N}\left(\frac{\theta_1(z_l-z_m,\tau)}{\eta(\tau)}\right)^{\beta},$$

(4.37)
$$Z_\beta(\tau,\bar\tau,\varphi,\bar\varphi) = \big(2\pi \mathrm{Im}\,\tau\big)^{-\frac{N_\phi}{2}}\cdot e^{-\frac{\pi i}{\beta(\tau-\bar\tau)}(\varphi-\bar\varphi)^2}.$$

Comparing Eqns. (4.15) and (4.20) we can write the relation between the normalization factors Z_β and $\mathcal{N}_0$ as follows

$$\mathcal{N}_0[g_0, B_0, \tau, \varphi] = e^{\mathcal{F}_\beta(g_0,B_0)} \cdot Z_\beta(\tau, \bar\tau, \varphi, \bar\varphi).$$

Remark. There is some ambiguity in the choice of the holomorphic part and of the Z_β-factor. Namely, one can redefine $Z_\beta \to |f(\tau,\varphi)|^2 Z_\beta$ and correspondingly $F_r \to f(\tau,\varphi)F_r$ by a holomorphic function of the moduli. Since $Jac(\Sigma)$ is compact f is a function of τ only. We can also take $f(\tau)$ to be non-vanishing on an open set of $\mathcal{M}_1$. In particular, $f(\tau)$ can be a power of $\eta(\tau)$, since the latter is non-vanishing in the upper half plane $\tau \in \mathbb{H}$ with a zero as $\tau \to i\infty$, where $\eta(\tau) \sim q^{1/24}$, $q = e^{2\pi i\tau}$. This will modify the adiabatic connection (2.27) and add the delta-function term, localized at $i\infty$, to the adiabatic curvature, but also modify the monodromies of the Laughlin states on the moduli space (which we review below), while preserving the adiabatic phases. In general, it should be possible to study the behavior of the normalized Laughlin states near the boundary of the moduli space of complex structures, by applying techniques of Ref. [11].

Next we note that the wave functions $\Psi_r^{\varepsilon,\delta}$ have the same general form as (2.24), so the relations (2.27) and (2.28) apply to the Laughlin states, with the substitution $Z(y,\bar y) = e^{\mathcal{F}_\beta} \cdot Z_\beta$. In particular, the adiabatic connection is projectively flat and adiabatic curvature is a scalar matrix. The factor $e^{\mathcal{F}_\beta}$ is given by a nontrivial path integral expression Eq. (4.19) and we will study it in the next section.

Let us now discuss the action of lattice shifts in the Jacobian $T_{[\varphi]}$ and the modular transformations. This is completely analogous to the action on one-particle states, worked out in §2.5. The group of lattice shifts $\varphi \to \varphi + t_1 + t_t\tau$ acts in the unitary representation

$$\begin{aligned}
&F_r^{\varepsilon,\delta}(\{z_l\}|\varphi + t_1 + t_2\tau, \tau) = e^{-\frac{\pi i}{\beta}t_2^2\tau - \frac{2\pi i}{\beta}t_2(\beta z_{\rm cm}+\varphi)} \cdot \sum_{r'=1}^{\beta} U_{rr'} F_{r'}^{\varepsilon,\delta}(\{z_l\}|\varphi,\tau),\\
&\text{where}\quad U_{rr'} = e^{\frac{2\pi i}{\beta}\left(t_1 r + t_1(\varepsilon+\beta(\frac12-\lambda)) - t_2(\delta+(\frac12-\lambda))\right)}\delta_{r,r'-t_2},\\
&\prod_{l=1}^N h_0^{N_\phi}(z_l,\bar z_l|\varphi + t_1 + t_2\tau,\tau) = e^{2\pi i t_2(z_{\rm cm}-\bar z_{\rm cm})}\prod_{l=1}^N h_0^{N_\phi}(z_l,\bar z_l|\varphi,\tau),\\
&Z_\beta(\varphi + t_1 + t_2\tau, \bar\varphi + t_1 + t_2\bar\tau, \tau, \bar\tau) = e^{-\frac{\pi i}{\beta}t_2^2(\tau-\bar\tau) - \frac{2\pi i}{\beta}t_2(\varphi-\bar\varphi)} \cdot Z_\beta(\varphi,\bar\varphi,\tau,\bar\tau),
\end{aligned}$$

and, as was already the case for the LLL states on the torus, t_1-shifts act diagonally and t_2-shift action is non-diagonal.

The formulas for the action of the modular group, for $\beta \in$ odd, are listed in the Appendix. The action of the modular group on the basis of Laughlin states is very similar the action on the basis of one particle states (2.38, 2.39), formally interchanging β and k. In particular the action on spin-structures is the same as in Fig. 3. For even number of particles $\lambda = 0$ the $(0,0)$ spin structure is conserved and for $\lambda = \frac{1}{2}$ the $(\frac{1}{2}, \frac{1}{2})$ spin structure is conserved. In these cases we have

$$(U^S U^T)^3 = e^{2\pi i \theta N_\phi} C, \ (U^S)^2 = C,$$

where $C^2 = 1$ and $\theta = \frac{1}{8}$, and thus Laughlin states transform in projective unitary representation of the modular group.

5 Geometric adiabatic transport and anomaly formulas

5.1 Generating functional for Laughlin states

The generating functional, which was defined for integer QH state (3.8), can be defined for the Laughlin states as well. The definition is analogous to the one given in (3.8). We start with the background configuration (Σ, g_0) and $(L^{N_\phi}, h_0^{N_\phi})$ and choose the corresponding L^2 normalized basis of holomorphic states $F_{0r}^{\varepsilon,\delta}$ (4.7), (4.8),

$$||F_{0r}^{\varepsilon,\delta}(z_1, ..., z_N)||^2 = |F_{0r}^{\varepsilon,\delta}(z_1, ..., z_N)|^2 \prod_{l=1}^N h_0^{N_\phi}(z_l, \bar{z}_l) g_{0z\bar{z}}^{-\mathrm{j}}(z_l, \bar{z}_l), \tag{5.1}$$

$$\langle \Psi_{0r}^{\varepsilon,\delta} | \Psi_{0r}^{\varepsilon,\delta} \rangle_{L^2} = \frac{1}{\mathcal{N}_0} \frac{1}{(2\pi)^N} \int_{\Sigma^N} ||F_{0r}^{\varepsilon,\delta}(z_1, ..., z_N)||^2 \prod_{l=1}^N \sqrt{g_0} d^2 z_l = 1,$$

where the normalization factor $\mathcal{N}_0$ is r-independent, which is the case e.g., for the flat torus with constant magnetic field (4.15). Next, we consider the curved metric (Σ, g) and magnetic field (L^{N_ϕ}, h^{N_ϕ}), where the g and F are in the same Kähler class

$$g = g_0 + \partial_z \partial_{\bar{z}} \phi, \tag{5.2}$$

$$h^{N_\phi} = h_0^{N_\phi} e^{-N_\phi \psi}, \tag{5.3}$$

$$F_{z\bar{z}} = F_{0z\bar{z}} + N_\phi \partial_z \bar{\partial}_{\bar{z}} \psi. \tag{5.4}$$

The partition function is then defined as

$$Z_{N_\phi}[g_0, B_0, g, B] = \frac{1}{\mathcal{N}_0} \frac{1}{(2\pi)^N} \int_{\Sigma^N} \frac{1}{2^{\mathrm{g}} n_{\beta,\mathrm{g}}} \sum_{r,\varepsilon,\delta} c_{\varepsilon,\delta} ||F_{0r}^{\varepsilon,\delta}(z_1, ..., z_N)||^2 \cdot \prod_{l=1}^N h^{N_\phi}(z_l, \bar{z}_l) g_{z\bar{z}}^{-\mathrm{j}}(z_l, \bar{z}_l) \sqrt{g} d^2 z_l. \tag{5.5}$$

In other words, we change the Hermitian metric on the line bundle and the metric on the surface, staying in the same Kähler class, and compute the sum of the norms of the wave functions in the new metric. It follows that the partition function is normalized as

$$Z_{N_\phi}[g_0, B_0, g_0, B_0] = 1.$$

The expression (5.5) is written on the surface of genus $\mathrm{g} > 0$ and includes sum over all degenerate Laughlin states and also over the spin-structures. On the sphere this formula simplifies, since there is only one Laughlin state. In the notations of Eq. (4.9), we can write on the sphere

(5.6)

$$Z_{N_\phi}[g_0, B_0, g, B] = \frac{1}{\mathcal{N}_0} \frac{1}{(2\pi)^N} \int_{(S^2)^N} |\det s_l(z_m)|^{2\beta} \prod_{l=1}^{N} h^{N_\phi}(z_l, \bar{z}_l) g_{z\bar{z}}^{-\mathrm{j}}(z_l, \bar{z}_l) \sqrt{g} d^2 z_l.$$

Taking into account (4.20), we can rewrite (5.5) via the correlator of vertex operators

$$\begin{aligned} Z_{N_\phi}[g_0, B_0, g, B] =& \frac{1}{(2\pi)^N} \int_{\Sigma^N} \frac{1}{2^{\mathrm{g}} n_{\beta,\mathrm{g}}} \sum_{r,\varepsilon,\delta} c_{\varepsilon,\delta} \left| \Psi_{0r}^{\varepsilon,\delta}(z_1, ..., z_N) \right|^2 \\ & \cdot e^{-\sum_{l=1}^N \left(N_\phi \psi(z_l, \bar{z}_l) + \mathrm{j} \log \frac{\sqrt{g}}{\sqrt{g_0}}|_{z_l} \right)} \prod_{l=1}^{N} \sqrt{g} d^2 z_l \\ = \frac{1}{e^{\mathcal{F}_\beta(g_0, B_0)}} \frac{1}{(2\pi)^N} & \int_{\Sigma^N} \mathcal{V}\big(g_0, B_0, \{z_l\}\big) e^{-\sum_{l=1}^N \left(N_\phi \psi(z_l, \bar{z}_l) + \mathrm{j} \log \frac{\sqrt{g}}{\sqrt{g_0}}|_{z_l} \right)} \prod_{l=1}^{N} \sqrt{g} d^2 z_l. \end{aligned} \tag{5.7}$$

As usual the logarithm of the partition function is called the generating functional. In this section we show that $\log Z_{N_\phi}$ admits an expansion for large magnetic field, which is analogous to the expansion of the generating functional for the integer QH state

$$\log Z_{N_\phi}[g_0, B_0, g, B] = \log \frac{Z_{H,\beta}}{Z_{H0,\beta}} + \mathcal{F}_\beta[g, B] - \mathcal{F}_\beta[g_0, B_0],$$

where $\mathcal{F}_\beta$ is local functional of g and B and $\log Z_{H,\beta}$ is non-local functional representing anomaly, where all the terms depend nontrivially on β.

In the integer QHE case we were able to compute the asymptotic expansion of $\log Z_{N_\phi}$ due to the determinantal representation of partition function (3.10), which allowed us to reduce the computation to the Bergman kernel expansion for high powers of line bundle. However, the partition function (5.5) does not admit the determinantal representation and novel methods are required. Here we review the path integral derivation of the asymptotic expansion following [37]; another derivation of this result can be found in [21, 22], where the Ward identity method of Refs. [108, 110, 109] was employed.

5.2 Effective action and gravitational anomaly

The calculation of asymptotic expansion of Z_{N_ϕ} is performed in two steps. At the first step we start from the path integral expression for $\mathcal{V}\big(g, B, \{z_l\}\big)$ in Eq. (4.18) and compute its transformation formula under the change of metrics (g, h) to (g_0, h_0) in the same Kähler class, Eqns. (5.2, 5.3). As we have seen in Eq. (4.31), for the surfaces of genus $\mathrm{g} > 0$ the path integral is the product of the classical and quantum parts $\mathcal{V} = Z_{\rm cl} Z_{\rm qu}$, where the classical part $Z_{\rm cl}$ essentially depends only on the Kähler class of the metric and not on a particular choice of the metric in that class. Therefore it suffices to derive the transformation formulas for the change of metrics in the quantum part of the path integral $Z_{\rm qu}$. The latter is given by the same formal expression Eq. (4.26) for the surfaces of any genus, including sphere. It remains to compute the transformation rules for different objects in that expression. The scalar curvatures and magnetic fields in background and curved metrics are related as

$$
\begin{aligned}
R\sqrt{g} &= R_0\sqrt{g_0} - \sqrt{g_0}\,\Delta_0 \log\frac{\sqrt{g}}{\sqrt{g_0}}, \\
B\sqrt{g} &= B_0\sqrt{g_0} + \frac{1}{2}N_\phi\sqrt{g_0}\,\Delta_0\psi.
\end{aligned}
\tag{5.8}
$$

For the metrics $g = g_0 + \partial\bar\partial\phi$ in the same Kähler class the regularized determinant of the laplacian transforms according to the Polyakov gravitational anomaly formula [85],

$$
\frac{\det'\Delta_g}{\det'\Delta_0} = e^{-\frac{1}{6}S_L(g_0,\phi)},
\tag{5.9}
$$

where $S_L(g_0, \phi)$ is the Liouville action Eq. (3.18).

The transformation formulas for other terms in (4.26) can be found using the identities for the transformation of Green functions and their integrals, derived in [38, §3] and [37, §4]. After a tedious but straightforward calculation, we arrive at

$$
\begin{aligned}
\log\frac{\mathcal{V}\big(g, B, \{z_l\}\big)}{\mathcal{V}\big(g_0, B_0, \{z_l\}\big)} &= -N_\phi\sum_{l=1}^{N}\psi(z_l,\bar z_l) - \mathrm{j}\sum_{l=1}^{N}\log\frac{\sqrt{g}}{\sqrt{g_0}}\Big|_{z_l} \\
&+ \frac{1}{\beta}N_\phi^2 S_2(g_0, B_0, \psi) - \frac{q}{2\sqrt{\beta}}N_\phi S_1(g_0, B_0, \psi, \phi) + \frac{1}{12}(1-3q^2)S_L(g_0,\phi).
\end{aligned}
\tag{5.10}
$$

Recall that the constant $q = \sqrt{\beta} - 2\mathrm{j}/\sqrt{\beta}$ here is defined in Eq. (4.22), and the functionals S_1 and S_2 are defined exactly as in Eqns. (3.16), (3.17) with $k \to N_\phi$, namely

$$
S_2(g_0, B_0, \psi) = \frac{1}{2\pi}\int_\Sigma\left(\frac{1}{4}\psi\Delta_0\psi + \frac{1}{N_\phi}B_0\psi\right)\sqrt{g_0}d^2z,
$$

$$S_1(g_0,B_0,\phi,\psi)=\frac{1}{2\pi}\int_\Sigma\left(-\frac12\psi R_0+\left(\frac{1}{N_\phi}B_0+\frac12\Delta_0\psi\right)\log\big(1+\frac12\Delta_0\phi\big)\right)\sqrt{g_0}d^2z.$$

Using Eq. (5.10) we can express the generating functional Eq. (5.7) as follows

$$\begin{aligned}(5.11)\quad \log Z_{N_\phi}[g_0,B_0,g,B]=&-\frac{1}{\beta}N_\phi^2S_2(g_0,B_0,\psi)+\frac{q}{2\sqrt{\beta}}N_\phi S_1(g_0,B_0,\psi,\phi)\\&-\frac{1}{12}(1-3q^2)S_L(g_0,\phi)+\mathcal{F}_\beta(g,B)-\mathcal{F}_\beta(g_0,B_0).\end{aligned}$$

The exact terms $\mathcal{F}_\beta$ are formally defined by the path integral Eq. (4.19) and we will come back to them shortly. The first three terms on the rhs in Eq. (5.11) contribute to the anomalous part of the generating functional $\log Z_H$. By analogy with Eqns. (3.20, 3.21) for the integer QH state, these can be written in two equivalent forms: as a double integral,

$$\begin{aligned}(5.12)\qquad &\log Z_{H,\beta}=\\&-\frac{1}{2\pi\beta}\int_{\Sigma\times\Sigma}\left(B+\frac{\beta-2\mathrm{j}}{4}R\right)\big|_z\Delta_g^{-1}(z,y)\left(B+\frac{\beta-2\mathrm{j}}{4}R\right)\big|_y\sqrt{g}d^2z\sqrt{g}d^2y\\&+\frac{1}{96\pi}\int_{\Sigma\times\Sigma}R(z)\Delta_g^{-1}(z,y)R(y)\sqrt{g}d^2z\sqrt{g}d^2y,\end{aligned}$$

and as a quadratic form in gauge and spin connections,

$$(5.13)\qquad \log Z_{H,\beta}=\frac{2}{\pi}\int_\Sigma\left[\sigma_HA_zA_{\bar z}+2\varsigma_H(A_z\omega_{\bar z}+\omega_zA_{\bar z})-\frac{1}{12}c_H\omega_z\omega_{\bar z}\right]d^2z,$$

where we introduced the following constants

$$(5.14)\qquad \sigma_H=\frac{1}{\beta},\quad \varsigma_H=\frac{q}{4\sqrt{\beta}},\quad c_H=1-3q^2,$$

and $q=\sqrt{\beta}-2\mathrm{j}/\sqrt{\beta}$. Recall that Eq. (5.13) assumes symmetric gauge Eq. (2.4) for the gauge and spin connections. As a consistency check, at $\beta=1$ we have $\mathrm{j}=s$ and $q=1-2s$ and the expressions above agree with Eqns. (3.20, 3.21).

At the moment Eq. (5.11) is a formal expression, valid for any $N_\phi\geqslant 0$. It turns out that the terms $\mathcal{F}_\beta$ (4.19) can be better understood for N_ϕ large. We can already see that the first three terms in Eq. (5.11) are written in the form of the large N_ϕ expansion, similar to anomalous terms (3.15) in the integer case. We argue that $\mathcal{F}_\beta$ admits asymptotic expansion for large N_ϕ with coefficients given by local functional of the metric and the magnetic field, which is similar to Eq. (3.19) in the integer case, but now β-deformed. The argument is as follows [37]. Starting from the representation (4.19) we can rewrite the remainder term in the form

$$e^{\mathcal{F}_\beta(g,B)}=\int e^{-S(g,B,\sigma)+N\log\frac{1}{2\pi}\int_\Sigma e^{i\sqrt{\beta}\sigma(z)}\sqrt{g}d^2z}\mathcal{D}_g\sigma.$$

This is the path integral of the interacting scalar field, where the number of particles N plays the role of a large parameter. Therefore one can apply the stationary phase method. The standard analysis of perturbation theory in $1/N$ reveals [37] that the contributions from the Feynman diagrams reduce to local integrals of polynomials in curvature and magnetic field and their derivatives. In particular, the leading term in the large magnetic field expansion of $\mathcal{F}_\beta$ has the form [68],

$$\mathcal{F}_\beta = \frac{\beta-2}{4\pi\beta}\int_\Sigma B\left(\log\frac{B}{2\pi}\right)\sqrt{g}d^2z + \mathcal{O}(\log B). \tag{5.15}$$

At $\beta = 1$ this coincides with the integer QH result for $\mathcal{F}$, cf. first term in Eq. (3.19).

5.3 Quillen metric and geometric adiabatic transport

First we discuss the geometric adiabatic transport in the integer QH state and then turn to the Laughlin states. We consider how the wave functions vary over the parameter space $Y = \mathcal{M}_{\rm g}\times Jac(\Sigma)$. Following the discussion in §2.3 it is especially convenient to put the wave function in the form Eq. (2.24), which emphasizes the holomorphic structure. For the integer QH state we can always choose the basis $s_l(z|y)$ in the space of holomorphic sections $H_0(\Sigma, L^k\otimes K^s)$, so that it depends holomorphically on local complex coordinate $y\in Y$. Then the integer QH state $\mathcal{S}$ (3.1) transforms as a holomorphic section of the Quillen's determinant line bundle $\mathcal{L} = \det H^0(\Sigma_y, L^k\otimes K_y^s)$ over Y. The point-wise Hermitian norm of the section $\det s_l(z_m)$ and the L^2 norm is defined as before in Eq. (3.8). The adiabatic connection and adiabatic curvature are then given by Eqns. (2.25), (2.26) and thus the adiabatic curvature can be expressed in terms of the L^2 norm of the IQHE state as

$$\mathcal{R} = -(\partial_y\partial_{\bar y}\log Z_k) idy\wedge d\bar y. \tag{5.16}$$

Here $\mathcal{R}$ carries no indices, since the integer QH state is not degenerate and is a section of the line bundle. We can now compute (5.16) using the following observation originally due to Avron-Seiler-Zograf [7]. We rewrite $\mathcal{R}$ as

$$\mathcal{R} = -\left(\partial_y\partial_{\bar y}\log\frac{Z_k}{\det'\Delta_{\rm L}}\right) idy\wedge d\bar y - \left(\partial_y\partial_{\bar y}\log\det{}'\Delta_{\rm L}\right) idy\wedge d\bar y, \tag{5.17}$$

and note that the first term here is the curvature $\mathcal{R}^{\mathcal{L}}$ of the Quillen metric on $\mathcal{L}$, where the latter is defined in Eq. (3.26). The formula for the curvature of the Quillen metric is known in physics literature as the Quillen anomaly formula, and it was first computed as part of the proof of the holomorphic factorisation of string theory integration measure in Ref. [11], see also [87, 113] for mathematical references. For the Riemann surfaces the curvature of Quillen metric can be written explicitly,

$$\mathcal{R}^{\mathcal{L}} = 2\pi d\varphi\wedge(\Omega-\bar\Omega)^{-1}d\bar\varphi-\left(\frac{k}{4}(1-2s)-\frac{1}{12}\big(1-3(1-2s)^2\big)\chi(\Sigma)\right)\Omega_{WP}. \tag{5.18}$$

The first term here is the 2-form on $Jac(\Sigma)$ written in complex coordinates (2.7) and summation over indices labelling 1-cycles is understood. In real coordinates on the Jacobian (2.5) the first term reads

$$2\pi d\varphi\wedge(\Omega-\bar\Omega)^{-1}d\bar\varphi = 2\pi\sum_{a=1}^{\mathrm{g}} d\varphi_1^a\wedge\varphi_2^a,$$

and thus corresponds to the flat Euclidean metric on the 2g dimensional torus. The Weil-Petersson form Ω_{WP} on the moduli space of complex structures $\mathcal{M}_{\mathrm{g}}$, for constant scalar curvature metrics on Σ, enters the second term in Eq. (5.18) and is defined as follows. The deformations of the metric, preserving the area of Σ, along the moduli space have the form

$$\delta(g_{z\bar z}dzd\bar z) = \frac{1}{1-|\delta\mu|^2}g_{z\bar z}|dz+\delta\bar\mu d\bar z|^2 - g_{z\bar z}dzd\bar z,$$

where $\delta\mu = \delta\mu^z{}_{\bar z}d\bar z(dz)^{-1}$ is the Beltrami differential with the weight $(-1,1)$. There exists [28] $3\mathrm{g}-3$ independent holomorphic quadratic differentials η on a surface of a genus $\mathrm{g}>1$ and the corresponding Beltrami differential $\delta\mu = g^{z\bar z}\sum_{\nu=1}^{3\mathrm{g}-3}\bar\eta_\nu dy_\nu$ is characterized by $3\mathrm{g}-3$ local complex coordinates y_ν. The Kähler $(1,1)$ form on $\mathcal{M}_{\mathrm{g}}$ corresponding to the Weil-Petersson metric can be written as

$$\Omega_{WP} = \frac{1}{2\pi}\int_\Sigma (i\delta\mu\wedge\delta\bar\mu)\sqrt{g}d^2z = \frac{1}{\pi}\int_\Sigma g^{z\bar z}\bar\eta_\nu\eta_\mu d^2z\; idy^\nu\wedge d\bar y^\mu.$$

We note that in the standard definition the scalar curvature R enters the integrand, which is constant in our case $R=2\chi(\Sigma)$, and the factor of $\chi(\Sigma)$ is already present in the formula (5.18). On the sphere there are no solenoid phases and the choice of complex structure is unique, so the corresponding moduli space is just a point. On the torus Eq. (5.18) is still valid with the replacement of Weil-Petersson form Ω_{WP} by the Poincaré metric on $\mathcal{M}_1$, and setting $s=0$. Indeed, using the determinantal formula (3.10) we immediately obtain for the flat torus and constant magnetic field,

$$Z_k = (Z)^k,$$

where $Z = Z(\tau,\bar\tau,\varphi,\bar\varphi)$ is the normalization Z-factor for one particle LLL states on the torus Eq. (2.37). Next, the determinant of the laplacian for the line bundle L^k is moduli-independent constant (3.30), so the adiabatic curvature $\mathcal{R}$ and Quillen curvature (5.18) coincide. Hence in the case of torus we obtain

$$\mathcal{R}_{T^2} = \mathcal{R}^{\mathcal{L}}_{T^2} = (-\partial_y\partial_{\bar y}\log Z_k)\,idy\wedge d\bar y = 2\pi\frac{d\varphi\wedge d\bar\varphi}{\tau-\bar\tau}+\frac{k}{4}\frac{2id\tau\wedge d\bar\tau}{(\tau-\bar\tau)^2},$$

in complete agreement with (2.45).

On higher genus surfaces the determinant of the laplacian depends on the moduli in a nontrivial way. Namely for the constant scalar curvature metric on $\Sigma_{\rm g}$ with ${\rm g} > 1$ and for canonical line bundle $L = K$ we have, according to Refs. [29, 7],

$$\det{}'\Delta_{L^k} = e^{-c_k\chi(\Sigma)}\prod_{\gamma}\prod_{j=1}^{\infty}\big[1 - e^{i\sum_{a=1}^{2{\rm g}}\varphi_a n_a(\gamma)}e^{-(j+k)l(\gamma)}\big]. \tag{5.19}$$

Here the surface Σ is realized as an orbit space for a discrete subgroup Γ of $SL(2,\mathbb{R})$ acting on upper half plane and $\gamma \in \Gamma$ are primitive hyperbolic elements of Γ representing conjugacy classes corresponding to closed geodesics on Σ. Next, n_a counts the number of times the closed geodesic goes around the ath fundamental loop, $l(\gamma)$ is the length of geodesic, and c_k is a constant. In the large k limit the leading term (apart form the constant, irrelevant for the computation of curvature) in $\log\det{}'\Delta_{L^k}$ decays exponentially as $e^{-kl(\gamma_{\rm min})}$, where $\gamma_{\rm min}$ is the length of the shortest geodesic. When the latter is bounded from zero, i.e., away from the boundary of the moduli space, the second term in (5.16) represents small fluctuations and $\mathcal{R} \approx \mathcal{R}^{\mathcal{L}} + \mathcal{O}(e^{-kl(\gamma_{\rm min})})$ with exponential precision. However, the exponential asymptotic of determinant changes to polynomial near the boundary of the moduli space, see [28, §V.F], where the second term in (5.16) starts to play a more prominent role. This regime, where the Riemann surface becomes singular, deserves to be understood in more detail. For recent work in QH states on singular surfaces we refer to [72, 47], see also [4].

The formula (5.18) is can be read off directly from the anomalous part of the generating functional written as the quadratic form (3.21). We write the variation of the gauge connection along the moduli space as

$$\delta(A_zdz) = 2\pi\delta\varphi(\Omega-\bar\Omega)^{-1}\bar\omega, \quad \delta(A_{\bar z}d\bar z) = -2\pi\delta\bar\varphi(\Omega-\bar\Omega)^{-1}\omega,$$

and the second variation of the spin connection (the first variation vanishes) as

$$\delta\wedge\bar\delta(\omega_zdz) = \frac{1}{2}\partial_z(\delta\mu\wedge\delta\bar\mu), \quad \delta\wedge\bar\delta(\omega_{\bar z}d\bar z)dz = -\frac{1}{2}\partial_{\bar z}(\delta\mu\wedge\delta\bar\mu)d\bar z.$$

Computing the second variation of $\log Z_H$ and plugging the above result is another way to obtain Eq. (5.18),

$$\begin{aligned}
&-(\partial_y\partial_{\bar y}\log Z_H)idy\wedge d\bar y\\
&=\frac{1}{\pi}\int_\Sigma\Big(\delta(A_zdz)\wedge\delta(A_{\bar z}d\bar z) + \frac{(1-2s)}{2}\big(A_zdz\,\delta\wedge\bar\delta(\omega_{\bar z}d\bar z) + c.c.\big)\\
&-\frac{1}{12}\big(1-3(1-2s)^2\big)(\omega_zdz\,\delta\wedge\bar\delta(\omega_{\bar z}d\bar z) + c.c.)\Big) = \mathcal{R}^{\mathcal{L}}.
\end{aligned} \tag{5.20}$$

Here in the second and third term we applied integration by parts and then projected the result onto the constant magnetic field and constant scalar curvature metric.

5.4 Geometric adiabatic transport for Laughlin states

Now we turn to the adiabatic curvature for the Laughlin states. On a Riemann surface Σ of genus $\mathrm{g} > 0$ the number of Laughlin states is $n_{\beta,\mathrm{g}} = \beta^{\mathrm{g}}$, as was mentioned in §4.2. Thus Laughlin states transform as the sections of a vector bundle of degree $n_{\beta,\mathrm{g}}$ over the parameter space Y. The adiabatic connection on this vector bundle is projectively flat (2.27), which was demonstrated for the Laughlin states on the torus in §4.2 (this is assumed to be the case for the higher-genus surfaces as well). Then the adiabatic curvature can be determined from the norm of the wave functions as in Eq. (2.28), or by analogy with the integer case, from the generating functional

$$\mathcal{R}_{rr'} = \mathcal{R}\delta_{rr'} = -\delta_{rr'}(\partial_y\partial_{\bar{y}}\log Z_{N_\phi})idy \wedge d\bar{y}. \tag{5.21}$$

Now we can determine adiabatic curvature $\mathcal{R}$ applying the variational method of Eq. (5.20) to the anomalous part of the generating functional (5.13). We immediately obtain

$$\begin{aligned}\mathcal{R} = {}& 2\pi\sigma_H d\varphi \wedge (\Omega - \bar{\Omega})^{-1}d\bar{\varphi} \\ & - \left(\varsigma_H N_\phi - \frac{1}{12}c_H\chi(\Sigma)\right)\Omega_{WP} - (\partial_y\partial_{\bar{y}}\mathcal{F}_\beta)\, idy \wedge d\bar{y},\end{aligned} \tag{5.22}$$

where the constants $\sigma_H, \varsigma_H, c_H$ are given in Eq. (5.14). The first two terms here differ from Eq. (5.18) only in overall coefficients. The last term in (5.22) reduces to the logarithm of the regularized determinant $\mathcal{F}_{\beta=1} = \log\det{}'\Delta_{\mathrm{L}}$ due to the bosonisation formula, as was discussed before Eq. (4.27). The quantization argument for the Hall conductance in the integer case Eq. (2.44) will go through for the Laughlin states if we can show that the last term is an exact $(1,1)$ from, corresponding to small fluctuations at large N_ϕ. We have already seen from the $1/N$ perturbation theory arguments [37] that $\mathcal{F}_\beta$ admits asymptotic expansion in large magnetic field with coefficients given by local curvature invariants (5.15). Since local terms are moduli-independent, $\partial_y\partial_{\bar{y}}\mathcal{F}_\beta$ is zero perturbatively, i.e., for all terms in asymptotic $1/N_\phi$ expansion. However exponential corrections of the form $e^{-N_\phi f}$ are possible, where f can be a nontrivial function of the moduli (in fact they appear already at $\beta = 1$ in the log determinant on higher-genus surfaces(5.19)). Since f is a function $\partial_y\partial_{\bar{y}}\mathcal{F}_\beta$ is exact, and the exponential suppression means that the last term in (5.22) represents exponentially small fluctuations of the adiabatic curvature. It would be interesting to check this indirect argument, e.g. by computing $\mathcal{F}_\beta$ from its path integral representation (4.19).

We can now apply the general formula Eq. (5.22) to the torus, where we worked out explicit expression for Z_β in Eq. (4.37). Plugging $Z_{N_\phi} = e^{\mathcal{F}_\beta}Z_\beta$ in (5.21) we obtain

$$\mathcal{R} = 2\pi\sigma_H\frac{d\varphi \wedge d\bar{\varphi}}{\tau - \bar{\tau}} + \frac{N_\phi}{4}\frac{2id\tau \wedge d\bar{\tau}}{(\tau - \bar{\tau})^2} - (\partial_y\partial_{\bar{y}}\mathcal{F}_\beta)\, idy \wedge d\bar{y}, \tag{5.23}$$

in agreement with (5.22) at $\chi(\Sigma) = 0$ and j = 0. Now, by analogy with (2.44) the first Chern class of the vector bundle of Laughlin states restricted to the Jacobian $E|_{T_{[\varphi]}}$ equals one,

$$\int_{T_{[\varphi]}} c_1(E|_{T_{[\varphi]}}) = \int_{T_{[\varphi]}} \frac{1}{2\pi}\mathrm{Tr}\,\mathcal{R}_{rr'} = \beta\sigma_H = 1, \tag{5.24}$$

and thus the Hall conductance $\sigma_H = 1/\beta$ is a fraction in this case. This argument in the fractional QHE was suggested in [95, 81].

As before in the integer case, the adiabatic transport on the moduli space of complex structure of the torus gives rise to the anomalous viscosity and the Hall viscosity coefficient is also proportional to the magnetic field flux $\eta_H = N_\phi/4$ (5.23), see Refs. [98, 99] and [89]. On the surfaces of genus g > 1 the corresponding coefficient acquires the finite-size correction (5.22), proportional to c_H, for "Hall central charge", since it appears also as the coefficient in front of the Liouville action (5.11).

5.5 Adiabatic phase and Chern-Simons action

We have already noticed the resemblance of the generating functional for the integer QH state in the form Eq. (3.21) to the 2+1d Chern-Simons action. Effective long-distance description of the quantum Hall effect in terms of Chern-Simons theory goes back to [106, 40, 41]. The gravitational Chern-Simons term in 2+1d, corresponding to the 2d gravitational anomaly term in Eq. (3.21), was derived only recently [1, 44, 46]. Following Ref. [66] we recall how Chern-Simons functional arises from the adiabatic curvature (5.16), and more precisely from the Quillen anomaly part of the adiabatic curvature $\mathcal{R}^{\mathcal{L}}$ (5.18).

We shall now consider the family of surfaces Σ_y parameterized by $y \in Y$ and the space M which is the union of all Σ_y over Y (sometimes M is called "the universal curve"). In particular, the dimension of M equals $\dim M = \dim Y + 2$ where 2 is the dimension of the Riemann surface. We consider also the family of line bundles $L_y^k \otimes K_y^s \to \Sigma_y$, $y \in Y$. The union of all such line bundles extends to the holomorphic line bundle E over M, and Hermitian metric h^k extends to the Hermitian metric h^E on E. We denote the curvature of the metric h^E as F^E. We also consider the extension of the union of tangent bundles $T\Sigma_y$ to the bundle $TM|Y$, which is still a line bundle (as opposed to the usual tangent bundle TM). Let $g^{TM|Y}$ be a smooth Hermitian metric on $TM|Y$ and $R_{TM|Y}$ be its curvature 2-form.

The following formula for the curvature of the Quillen determinant line bundle $\mathcal{R}^{\mathcal{L}}$ is due to Bismut-Gillet-Soulé [17, Thm. 1.27], see also [16],

$$\mathcal{R}^{\mathcal{L}} = -2\pi i \int_{M|Y} \big[\mathrm{Ch}(E)\mathrm{Td}(TM|Y)\big]_{(4)}. \tag{5.25}$$

The notation $M|Y$ means that the integration goes over the fibers in the fibration $\sigma : M \to Y$, i.e., over the spaces Σ_y at y fixed. The expression in the brackets is a form of mixed degree on M, Ch is the Chern character an Td is the Todd class, see e.g. [43] for standard definitions. The subscript (4) means that only the 4-form component of the integrand is retained, and after the integration we end up with the 2-form on Y. In order to apply the results of Ref. [17] in our context we need to check that the fibration $\sigma : M \to Y$ is locally Kähler, which turns out to be the case, as explained in Ref. [66].

Next, we choose an adiabatic process, which is a smooth closed contour $\mathcal{C} \in Y$. When the wave function (in this context we are talking only about the integer QH state) is transported around the contour, we can compute the geometric part of the adiabatic phase (corresponding to $\mathcal{R}^{\mathcal{L}}$) as $\int_{\mathcal{C}} \mathcal{A}^{\mathcal{L}}$, where the connection 1-form $\mathcal{A}^{\mathcal{L}}$ on $\mathcal{L}$ can be computed using the formula (5.25) locally as $\mathcal{R}^{\mathcal{L}} = d\mathcal{A}$.

First, we need to write down the integrand in Eq. (5.25) explicitly in our case. Since we are interested only in the 4-form part in the integrand, we expand the Chern character form $\mathrm{Ch}(E)$ and the Todd form $\mathrm{Td}(TM|Y)$ up to the 4-form order and restrict to the line bundle case (i.e., setting $c_2(E) = c_2(TM|Y) = 0$),

$$\begin{aligned}\mathrm{Ch}(E) &= 1 + c_1(E) + \frac{1}{2}c_1^2(E) + ...,\\ \mathrm{Td}(TM|Y) &= 1 + \frac{1}{2}c_1(TM|Y) + \frac{1}{12}c_1^2(TM|Y) + ...,\end{aligned}$$

where the forms representing first Chern classes read

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}\,\mathrm{F}^E, \quad c_1(TM) = \frac{i}{2\pi}\mathrm{Tr}\,\mathrm{R}^{TM}$$

Also we split the curvature 2-form of the bundle E as: $\mathrm{F}^E = \mathrm{F} - s\mathrm{R}_{TM|Y}$ where F now refers to the part of the curvature 2-form corresponding to the line bundle $\tilde{L}^k \to M$, which is the union of all bundles $L_y^k \to \Sigma_y$. Using the composition property $\mathrm{Ch}(E \otimes E') = \mathrm{Ch}(E) \cdot \mathrm{Ch}(E')$ for the product of two bundles E, E', we obtain

$$\begin{aligned}\mathrm{Ch}(L^k \otimes K^s) =& 1 + c_1(L^k) - sc_1(TM|Y) - sc_1(L^k)c_1(TM|Y)\\ &+ \frac{1}{2}\big(c_1^2(L^k) + s^2c_1^2(TM|Y)\big) + ...\end{aligned}$$

Then the curvature formula (5.25) specified to our case reads

$$\begin{aligned}\mathcal{R}^{\mathcal{L}} = \frac{i}{4\pi}\int_{M|Y} \Big[&\mathrm{F}\wedge\mathrm{F} + (1-2s)\,\mathrm{F}\wedge\mathrm{R}_{TM|Y}\\ &+ \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\mathrm{R}_{TM|Y}\wedge\mathrm{R}_{TM|Y}\Big].\end{aligned} \tag{5.26}$$

Now we introduce notations for the one forms $F = dA_{(M)}$ and $R_{TM|Y} = d\omega_{(M)}$ on M. Locally we can write the integrand as the derivative of the Chern-Simons term

$$\mathcal{R}^{\mathcal{L}} = \frac{i}{4\pi}\int_{M|Y} d_M CS(A_{(M)}, \omega_{(M)}),$$

and use the formula for the commutation of the exterior derivative with the integral along the fiber [43, Eq. (1.17)], $d_Y \int_{M|Y} \alpha = d_M\alpha$, in order to show that

$$\int_{\mathcal{C}} \mathcal{A}^{\mathcal{L}} = \frac{1}{4\pi}\int_{\sigma^{-1}(\mathcal{C})} CS(A_{(M)}, \omega_{(M)}) \tag{5.27}$$

$$= \frac{1}{4\pi}\int_{\sigma^{-1}(\mathcal{C})} A\wedge dA + \frac{1-2s}{2}(A\wedge d\omega + dA\wedge\omega) + \left(\frac{(1-2s)^2}{4} - \frac{1}{12}\right)\omega\wedge d\omega\,.$$

The notation $\sigma^{-1}(\mathcal{C})$ means that the integration goes over the $2+1d$ space with one dimension along the contour and two dimensions along the fiber Σ_y at the point $y \in Y$. Here A and ω are 2 + 1d connection 1-forms with components along the fiber retained and third component A_0 is the projection on the contour $A_0 dt = A_y dy + A_{\bar y} d\bar y$ where t is a parameter along the contour.

Thus Eq. (5.27) is a formal derivation of the Chern-Simons action, which here has the meaning of the adiabatic phase acquired upon the transport of integer QH state along a contour $\mathcal{C} \in$ in the parameter space. We note that the full formula for adiabatic curvature Eq. (5.17) has also the exact form contribution due to determinant, which will also lead to a contribution to the phase, which is not reflected in Eq. (5.27) (by analogy with the discussion around Eq. (5.19), we can argue that this part is exponentially small for large magnetic fields).

6 Appendix

Out notations for theta functions follow Mumford [80]. Theta function with characteristics

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{n\in\mathbb{Z}} \exp\left(\pi i(n+a)^2\tau + 2\pi i(n+a)(z+b)\right), \tag{6.1}$$

where $a, b \in \mathbb{R}$. Their transformation property under the lattice shifts

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z+t_1+t_2\tau,\tau) = e^{-i\pi t_2^2\tau - 2\pi i t_2 z + 2\pi i(at_1 - bt_2)}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau), \quad t_1, t_2 \in \mathbb{Z}. \tag{6.2}$$

We use the standard notation

$$\vartheta_1(z,\tau) = \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(z,\tau).$$

Another useful lattice shift formula

$$\prod_{j<l}^{N}\big(\vartheta_1(z_j-z_l,\tau)\big)^{\beta}\big|_{z_m\to z_m+t_1+t_2\tau}$$

$$= e^{-i\pi\tau t_2^2\beta(N-1)+\pi i(t_1+t_2)\beta(N+1)-2\pi i t_2\beta N z_m+2\pi i t_2\beta z_{\rm cm}}\prod_{j<l}^{N}\big(\vartheta_1(z_j-z_l,\tau)\big)^{\beta} \tag{6.3}$$

Poisson summation formula

$$\frac{1}{\sqrt{A}}\sum_{m'\in\mathbb{Z}} e^{-\frac{\pi}{A}\left(m'+\frac{B}{2\pi i}\right)^2}=\sum_{n\in\mathbb{Z}} e^{-\pi A n^2+Bn}. \tag{6.4}$$

Modular transformation formulas:

$$\vartheta\left[\begin{array}{c} a\\ b\end{array}\right](mz,m(\tau+1))=e^{-\pi i m a(a+1)}\vartheta\left[\begin{array}{c} a\\ b+m(a+\frac12)\end{array}\right](mz,m\tau), \tag{6.5}$$

$$\vartheta\left[\begin{array}{c} a\\ b\end{array}\right]\left(m\frac{z}{\tau},-m\frac{1}{\tau}\right)=\sqrt{\frac{-i\tau}{m}}\,e^{\pi i m\frac{z^2}{\tau}+2\pi i ab}\sum_{c=1}^{m}\vartheta\left[\begin{array}{c} \frac{b+c-1}{m}\\ -ma\end{array}\right](mz,m\tau), \tag{6.6}$$

$$\eta(\tau+1)=e^{\frac{\pi i}{12}}\eta(\tau), \tag{6.7}$$

$$\eta(-1/\tau)=\sqrt{-i\tau}\,\eta(\tau), \tag{6.8}$$

$$\prod_{j<l}^{N}\big(\vartheta_1(z_j-z_l,\tau+1)\big)^{\beta}=e^{\frac{\pi i}{8}\beta N(N-1)}\prod_{j<l}^{N}\big(\vartheta_1(z_j-z_l,\tau)\big)^{\beta}, \tag{6.9}$$

$$\prod_{j<l}^{N}\left(\vartheta_1\left(\frac{z_j-z_l}{\tau},-\frac{1}{\tau}\right)\right)^{\beta} \tag{6.10}$$
$$=\left(\sqrt{-i\tau}\right)^{\beta\frac{N(N-1)}{2}}e^{-\frac{\pi i}{4}\beta N(N-1)+\frac{\pi i N_\phi}{\tau}\sum_l z_l^2-\frac{\pi i\beta}{\tau}z_{\rm cm}^2}\prod_{j<l}^{N}\left(\vartheta_1(z_j-z_l,\tau)\right)^{\beta}.$$

Modular group action on Laughlin states (4.35-4.37),

$$T\circ F_r^{\varepsilon,\delta}(\{z_l\}|\varphi,\tau)=U^T_{rr'}F_{r'}^{\varepsilon,\delta+\varepsilon-\lambda}(\{z_l\}|\varphi,\tau),$$
where $U^T_{rr'}=\delta_{rr'}e^{\frac{\pi i}{12}(NN_\phi-1)+\frac{\pi i}{\beta}(r+\varepsilon-\beta\lambda+\frac{\beta}{2})(r-\varepsilon+(2-\beta)\lambda+\frac{\beta}{2})}$,
$$T\circ\prod_{l=1}^{N}h_0^{N_\phi}(z_l,\bar z_l)=\prod_{l=1}^{N}h_0^{N_\phi}(z_l,\bar z_l),$$
$$T\circ Z_\beta(\varphi,\bar\varphi,\tau,\bar\tau)=Z_\beta(\varphi,\bar\varphi,\tau,\bar\tau),$$
$$S\circ F_r^{\varepsilon,\delta}(\{z_l\}|\varphi,\tau)=(\sqrt{-i\tau})^{N_\phi}\cdot e^{\frac{\pi i N_\phi}{\tau}\sum_l z_l^2+\frac{2\pi i}{\tau}z_{\rm zm}\varphi+\frac{\pi i}{\beta}\frac{\varphi^2}{\tau}}\sum_{r'=1}^{\beta}U^S_{rr'}F_{r'}^{\delta,\varepsilon}(\{z_l\}|\varphi,\tau),$$

$$\text{where} \quad U^S_{rr'} = \frac{1}{\sqrt{\beta}} e^{-\frac{\pi i}{4} N_\phi (N-1) - \frac{2\pi i}{\beta}\left(\varepsilon + \beta(\frac{1}{2}-\lambda)\right)\left(\delta+\beta(\frac{1}{2}-\lambda)\right) - \frac{2\pi i}{\beta} r'(r+2\varepsilon)},$$

$$S \circ \prod_{l=1}^{N} h_0^{N_\phi}(z_l, \bar{z}_l) = e^{-\frac{\pi i N_\phi}{\tau}\sum_l z_l^2 + \frac{\pi i N_\phi}{\bar{\tau}}\sum_l \bar{z}_l^2 - \frac{2\pi i}{\tau} z_{\rm cm}\varphi + \frac{2\pi i}{\bar{\tau}}\bar{z}_{\rm cm}\bar{\varphi}} \prod_{l=1}^{N} h_0^{N_\phi}(z_l, \bar{z}_l),$$

$$S \circ Z_\beta(\varphi, \bar{\varphi}, \tau, \bar{\tau}) = (\sqrt{\tau\bar{\tau}})^{N_\phi} \cdot e^{\frac{\pi i}{\beta}\frac{\varphi^2}{\tau} - \frac{\pi i}{\beta}\frac{\bar{\varphi}^2}{\bar{\tau}}} \cdot Z_\beta(\varphi, \bar{\varphi}, \tau, \bar{\tau}).$$

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Companion cluster algebras to a generalized cluster algebra

by Tomoki Nakanishi and Dylan Rupel

Abstract

We study the c-vectors, g-vectors, and F-polynomials for generalized cluster algebras satisfying a normalization condition and a power condition recovering classical recursions and separation of additions formulas. We establish a relationship between the c-vectors, g-vectors, and F-polynomials of such a generalized cluster algebra and its (left- and right-) companion cluster algebras. Our main result states that the cluster variables and coefficients of the (left- and right-) companion cluster algebras can be recovered via a specialization of the F-polynomials.

1 Introduction

Cluster algebras have risen to prominence as the correct algebraic/combinatorial language for describing a certain class of recursive calculations. These recursions appear in many forms across various disciplines including Poisson geometry [GSV], combinatorics [MP], hyperbolic geometry [FG, FST, MSW], representation theory of associative algebras [CC, CK, BMRRT, R1, Q, R2], mathematical physics [EF], and quantum groups [K, GLS, KQ, BR]. In the current standard theory a product of cluster variables, one known and one unknown, is equal to a binomial in other known quantities. Recently examples have emerged in the context of hyperbolic orbifolds [CS], exact WKB analysis [IN], and quantum groups [G, BGR] that require a more general setup: these *binomial* exchange relations should be replaced by *polynomial* exchange relations.

The general study of such *generalized* cluster algebras was initiated by Chekhov and Shapiro [CS] where an analogue of the classical Laurent Phenomenon was established. Following these developments, the first author [N] studied the analogues of c-vectors, g-vectors, and F-polynomials for a class of generalized cluster algebras satisfying a *normalization condition* and a *reciprocity condition*. In that work, relationships between these c- and g-vectors with the corresponding quantities for certain *companion* cluster algebras were established. Our goal in the present paper is to extend these results to the case when the reciprocity condition is replaced by a weaker *power condition* and to clarify the corresponding relationships between F-polynomials, x-variables, and y-variables. The main message of this note, continuing from [N], is as follows: the generalized cluster algebras are as good and natural as ordinary cluster algebras. Also in this direction, analogues of the classical greedy bases from [LLZ] have been constructed for rank 2 generalized cluster algebras by the second author [R3].

In order to state our main theorem we will need to fix some notation. A cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y},B) \subset \mathcal{F}$ is defined recursively from the initial data of a seed $(\mathbf{x},\mathbf{y},B)$ where $\mathbf{y} = (y_1,\dots,y_n)$ is a collection of elements from a semifield $\mathbb{P}$, $\mathbf{x} = (x_1,\dots,x_n)$ is a collection of algebraically independent elements in a degree n purely transcendental extension $\mathcal{F}$ of $\mathbb{QP}$ (in particular, we may identify $\mathcal{F}$ with the rational function field $\mathbb{QP}(\mathbf{x})$) where $\mathbb{QP}$ is the field of fractions of the group ring $\mathbb{ZP}$, and $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ matrix. A generalized cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x},\mathbf{y},B,\mathbf{Z}) \subset \mathcal{F}$ requires the additional data of a collection of *exchange polynomials* $\mathbf{Z} = (Z_1,\dots,Z_n)$ where

$$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i}$$

with each $z_{i,s} \in \mathbb{P}$ and $z_{i,0} = z_{i,d_i} = 1$. Write $\mathbf{z} = (z_{i,s})$ $(1 \le i \le n,\ 0 \le s \le d_i)$.

Write $D = (d_i\delta_{ij})$ for the diagonal $n \times n$ matrix. Denote by $\mathbf{x}^{1/\mathbf{d}}$ the collection $(x_1^{1/d_1},\dots,x_n^{1/d_n})$ in the extension field $\mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$ of $\mathcal{F}$. Define the *left-companion cluster algebra* ${}^L\!\mathcal{A}$ of $\mathcal{A}$ to be the cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}},\mathbf{y},DB) \subset \mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$. Write $({}^L\mathbf{x}^t, {}^L\mathbf{y}^t, {}^LB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^L\!\mathcal{A}$ and denote by ${}^L\mathbf{c}_j^t$, ${}^L\mathbf{g}_j^t$, and ${}^LF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^L\!\mathcal{A}$.

Let $\mathbf{z}^{\mathrm{bin}} = (z_{i,s}^{\mathrm{bin}})$ where $z_{i,s}^{\mathrm{bin}} = \binom{d_i}{s}$. Then we write $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{z}^{\mathrm{bin}})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$. Our first main result is the following.

Theorem 1.1. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} = \left({}^Lx_i^t\right)^{d_i}$ *and* $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\mathrm{bin}}} = {}^Ly_i^t$.

Denote by $\mathbf{y}^{\mathbf{d}}$ for the collection $(y_1^{d_1},\dots,y_n^{d_n})$ in $\mathbb{P}$. Define the *right-companion cluster algebra* ${}^R\!\mathcal{A}$ of $\mathcal{A}$ to be the cluster algebra $\mathcal{A}(\mathbf{x},\mathbf{y}^{\mathbf{d}},BD) \subset \mathbb{QP}(\mathbf{x})$. Write $({}^R\mathbf{x}^t, {}^R\mathbf{y}^t, {}^RB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^R\!\mathcal{A}$ (see Section 2 for details).

Write $x_i^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{0})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$. Our second main result is the following.

Theorem 1.2. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{0}} = {}^Rx_i^t$ *and* $\left(y_j^t\big|_{\mathbf{z}=\mathbf{0}}\right)^{d_j} = {}^Ry_j^t$.

2 Cluster Algebras

A *semifield* is a multiplicative abelian group $(\mathbb{P}, \cdot)$ together with an auxiliary addition $\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P}$ which is associative, commutative and satisfies the usual distributivity with the multiplication of $\mathbb{P}$. Write $\mathbb{ZP}$ for the group ring of $\mathbb{P}$. Since $\mathbb{P}$ is necessarily torsion-free (see e.g. [FZ1, Sec. 5]), $\mathbb{ZP}$ is a domain [FZ1, Sec. 2] and we write $\mathbb{QP}$ for its field of fractions. There are two main examples of semifields that will be most relevant for our purposes.

Example 2.1.

1. The *universal semifield* $\mathbb{Q}_{\mathrm{sf}}(y_1, \ldots, y_n)$ is the set of rational functions in the variables $y_1, \ldots, y_n$ which can be written in a subtraction-free form. Addition and multiplication in the universal semifield are the ordinary operations on rational functions. The semifield $\mathbb{Q}_{\mathrm{sf}}(y_1, \ldots, y_n)$ is universal in the following sense. Each element of $\mathbb{Q}_{\mathrm{sf}}(y_1, \ldots, y_n)$ can be written as a ratio of positive polynomials in $\mathbb{Z}_{\geq 0}[y_1, \ldots, y_n]$ so that for any other semifield $\mathbb{P}$ there is a specialization homomorphism $\mathbb{Q}_{\mathrm{sf}}(y_1, \ldots, y_n) \to \mathbb{P}$, given by $y_i \mapsto p_i$ and $1 \mapsto 1$, which respects the semifield structure for any choice of $p_1, \ldots, p_n \in \mathbb{P}$.

2. The *tropical semifield* $\mathrm{Trop}(y_1, \ldots, y_n)$ is the free (multiplicative) abelian group generated by $y_1, \ldots, y_n$ with auxiliary addition $\oplus$ defined by

$$\prod_{j=1}^{n} y_j^{a_j} \oplus \prod_{j=1}^{n} y_j^{b_j} = \prod_{j=1}^{n} y_j^{\min(a_j, b_j)}.$$

   The group ring of $\mathbb{P} = \mathrm{Trop}(y_1, \ldots, y_n)$ is the Laurent polynomial ring $\mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$ while $\mathbb{QP} = \mathbb{Q}(y_1, \ldots, y_n)$.

Fix a semifield $\mathbb{P}$ and write $\mathcal{F} = \mathbb{QP}(w_1, \ldots, w_n)$ for the field of rational functions in algebraically independent variables $w_1, \ldots, w_n$. A *(labeled) seed* $(\mathbf{x}, \mathbf{y}, B)$ *over* $\mathbb{P}$ consists of the following data:

- an algebraically independent collection $\mathbf{x} = (x_1, \ldots, x_n)$, called a *cluster*, consisting of elements from $\mathcal{F}$ called *cluster variables* or *x-variables*;

- a collection $\mathbf{y} = (y_1, \ldots, y_n)$ of elements from $\mathbb{P}$ called *coefficients* or *y-variables*;

- an $n \times n$ skew-symmetrizable matrix $B = (b_{ij})$ called the *exchange matrix*.

The main ingredient in the definition of a cluster algebra is the notion of mutation for seeds. For notational convenience we abbreviate $[b]_+ = \max(b, 0)$.

Definition 2.2. For $1 \leq k \leq n$ we define the *seed mutation in direction k* by $\mu_k(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ where

- the cluster $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by $x'_i = x_i$ for $i \neq k$ and x'_k is determined using the *exchange relation*:

$$x'_k x_k = \left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \right) \frac{1 + \hat{y}_k}{1 \oplus y_k}, \quad \hat{y}_k = y_k \prod_{i=1}^{n} x_i^{b_{ik}}; \tag{2.1}$$

- the coefficient tuple $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is given by $y'_k = y_k^{-1}$ and for $j \neq k$ we set

$$y'_j = y_j y_k^{[b_{kj}]_+} \big(1 \oplus y_k\big)^{-b_{kj}}; \tag{2.2}$$

- the matrix $B' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{2.3}$$

Write $\mathbb{T}_n$ for the n-regular tree with edges labeled by the set $\{1, \ldots, n\}$ so that the n edges emanating from each vertex receive different labels. We write $t \overset{k}{\text{---}} t'$ to denote two vertices t and t' of $\mathbb{T}_n$ connected by an edge labeled by k. A *cluster pattern* Σ *over* $\mathbb{P}$ is an assignment of a seed Σ^t to each vertex $t \in \mathbb{T}_n$ such that whenever $t \overset{k}{\text{---}} t'$ we have $\mu_k \Sigma^t = \Sigma^{t'}$, that is Σ^t and $\Sigma^{t'}$ are related by the seed mutation in direction k whenever t and t' are adjoined by an edge labeled by k. Fix a choice of initial vertex t_0, we will write $\Sigma^{t_0} = (\mathbf{x}, \mathbf{y}, B)$ while for an arbitrary vertex $t \in \mathbb{T}_n$ we write $\Sigma^t = (\mathbf{x}^t, \mathbf{y}^t, B^t)$ where

$$\mathbf{x}^t = (x_1^t, \ldots, x_n^t), \qquad \mathbf{y}^t = (y_1^t, \ldots, y_n^t), \qquad B^t = (b_{ij}^t).$$

Note that every seed Σ^t for $t \in \mathbb{T}_n$ is uniquely determined once we have specified Σ^{t_0}. Moreover, it is important to note that the exchange matrices B^t are independent of the initial choice of $\mathbf{x}$ and $\mathbf{y}$.

Definition 2.3. The *cluster algebra* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds appearing in the cluster pattern Σ, more precisely

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, B) = \mathbb{ZP}[x_i^t : t \in \mathbb{T}_n, 1 \leq i \leq n] \subset \mathcal{F}.$$

A priori the most one can say about these constructions is that the cluster variables x_i^t admit a description as subtraction-free rational expressions in the cluster variables of $\mathbf{x}$ with coefficients in $\mathbb{ZP}$ and that the coefficients y_j^t admit a description as subtraction-free rational expressions in $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$. More precisely, to see this claim for x_i^t we may, for each initial seed $(\mathbf{x}, \mathbf{y}, B)$,

- replace the x- and y-variables by formal indeterminants (which by abuse of notation we denote by the same symbols);

- replace the semifield $\mathbb{P}$ by the tropical semifield $\mathrm{Trop}(\mathbf{y})$;
- replace $\mathcal{F}$ by $\mathbb{Q}(\mathbf{x},\mathbf{y})$ and opt to perform all calculations here.

Since no subtraction occurs in the recursions (2.1), we obtain in this way *X-functions* $X_i^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{x},\mathbf{y})$. Alternatively performing the y-mutations (2.2) inside $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ we obtain *Y-functions* $Y_j^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$. By the universality of the semifield $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ we may recover the original coefficient y_j^t by the specialization $Y_j^t\big|_{\mathbb{P}}$. Taking this specialization where $\mathbb{P} = \mathrm{Trop}(\mathbf{y})$ we obtain monomials $Y_j^t\big|_{\mathrm{Trop}(\mathbf{y})} = \prod_{i=1}^n y_i^{c_{ij}^t}$ where we write C^t for the resulting matrix whose columns $\mathbf{c}_j^t \in \mathbb{Z}^n$ are called *c-vectors*. Note that the c-vectors only depend on the initial exchange matrix B and not on the choice of initial cluster $\mathbf{x}$.

Proposition 2.4. *[FZ4, Eq. 5.9] The c-vectors satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t & \text{if } j = k; \\ c_{ij}^t + c_{ik}^t[b_{kj}^t]_+ + [-c_{ik}^t]_+ b_{kj}^t & \text{if } j \neq k. \end{cases} \tag{2.4}$$

Obtaining the cluster variable x_i^t from X_i^t is more interesting and will be discussed further below. As a first step toward this goal, we note that the cluster algebra $\mathcal{A}$ admits the following remarkable "Laurent Phenomenon".

Theorem 2.5. *[FZ1, Th. 3.1] Fix an initial seed $(\mathbf{x},\mathbf{y},B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each cluster variable x_i^t can be expressed as a Laurent polynomial in $\mathbf{x}$ with coefficients in $\mathbb{Z}\mathbb{P}$.*

For a seed $(\mathbf{x},\mathbf{y},B)$ over $\mathbb{P} = \mathrm{Trop}(\mathbf{y})$ we may apply Theorem 2.5 to write each X-function as an element of $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}^{\pm 1}]$. Moreover, y-variables never appear in the denominators of the X-functions.

Proposition 2.6. *[FZ4, Prop. 3.6] Each X-function X_i^t is contained in $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}]$.*

In fact, the X-functions are homogeneous with respect to a certain $\mathbb{Z}^n$-grading on $\mathbb{Z}[\mathbf{x}^{\pm 1},\mathbf{y}]$. Write $\mathbf{b}_j \in \mathbb{Z}^n$ for the j^{th} column of B.

Proposition 2.7. *[FZ4, Prop. 6.1, Prop. 6.6] Under the $\mathbb{Z}^n$-grading*

$$\deg(x_i) = \mathbf{e}_i \qquad \text{and} \qquad \deg(y_j) = -\mathbf{b}_j,$$

each X-function is homogeneous and we write $\deg\left(X_j^t\right) = \mathbf{g}_j^t = \sum_{i=1}^n g_{ij}^t \mathbf{e}_i$. *Moreover, these g-*vectors *satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t & \text{if } j \neq k; \\ -g_{ik}^t + \sum_{\ell=1}^n g_{i\ell}^t[-b_{\ell k}^t]_+ - \sum_{\ell=1}^n b_{i\ell}^t[-c_{\ell k}^t]_+ & \text{if } j = k. \end{cases} \tag{2.5}$$

Following Proposition 2.6 we may define *F-polynomials* $F_i^t(\mathbf{y}) \in \mathbb{Z}[\mathbf{y}]$ via the specialization $F_i^t(\mathbf{y}) = X_i^t(\mathbf{1}, \mathbf{y})$, i.e. by setting all initial cluster variables x_j to 1. The F-polynomials satisfy a recurrence relation analogous to (2.1).

Proposition 2.8. *[FZ4, Prop. 5.1] The F-polynomials satisfy the following recurrence relation for $t \overset{k}{—} t'$:*

$$F_j^{t'} = \begin{cases} F_j^t & \text{if } j \neq k; \\ \left(F_k^t\right)^{-1} \left(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+} \left(F_i^t\right)^{[-b_{ik}^t]_+} \right) \left(1 + \prod_{i=1}^n y_i^{c_{ik}^t} \left(F_i^t\right)^{b_{ik}^t} \right) & \text{if } j = k. \end{cases} \tag{2.6}$$

Notice that each F-polynomial admits an expression as a subtraction-free rational expression and thus may be considered as an element of $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$, in particular the specialization $F_i^t\big|_{\mathbb{P}}$ makes sense for any semifield $\mathbb{P}$. With this we may obtain a description of the y-variables in terms of the c-vectors and the specializations of the F-polynomials.

Theorem 2.9. *[FZ4, Prop. 3.13] Fix an initial seed $(\mathbf{x}, \mathbf{y}, B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each coefficient y_j^t of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ can be computed as*

$$y_j^t = \left(\prod_{i=1}^n y_i^{c_{ij}^t} \right) \prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y})^{b_{ij}^t}.$$

Finally, we obtain a "separation of additions" formula for the cluster variables x_i^t in terms of the g-vectors and the F-polynomials.

Theorem 2.10. *[FZ4, Cor. 6.3] Fix an initial seed $(\mathbf{x}, \mathbf{y}, B)$ over a semifield $\mathbb{P}$. For any vertex $t \in \mathbb{T}_n$ each cluster variable x_j^t of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B)$ can be computed as*

$$x_j^t = \left(\prod_{i=1}^n x_i^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y})}.$$

3 Generalized Cluster Algebras

Let $(\mathbf{x}, \mathbf{y}, B)$ be a seed over the semifield $\mathbb{P}$. Fix a collection $\mathbf{Z} = (Z_1, \ldots, Z_n)$ of positive degree *exchange polynomials*

$$Z_i(u) = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i} \in \mathbb{ZP}[u]$$

such that $z_{i,s} \in \mathbb{P}$ for $0 \leq s \leq d_i$ and $z_{i,0} = z_{i,d_i} = 1$. It will often be convenient to write $\mathbf{z} = (z_{i,s})$ with $1 \leq i \leq n$ and $0 \leq s \leq d_i$ for the coefficients of the polynomials Z_i. Write $\overline{Z_i}(u) = u^{d_i} Z_i(u^{-1})$ for the exchange polynomial with coefficients reversed. Together we call $\Sigma = (\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ a *generalized seed over* $\mathbb{P}$. The additional data of the polynomials $\mathbf{Z}$ allows to generalize the notion of seed mutation in such a way that all nice properties and constructions related to cluster algebras in section 2 carry over to the new setting.

Definition 3.1. For $1 \le k \le n$ we define the *generalized seed mutation in direction* k by $\mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) = (\mathbf{x}', \mathbf{y}', B', \mathbf{Z}')$ where

- the cluster $\mathbf{x}' = (x'_1, \dots, x'_n)$ is given by $x'_i = x_i$ for $i \ne k$ and x'_k is determined using the *exchange relation*:

$$x'_k x_k = \left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)}, \quad \hat{y}_k = y_k \prod_{i=1}^{n} x_i^{b_{ik}}; \tag{3.1}$$

- the coefficient tuple $\mathbf{y}' = (y'_1, \dots, y'_n)$ is given by $y'_k = y_k^{-1}$ and for $j \ne k$ we set

$$y'_j = y_j \big(y_k^{d_k}\big)^{[b_{kj}]_+} Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}; \tag{3.2}$$

- the matrix $B' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + [b_{ik}]_+ d_k b_{kj} + b_{ik} d_k [-b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{3.3}$$

- the exchange polynomials $\mathbf{Z}' = (Z'_1, \dots, Z'_n)$ are given by $Z'_i = Z_i$ for $i \ne k$ and $Z'_k = \overline{Z}_k$, writing this relation purely in terms of coefficients gives $z'_{i,s} = z_{i,s}$ for $i \ne k$ and $z'_{k,s} = z_{k,d_k - s}$.

One may easily check that the $\hat{y}$-variables mutate in the same way as the y-variables, namely $\hat{y}'_k = \hat{y}_k^{-1}$ and for $j \ne k$ we have

$$\hat{y}'_j = \hat{y}_j \big(\hat{y}_k^{d_k}\big)^{[b_{kj}]_+} Z_k(\hat{y}_k)^{-b_{kj}}. \tag{3.4}$$

As a first indication that this definition is correct we verify that $\mu_k^2 \Sigma = \Sigma$.

Proposition 3.2. *The generalized seed mutation* μ_k *is involutive.*

Proof. Consider the generalized seed mutations

$$(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}') = \mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) \quad \text{and} \quad (\mathbf{x}'', \mathbf{y}'', B'', \mathbf{Z}'') = \mu_k(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}').$$

To begin note that $x''_i = x'_i = x_i$ for $i \ne k$ and $\big(\hat{y}'_k\big)^{-1} = \hat{y}_k$. Then x''_k is given by

$$\begin{aligned} x''_k &= \frac{1}{x'_k} \left(\prod_{i=1}^{n} (x'_i)^{[-b'_{ik}]_+} \right)^{d_k} \frac{\overline{Z}_k\big(\hat{y}'_k\big)}{\overline{Z}_k\big|_{\mathbb{P}}(y'_k)} = \frac{1}{x'_k} \left(\prod_{i=1}^{n} x_i^{[b_{ik}]_+} \right)^{d_k} \frac{\hat{y}_k^{-d_k} Z_k\big(\hat{y}_k\big)}{y_k^{-d_k} Z_k\big|_{\mathbb{P}}(y_k)} \\ &= \frac{1}{x'_k} \left(\prod_{i=1}^{n} x_i^{[b_{ik}]_+ - b_{ik}} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)} = \frac{1}{x'_k} \left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+} \right)^{d_k} \frac{Z_k\big(\hat{y}_k\big)}{Z_k\big|_{\mathbb{P}}(y_k)} = x_k. \end{aligned}$$

Also $y''_k = (y'_k)^{-1} = y_k$, while for $j \neq k$ we have

$$\begin{aligned} y''_j &= y'_j\big((y'_k)^{d_k}\big)^{[b'_{kj}]_+}\overline{Z}_k\big|_{\mathbb{P}}(y'_k)^{-b'_{kj}} \\ &= y_j\big(y_k^{d_k}\big)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\big(y_k^{d_k}\big)^{-[-b_{kj}]_+}\Big(y_k^{-d_k}Z_k\big|_{\mathbb{P}}(y_k)\Big)^{b_{kj}} = y_j. \end{aligned}$$

To see that the matrix mutation is involutive notice that we may apply the classical matrix mutation (2.3) to obtain exchange matrices $(DB)'$ and $(BD)'$ where $D = (d_i\delta_{ij})$. Then it is immediate from (3.3) that we have $DB' = (DB)'$ and $B'D = (BD)'$, the involutivity of matrix mutation (3.3) follows. Finally the equality $\mathbf{Z}'' = \mathbf{Z}$ is immediate from the definitions. □

The generalized seeds and their mutations we have defined here are a specialization of the setup in [CS]. There a generalized seed over $\mathbb{P}$ is a triple $(\mathbf{x}, \mathbf{p}, B)$ where $\mathbf{x}$ is a cluster, B is an exchange matrix, and $\mathbf{p} = (p_{i,s})$, where $1 \le i \le n$ and $0 \le s \le d_i$, is a collection of elements of $\mathbb{P}$. The mutation $\mu_k(\mathbf{x}, \mathbf{p}, B) = (\mathbf{x}', \mathbf{p}', B')$ is given by replacing (3.1) with

$$x'_k x_k = \left(\prod_{i=1}^{n} x_i^{[-b_{ik}]_+}\right)^{d_k} \sum_{s=0}^{d_k} p_{k,s} w_k^s, \quad w_k = \prod_{i=1}^{n} x_i^{b_{ik}} \tag{3.5}$$

and by replacing (3.2) with

$$p'_{k,s} = p_{k,d_k-s} \quad \text{and} \quad \frac{p'_{j,s}}{p'_{j,0}} = \frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+} p_{k,0}^{sb_{kj}}.$$

Our generalized seed mutations can be related to the more general setting of [CS] by defining

$$p_{i,s} = \frac{z_{i,s}y_i^s}{Z_i\big|_{\mathbb{P}}(y_i)} \tag{3.6}$$

where we note the identities

$$\bigoplus_{s=0}^{d_i} p_{i,s} = 1 \quad \text{and} \quad \frac{p_{i,d_i}}{p_{i,0}} = y_i^{d_i}.$$

Proposition 3.3. *Generalized seeds of the form* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *are in bijection with generalized seeds of the form* $(\mathbf{x}, \mathbf{p}, B)$ *satisfying*

1. *(normalization condition)* $\bigoplus_{s=0}^{d_i} p_{i,s} = 1$*;*
2. *(power condition)* $\frac{p_{i,d_i}}{p_{i,0}} = y_i^{d_i}$ *for some* $y_i \in \mathbb{P}$*.*

Moreover, this bijection is compatible with mutations.

Remark 3.4. Such y_i as in (2) is unique since $\mathbb{P}$ is torsion-free, i.e. if $(y'_i)^{d_i} = y_i^{d_i}$ then $\left(\frac{y'_i}{y_i}\right)^{d_i} = 1$ and so $\frac{y'_i}{y_i} = 1$.

Proof. For a generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ define $p_{i,s}$ as in (3.6). Write $(\mathbf{x}', \mathbf{y}', B', \mathbf{Z}') = \mu_k(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ and again use (3.6) to define $p'_{i,s}$ in terms of this seed. Then we have

$$p'_{k,s} = \frac{z'_{k,s}(y'_k)^s}{\overline{Z}_k\big|_{\mathbb{P}}(y'_k)} = \frac{z_{k,d_k-s}y_k^{-s}}{y_k^{-d_k}Z_k\big|_{\mathbb{P}}(y_k)} = \frac{z_{k,d_k-s}y_k^{d_k-s}}{Z_k\big|_{\mathbb{P}}(y_k)} = p_{k,d_k-s}$$

while for $j \neq k$ we have

$$\frac{p'_{j,s}}{p'_{j,0}} = \frac{z'_{j,s}(y'_j)^s}{\overline{Z}_j\big|_{\mathbb{P}}(y'_j)}\frac{\overline{Z}_j\big|_{\mathbb{P}}(y'_j)}{z'_{j,0}} = z_{j,s}\left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^s = \frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+}p_{k,0}^{sb_{kj}}$$

as desired.

Conversely, let $(\mathbf{x}, \mathbf{p}, B)$ be a generalized seed satisfying (1) and (2) where we define y_i using (2). Set $z_{i,s} = y_i^{-s}\frac{p_{i,s}}{p_{i,0}}$. Notice that the definitions immediately imply $z_{i,0} = z_{i,d_i} = 1$. Since $p_{i,s} = z_{i,s}y_i^s p_{i,0}$, by the normalization condition we have $p_{i,0}^{-1} = Z_i|_{\mathbb{P}}(y_i)$ where we write $Z_i = z_{i,0} + z_{i,1}u + \cdots + z_{i,d_i-1}u^{d_i-1} + z_{i,d_i}u^{d_i}$. Write $(\mathbf{x}', \mathbf{p}', B') = \mu_k(\mathbf{x}, \mathbf{p}, B)$ so that we may define y'_i and $z'_{i,s}$ as above using this generalized seed. Then we have

$$(y'_k)^{d_k} = \frac{p'_{k,d_k}}{p'_{k,0}} = \frac{p_{k,0}}{p_{k,d_k}} = y_k^{-d_k}$$

while for $j \neq k$ we have

$$(y'_j)^{d_j} = \frac{p'_{j,d_j}}{p'_{j,0}} = \frac{p_{j,d_j}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{d_j[b_{kj}]_+}p_{k,0}^{d_jb_{kj}} = \left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^{d_j},$$

so the coefficients mutate as desired. Similarly we have

$$z'_{k,s} = (y'_k)^{-s}\frac{p'_{k,s}}{p'_{k,0}} = y_k^s\frac{p_{k,d_k-s}}{p_{k,d_k}} = y_k^s\frac{p_{k,d_k-s}}{p_{k,0}}\frac{p_{k,0}}{p_{k,d_k}} = y_k^s y_k^{d_k-s}z_{k,d_k-s}y_k^{-d_k} = z_{k,d_k-s}$$

and for $j \neq k$ we have

$$z'_{j,s} = (y'_j)^{-s}\frac{p'_{j,s}}{p'_{j,0}} = \left(y_j\left(y_k^{d_k}\right)^{[b_{kj}]_+}Z_k\big|_{\mathbb{P}}(y_k)^{-b_{kj}}\right)^{-s}\frac{p_{j,s}}{p_{j,0}}\left(\frac{p_{k,d_k}}{p_{k,0}}\right)^{s[b_{kj}]_+}p_{k,0}^{sb_{kj}} = z_{j,s}$$

as desired. □

A *generalized cluster pattern* Σ *over* $\mathbb{P}$ is in assignment of a generalized seed Σ^t to each vertex $t \in \mathbb{T}_n$ such that whenever $t \overset{k}{—} t'$ we have $\mu_k \Sigma^t = \Sigma^{t'}$. As for cluster algebras, the entire generalized cluster pattern Σ is uniquely determined from any choice of initial seed $\Sigma^{t_0} = (\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$. We maintain the notation $\Sigma^t = (\mathbf{x}^t, \mathbf{y}^t, B^t, \mathbf{Z}^t)$ from above where we write $\mathbf{Z}^t = (Z_1^t, \dots, Z_n^t)$.

Definition 3.5. The *generalized cluster algebra* $\mathcal{A} = \mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ is the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by all cluster variables from seeds appearing in the generalized cluster pattern Σ, more precisely

$$\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z}) = \mathbb{ZP}[x_i^t : t \in \mathbb{T}_n, 1 \leq i \leq n] \subset \mathcal{F}.$$

The main feature of cluster algebras to which one might attribute their ubiquity is the Laurent Phenomenon, a first indication that generalized cluster algebras will find themselves as useful is the following consequence of Proposition 3.3 and [CS, Th. 2.5].

Corollary 3.6. *Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$. *For any vertex* $t \in \mathbb{T}_n$ *each cluster variable* x_i^t *can be expressed as a Laurent polynomial of* $\mathbf{x}$ *with coefficients in* $\mathbb{ZP}$.

Example 3.7. Consider the rank 2 generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ over $\mathbb{P}$ where $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2)$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{Z} = (Z_1, Z_2)$ where $Z_1(u) = 1 + z_1 u + z_2 u^2 + u^3$ and $Z_2(u) = 1 + u$. In this case we have $\hat{y}_1 = y_1 x_2$ and $\hat{y}_2 = y_2 x_1^{-1}$. Write $\Sigma(1) = (\mathbf{x}(1), \mathbf{y}(1), B(1), \mathbf{Z}(1))$ for the initial generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ and define seeds $\Sigma(t)$ for $t = 2, \dots, 9$ inductively via the alternating mutation sequence below:

(3.7)

$$\Sigma(1) \overset{\mu_1}{\longleftrightarrow} \Sigma(2) \overset{\mu_2}{\longleftrightarrow} \Sigma(3) \overset{\mu_1}{\longleftrightarrow} \Sigma(4) \overset{\mu_2}{\longleftrightarrow} \Sigma(5) \overset{\mu_1}{\longleftrightarrow} \Sigma(6) \overset{\mu_2}{\longleftrightarrow} \Sigma(7) \overset{\mu_1}{\longleftrightarrow} \Sigma(8) \overset{\mu_2}{\longleftrightarrow} \Sigma(9).$$

Then the exchange matrices and exchange polynomials of these generalized seeds are given by

$$B(t) = (-1)^{t+1} B, \qquad Z_2(t) = Z_2, \qquad \text{and} \qquad Z_1(t) = \begin{cases} Z_1 & \text{if } t \text{ is odd;} \\ \overline{Z}_1 & \text{if } t \text{ is even.} \end{cases}$$

The resulting cluster variables and coefficients are presented in Table 1.

Following the same formal procedure as in section 2, we may define *X-functions* $X_i^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and *Y-functions* $Y_j^t \in \mathbb{Q}_{\mathrm{sf}}(\mathbf{y}, \mathbf{z})$ by computing x_i^t and y_i^t, respectively, in the field $\mathbb{Q}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ represent collections of formal indeterminants. Using that $z_{i,0} = z_{i,d_i} = 1$, the specialization of the Y-functions in the tropical semifield $\mathbb{P} = \mathrm{Trop}(\mathbf{y}, \mathbf{z})$ again produces monomials $Y_j^t\big|_{\mathrm{Trop}(\mathbf{y},\mathbf{z})} = \prod_{i=1}^{n} y_i^{c_{ij}^t}$ where we write C^t for the resulting matrix whose columns $\mathbf{c}_j^t \in \mathbb{Z}^n$ we continue to call *c-vectors*.

$$\begin{cases} x_1(1) = x_1 \\ x_2(1) = x_2 \end{cases} \qquad \begin{cases} y_1(1) = y_1 \\ y_2(1) = y_2 \end{cases}$$

$$\begin{cases} x_1(2) = x_1^{-1}\frac{1+z_1\hat{y}_1+z_2\hat{y}_1^2+\hat{y}_1^3}{1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3} \\ x_2(2) = x_2 \end{cases} \qquad \begin{cases} y_1(2) = y_1^{-1} \\ y_2(2) = y_2(1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3) \end{cases}$$

$$\begin{cases} x_1(3) = x_1^{-1}\frac{1+z_1\hat{y}_1+z_2\hat{y}_1^2+\hat{y}_1^3}{1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3} \\ x_2(3) = x_2^{-1}\frac{1+\hat{y}_2+z_1\hat{y}_1\hat{y}_2+z_2\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2}{1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2} \end{cases} \qquad \begin{cases} y_1(3) = y_1^{-1}(1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2) \\ y_2(3) = y_2^{-1}(1\oplus z_1y_1\oplus z_2y_1^2\oplus y_1^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(4) = x_1x_2^{-3}\frac{\substack{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+2z_1\hat{y}_1\hat{y}_2+4z_1\hat{y}_1\hat{y}_2^2+2z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2+z_1^2\hat{y}_1^2\hat{y}_2^2+3z_2\hat{y}_1^2\hat{y}_2^2+z_1^2\hat{y}_1^2\hat{y}_2^3+2z_2\hat{y}_1^2\hat{y}_2^3 \\ +3\hat{y}_1^3\hat{y}_2^2+z_1z_2\hat{y}_1^3\hat{y}_2^2+2\hat{y}_1^3\hat{y}_2^3+2z_1z_2\hat{y}_1^3\hat{y}_2^3+z_1\hat{y}_1^4\hat{y}_2^2+2z_1\hat{y}_1^4\hat{y}_2^3+z_2^2\hat{y}_1^4\hat{y}_2^3+2z_2\hat{y}_1^5\hat{y}_2^3+\hat{y}_1^6\hat{y}_2^3}}{\substack{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3\oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3\oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3}} \\ x_2(4) = x_2^{-1}\frac{1+\hat{y}_2+z_1\hat{y}_1\hat{y}_2+z_2\hat{y}_1^2\hat{y}_2+\hat{y}_1^3\hat{y}_2}{1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2} \end{cases}$$

$$\begin{cases} y_1(4) = y_1(1\oplus y_2\oplus z_1y_1y_2\oplus z_2y_1^2y_2\oplus y_1^3y_2)^{-1} \\ y_2(4) = y_1^{-3}y_2^{-1}(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \qquad \oplus z_1z_2y_1^3y_2^2\oplus 2z_1z_2y_1^3y_2^3\oplus 3y_1^3y_2^2\oplus 2y_1^3y_2^3 \\ \qquad \oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3) \end{cases}$$

$$\begin{cases} x_1(5) = x_1x_2^{-3}\frac{\substack{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+2z_1\hat{y}_1\hat{y}_2+4z_1\hat{y}_1\hat{y}_2^2+2z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2+z_1^2\hat{y}_1^2\hat{y}_2^2+3z_2\hat{y}_1^2\hat{y}_2^2+z_1^2\hat{y}_1^2\hat{y}_2^3+2z_2\hat{y}_1^2\hat{y}_2^3 \\ +3\hat{y}_1^3\hat{y}_2^2+z_1z_2\hat{y}_1^3\hat{y}_2^2+2\hat{y}_1^3\hat{y}_2^3+2z_1z_2\hat{y}_1^3\hat{y}_2^3+z_1\hat{y}_1^4\hat{y}_2^2+2z_1\hat{y}_1^4\hat{y}_2^3+z_2^2\hat{y}_1^4\hat{y}_2^3+2z_2\hat{y}_1^5\hat{y}_2^3+\hat{y}_1^6\hat{y}_2^3}}{\substack{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3\oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3\oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3}} \\ x_2(5) = x_1x_2^{-2}\frac{1+2\hat{y}_2+\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2+z_1\hat{y}_1\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2} \end{cases}$$

$$\begin{cases} y_1(5) = y_1^{-2}y_2^{-1}(1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2) \\ y_2(5) = y_1^3y_2(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus 2z_1y_1y_2\oplus 4z_1y_1y_2^2\oplus 2z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2\oplus z_1^2y_1^2y_2^2\oplus 3z_2y_1^2y_2^2\oplus z_1^2y_1^2y_2^3\oplus 2z_2y_1^2y_2^3 \\ \qquad \oplus 3y_1^3y_2^2\oplus z_1z_2y_1^3y_2^2\oplus 2y_1^3y_2^3\oplus 2z_1z_2y_1^3y_2^3 \\ \qquad \oplus z_1y_1^4y_2^2\oplus 2z_1y_1^4y_2^3\oplus z_2^2y_1^4y_2^3\oplus 2z_2y_1^5y_2^3\oplus y_1^6y_2^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(6) = x_1^2x_2^{-3}\frac{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+z_1\hat{y}_1\hat{y}_2+2z_1\hat{y}_1\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^3+\hat{y}_1^3\hat{y}_2^3}{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3\oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3} \\ x_2(6) = x_1x_2^{-2}\frac{1+2\hat{y}_2+\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2+z_1\hat{y}_1\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^2+\hat{y}_1^3\hat{y}_2^2}{1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2} \end{cases}$$

$$\begin{cases} y_1(6) = y_1^2y_2(1\oplus 2y_2\oplus y_2^2\oplus z_1y_1y_2\oplus z_1y_1y_2^2\oplus z_2y_1^2y_2^2\oplus y_1^3y_2^2)^{-1} \\ y_2(6) = y_1^{-3}y_2^{-2}(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3) \end{cases}$$

$$\begin{cases} x_1(7) = x_1^2x_2^{-3}\frac{1+3\hat{y}_2+3\hat{y}_2^2+\hat{y}_2^3+z_1\hat{y}_1\hat{y}_2+2z_1\hat{y}_1\hat{y}_2^2+z_1\hat{y}_1\hat{y}_2^3+z_2\hat{y}_1^2\hat{y}_2^2+z_2\hat{y}_1^2\hat{y}_2^3+\hat{y}_1^3\hat{y}_2^3}{1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3\oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3} \\ x_2(7) = x_1x_2^{-1}\frac{1+\hat{y}_2}{1\oplus y_2} \end{cases}$$

$$\begin{cases} y_1(7) = y_1^{-1}y_2^{-1}(1\oplus y_2) \\ y_2(7) = y_1^3y_2^2(1\oplus 3y_2\oplus 3y_2^2\oplus y_2^3\oplus z_1y_1y_2\oplus 2z_1y_1y_2^2\oplus z_1y_1y_2^3 \\ \qquad \oplus z_2y_1^2y_2^2\oplus z_2y_1^2y_2^3\oplus y_1^3y_2^3)^{-1} \end{cases}$$

$$\begin{cases} x_1(8) = x_1 \\ x_2(8) = x_1x_2^{-1}\frac{1+\hat{y}_2}{1\oplus y_2} \end{cases} \qquad \begin{cases} y_1(8) = y_1y_2(1\oplus y_2)^{-1} \\ y_2(8) = y_2^{-1} \end{cases}$$

$$\begin{cases} x_1(9) = x_1 \\ x_2(9) = x_2 \end{cases} \qquad \begin{cases} y_1(9) = y_1 \\ y_2(9) = y_2 \end{cases}$$

Table 1: Cluster variables and coefficients for the mutation sequence (3.7).

Proposition 3.8. *(cf. [N, Prop. 3.8]) The c-vectors satisfy the following recurrence relation for* $t \overset{k}{—} t'$*:*

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t & \text{if } j = k; \\ c_{ij}^t + c_{ik}^t[d_k b_{kj}^t]_+ + [-c_{ik}^t]_+ d_k b_{kj}^t & \text{if } j \neq k. \end{cases} \tag{3.8}$$

Remark 3.9. It immediately follows that the c-vectors of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ do not depend on the choice of exchange polynomials $\mathbf{Z}$, only their degrees.

As in section 2 the X-functions become particularly nice.

Proposition 3.10. *(cf. [N, Prop. 3.3]) Each* X*-function* X_i^t *is contained in* $\mathbb{Z}[\mathbf{x}^{\pm 1}, \mathbf{y}, \mathbf{z}]$.

Using essentially the same $\mathbb{Z}^n$-grading these X-functions will once again be homogeneous.

Proposition 3.11. *(cf. [N, Prop. 3.15]) Under the* $\mathbb{Z}^n$*-grading*

$$\deg(x_i) = \mathbf{e}_i, \qquad \deg(y_j) = -\mathbf{b}_j, \qquad \text{and} \qquad \deg(z_{i,s}) = \mathbf{0},$$

each X*-function is homogeneous and we write* $\deg\left(X_j^t\right) = \mathbf{g}_j^t = \sum_{i=1}^n g_{ij}^t \mathbf{e}_i$*. Moreover, these* g*-vectors satisfy the following recurrence relation for* $t \overset{k}{—} t'$*:*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t & \text{if } j \neq k; \\ -g_{ik}^t + \sum_{\ell=1}^n g_{i\ell}^t[-b_{\ell k}^t d_k]_+ - \sum_{\ell=1}^n b_{i\ell}^t[-c_{\ell k}^t d_k]_+ & \text{if } j = k. \end{cases}$$

Remark 3.12. It immediately follows that the g-vectors of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ also do not depend on the choice of exchange polynomials $\mathbf{Z}$, only their degrees.

Continuing to follow the developments of section 2 we may define F-polynomials $F_i^t(\mathbf{y}, \mathbf{z}) \in \mathbb{Z}[\mathbf{y}, \mathbf{z}]$ by specializing all cluster variables x_i to 1 in the X-functions, i.e. $F_i^t(\mathbf{y}, \mathbf{z}) = X_i^t(\mathbf{1}, \mathbf{y}, \mathbf{z})$.

Proposition 3.13. *(cf. [N, Prop. 3.12]) The* F*-polynomials satisfy the following recurrence relation for* $t \overset{k}{—} t'$*:*

$$F_j^{t'} = \begin{cases} F_j^t & \text{if } j \neq k; \\ \left(F_k^t\right)^{-1} \left(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+} \left(F_i^t\right)^{[-b_{ik}^t]_+} \right)^{d_k} Z_k \left(\prod_{i=1}^n y_i^{c_{ik}^t} \left(F_i^t\right)^{b_{ik}^t} \right) & \text{if } j = k. \end{cases} \tag{3.9}$$

The coefficients y_j^t can still be computed using the c-vectors and F-polynomials.

Theorem 3.14. *(cf. [N, Th. 3.23]) Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$*. For any vertex* $t \in \mathbb{T}_n$ *each coefficient* y_j^t *of* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *can be computed as*

$$y_j^t = \left(\prod_{i=1}^{n} y_i^{c_{ij}^t} \right) \prod_{i=1}^{n} F_i^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z})^{b_{ij}^t}. \tag{3.10}$$

Finally the separation of additions formula still holds for cluster variables of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$.

Theorem 3.15. *(cf. [N, Th. 3.24]) Fix an initial generalized seed* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *over a semifield* $\mathbb{P}$*. For any vertex* $t \in \mathbb{T}_n$ *each cluster variable* x_j^t *of* $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *can be computed as*

$$x_j^t = \left(\prod_{i=1}^{n} x_i^{g_{ij}^t} \right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z})}, \tag{3.11}$$

where $\hat{y}_k = y_k \prod_{i=1}^{n} x_i^{b_{ik}}$.

Example 3.16. Following Theorems 3.14 and 3.15 we may immediately extract the C-matrix, G-matrix, and F-polynomials associated to each of the seeds $\Sigma(t)$ in Example 3.7. Writing $C(t)$, $G(t)$, and $F(t)$ for these quantities associated to the generalized seed $\Sigma(t)$ we obtain Table 2.

4 Companion Cluster Algebras

Fix an initial generalized seed $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ over a semifield $\mathbb{P}$. Write $D = (d_i \delta_{ij})$ where d_i is the degree of the exchange polynomial Z_i.

Denote by ${}^L\mathbf{x} := \mathbf{x}^{1/\mathbf{d}}$ the collection $({}^Lx_1, \cdots, {}^Lx_n) := (x_1^{1/d_1}, \dots, x_n^{1/d_n})$ in the extension field $\mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$ of $\mathbb{QP}(\mathbf{x})$. For clarity we also write ${}^L\mathbf{y} = \mathbf{y}$, i.e. ${}^Ly_j = y_j$. Define the *left-companion cluster algebra* ${}^L\!\mathcal{A}$ of $\mathcal{A}$ to be $\mathcal{A}({}^L\mathbf{x}, {}^L\mathbf{y}, DB) \subset \mathbb{QP}(\mathbf{x}^{1/\mathbf{d}})$. Write $({}^L\mathbf{x}^t, {}^L\mathbf{y}^t, {}^LB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^L\!\mathcal{A}$ and denote by ${}^L\mathbf{c}_j^t$, ${}^L\mathbf{g}_j^t$, and ${}^LF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^L\!\mathcal{A}$.

Write ${}^R\mathbf{x} = \mathbf{x}$, i.e. ${}^Rx_i = x_i$, and denote the collection $({}^Ry_1, \dots, {}^Ry_n) = (y_1^{d_1}, \dots, y_n^{d_n})$ by ${}^R\mathbf{y} := \mathbf{y}^{\mathbf{d}}$. Define the *right-companion cluster algebra* ${}^R\!\mathcal{A}$ of $\mathcal{A}$ to be $\mathcal{A}({}^R\mathbf{x}, {}^R\mathbf{y}, BD) \subset \mathbb{QP}(\mathbf{x})$. Write $({}^R\mathbf{x}^t, {}^R\mathbf{y}^t, {}^RB^t)$ for the seed associated to vertex $t \in \mathbb{T}_n$ in the construction of ${}^R\!\mathcal{A}$ and denote by ${}^R\mathbf{c}_j^t$, ${}^R\mathbf{g}_j^t$, and ${}^RF_j^t$ the c-vectors, g-vectors, and F-polynomials of ${}^R\!\mathcal{A}$.

We immediately obtain the following result as a consequence of Proposition 2.4 and Proposition 3.8 (cf. [N, Props. 3.9 and 3.10]).

$$C(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad G(1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \begin{cases}F_1(1)=1\\F_2(1)=1\end{cases}$$

$$C(2)=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad G(2)=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad \begin{cases}F_1(2)=1+z_1y_1+z_2y_1^2+y_1^3\\F_2(2)=1\end{cases}$$

$$C(3)=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad G(3)=\begin{pmatrix}-1&0\\0&-1\end{pmatrix},\quad \begin{cases}F_1(3)=1+z_1y_1+z_2y_1^2+y_1^3\\F_2(3)=1+y_2+z_1y_1y_2+z_2y_1^2y_2+y_1^3y_2\end{cases}$$

$$C(4)=\begin{pmatrix}1&-3\\0&-1\end{pmatrix},\quad G(4)=\begin{pmatrix}1&0\\-3&-1\end{pmatrix},\quad \begin{cases}F_1(4)=1+3y_2+3y_2^2+y_2^3+2z_1y_1y_2+4z_1y_1y_2^2+2z_1y_1y_2^3\\ \qquad +z_2y_1^2y_2+z_1^2y_1^2y_2^2+3z_2y_1^2y_2^2+z_1^2y_1^2y_2^3+2z_2y_1^2y_2^3\\ \qquad +z_1z_2y_1^3y_2^2+2z_1z_2y_1^3y_2^3+3y_1^3y_2^2+2y_1^3y_2^3\\ \qquad +z_1y_1^4y_2^2+2z_1y_1^4y_2^3+z_2^2y_1^4y_2^3+2z_2y_1^5y_2^3+y_1^6y_2^3\\F_2(4)=1+y_2+z_1y_1y_2+z_2y_1^2y_2+y_1^3y_2\end{cases}$$

$$C(5)=\begin{pmatrix}-2&3\\-1&1\end{pmatrix},\quad G(5)=\begin{pmatrix}1&1\\-3&-2\end{pmatrix},\quad \begin{cases}F_1(5)=1+3y_2+3y_2^2+y_2^3+2z_1y_1y_2+4z_1y_1y_2^2+2z_1y_1y_2^3\\ \qquad +z_2y_1^2y_2+z_1^2y_1^2y_2^2+3z_2y_1^2y_2^2+z_1^2y_1^2y_2^3+2z_2y_1^2y_2^3\\ \qquad +z_1z_2y_1^3y_2^2+2z_1z_2y_1^3y_2^3+3y_1^3y_2^2+2y_1^3y_2^3\\ \qquad +z_1y_1^4y_2^2+2z_1y_1^4y_2^3+z_2^2y_1^4y_2^3+2z_2y_1^5y_2^3+y_1^6y_2^3\\F_2(5)=1+2y_2+y_2^2+z_1y_1y_2+z_1y_1y_2^2+z_2y_1^2y_2^2+y_1^3y_2^2\end{cases}$$

$$C(6)=\begin{pmatrix}2&-3\\1&-2\end{pmatrix},\quad G(6)=\begin{pmatrix}2&1\\-3&-2\end{pmatrix},\quad \begin{cases}F_1(6)=1+3y_2+3y_2^2+y_2^3+z_1y_1y_2+2z_1y_1y_2^2+z_1y_1y_2^3\\ \qquad +z_2y_1^2y_2^2+z_2y_1^2y_2^3+y_1^3y_2^3\\F_2(6)=1+2y_2+y_2^2+z_1y_1y_2+z_1y_1y_2^2+z_2y_1^2y_2^2+y_1^3y_2^2\end{cases}$$

$$C(7)=\begin{pmatrix}-1&3\\-1&2\end{pmatrix},\quad G(7)=\begin{pmatrix}2&1\\-3&-1\end{pmatrix},\quad \begin{cases}F_1(7)=1+3y_2+3y_2^2+y_2^3+z_1y_1y_2+2z_1y_1y_2^2+z_1y_1y_2^3\\ \qquad +z_2y_1^2y_2^2+z_2y_1^2y_2^3+y_1^3y_2^3\\F_2(7)=1+y_2\end{cases}$$

$$C(8)=\begin{pmatrix}1&0\\1&-1\end{pmatrix},\quad G(8)=\begin{pmatrix}1&1\\0&-1\end{pmatrix},\quad \begin{cases}F_1(8)=1\\F_2(8)=1+y_2\end{cases}$$

$$C(9)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad G(9)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \begin{cases}F_1(9)=1\\F_2(9)=1\end{cases}$$

Table 2: C-matrices, G-matrices, and F-polynomials for the mutation sequence (3.7).

Corollary 4.1. *The c-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ coincide with the c-vectors of its left-companion cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}}, \mathbf{y}, DB)$ while the c-vectors of its right-companion cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}^{\mathbf{d}}, BD)$ can be obtained from those of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ by the transformation ${}^R\mathbf{c}_j^t = d_i^{-1} c_{ij}^t d_j$.*

Similarly the following result is an immediate consequence of Proposition 2.7 and Proposition 3.11 (cf. [N, Props. 3.16 and 3.17]).

Corollary 4.2. *The g-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ coincide with the g-vectors of its right-companion cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}^{\mathbf{d}}, BD)$ while the g-vectors of its left-companion cluster algebra $\mathcal{A}(\mathbf{x}^{1/\mathbf{d}}, \mathbf{y}, DB)$ can be obtained from those of $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ by the transformation ${}^L\mathbf{g}_j^t = d_i g_{ij}^t d_j^{-1}$.*

We see from Corollary 4.1 and Corollary 4.2 that the c- and g-vectors of the generalized cluster algebra $\mathcal{A}(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ are intimately related to those of its left- and right-companion cluster algebras. The same is true for F-polynomials, however the precise relationship for left- and right-companions are very different.

We begin with the left-companion. For $1 \le i \le n$ and $0 \le s \le d_i$ we will write $\mathbf{z}^{\mathrm{bin}} = (z_{i,s}^{\mathrm{bin}})$ where $z_{i,s}^{\mathrm{bin}} = \binom{d_i}{s} \in \mathbb{Z}$.

Proposition 4.3. *Let $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ be a generalized seed over $\mathbb{P}$. For any $t \in \mathbb{T}_n$ and any $1 \le j \le n$ we have the following equalities in $\mathbb{Q}_{\mathrm{sf}}(\mathbf{y})$ and $\mathbb{Q}_{\mathrm{sf}}(\mathbf{x}, \mathbf{y})$ respectively:*

$$F_j^t(\mathbf{y}, \mathbf{z}^{\mathrm{bin}}) = {}^L F_j^t\big({}^L\mathbf{y}\big)^{d_j} \quad \text{and} \quad F_j^t(\hat{\mathbf{y}}, \mathbf{z}^{\mathrm{bin}}) = {}^L F_j^t\big({}^L\hat{\mathbf{y}}\big)^{d_j}, \tag{4.1}$$

where ${}^L y_i = y_i$ and ${}^L\hat{y}_i = \hat{y}_i$.

Proof. We will proceed by induction on the distance from t_0 to t in $\mathbb{T}_n$. To begin, note that by definition we have $\big({}^L x_j^{t_0}\big)^{d_j} = \big(x_j^{1/d_j}\big)^{d_j} = x_j = x_j^{t_0}$ so that $F_j^{t_0} = 1 = {}^L F_j^{t_0}$, in particular $F_j^{t_0} = \big({}^L F_j^{t_0}\big)^{d_j}$. Consider $t \overset{k}{\text{---}} t'$ with t' further from t_0 than t and suppose $F_j^t = \big({}^L F_j^t\big)^{d_j}$ for all j. Then by Proposition 2.8 and Proposition 3.13 we see for $j \ne k$ that $F_j^{t'} = F_j^t = \big({}^L F_j^t\big)^{d_j} = \big({}^L F_j^{t'}\big)^{d_j}$ while taking $j = k$ we have

$$\begin{aligned}
{}^L F_k^{t'}({}^L\mathbf{y})^{d_k} &= \bigg(({}^L F_k^t)^{-1}\Big(\prod_{i=1}^n {}^L y_i^{[-{}^L c_{ik}^t]_+}({}^L F_i^t)^{[-d_i b_{ik}^t]_+}\Big)\Big(1+\prod_{i=1}^n {}^L y_i^{{}^L c_{ik}^t}({}^L F_i^t)^{d_i b_{ik}^t}\Big)\bigg)^{d_k} \\
&= \Big(({}^L F_k^t)^{d_k}\Big)^{-1}\Big(\prod_{i=1}^n {}^L y_i^{[-{}^L c_{ik}^t]_+}\Big(({}^L F_i^t)^{d_i}\Big)^{[-b_{ik}^t]_+}\Big)^{d_k}\sum_{s=0}^{d_k}\binom{d_k}{s}\Big(\prod_{i=1}^n {}^L y_i^{{}^L c_{ik}^t}\Big(({}^L F_i^t)^{d_i}\Big)^{b_{ik}^t}\Big)^s \\
&= (F_k^t)^{-1}\Big(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+}(F_i^t)^{[-b_{ik}^t]_+}\Big)^{d_k}\sum_{s=0}^{d_k}\binom{d_k}{s}\Big(\prod_{i=1}^n y_i^{c_{ik}^t}(F_i^t)^{b_{ik}^t}\Big)^s \\
&= F_k^{t'}(\mathbf{y}, \mathbf{z}^{\mathrm{bin}}),
\end{aligned}$$

where we used Corollary 4.1 in the third equality above. It follows by induction that $F_j^t(\mathbf{y}, \mathbf{z}^{\mathrm{bin}}) = {}^L F_j^t\big({}^L\mathbf{y}\big)^{d_j}$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. Note that $\hat{y}_j = y_j \prod_{i=1}^n$

$x_i^{b_{ij}} = {}^L y_j \prod_{i=1}^n {}^L x_i^{d_i b_{ij}} = {}^L \hat{y}_j$ so that substituting the variables $\hat{y}_j$ into this identity gives $F_j^t(\hat{\mathbf{y}}, \mathbf{z}^{\text{bin}}) = {}^L F_j^t({}^L\hat{\mathbf{y}})^{d_j}$ for all $t \in \mathbb{T}_n$ and $1 \le j \le n$. □

Write $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\text{bin}}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\text{bin}}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y}, \mathbf{z}^{\text{bin}})$ in place of the generic F-polynomials $F_j^t(\mathbf{y}, \mathbf{z})$.

Theorem 4.4. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{z}^{\text{bin}}} = ({}^L x_i^t)^{d_i}$ *and* $y_j^t\big|_{\mathbf{z}=\mathbf{z}^{\text{bin}}} = {}^L y_i^t$.

Proof. For coefficients we apply Theorem 2.9 and Theorem 3.14 along with Corollary 4.1 to get

$$ {}^L y_j^t = \left(\prod_{i=1}^n {}^L y_i^{{}^L c_{ij}^t}\right) \prod_{i=1}^n {}^L F_i^t\big|_{\mathbb{P}}({}^L\mathbf{y})^{d_i b_{ij}^t} = \left(\prod_{i=1}^n y_i^{c_{ij}^t}\right) \prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\text{bin}})^{b_{ij}^t} = y_j^t. $$

To finish, we may apply Theorem 2.10 and Theorem 3.15 along with Corollary 4.2 to get

$$ \begin{aligned} ({}^L x_j^t)^{d_j} &= \left(\left(\prod_{i=1}^n {}^L x_i^{{}^L g_{ij}^t}\right) \frac{{}^L F_j^t\big|_{\mathcal{F}}({}^L\hat{\mathbf{y}})}{{}^L F_j^t\big|_{\mathbb{P}}({}^L\mathbf{y})}\right)^{d_j} = \left(\prod_{i=1}^n {}^L x_i^{{}^L g_{ij}^t d_j}\right) \frac{{}^L F_j^t\big|_{\mathcal{F}}({}^L\hat{\mathbf{y}})^{d_j}}{{}^L F_j^t\big|_{\mathbb{P}}({}^L\mathbf{y})^{d_j}} \\ &= \left(\prod_{i=1}^n \left({}^L x_i^{1/d_i}\right)^{g_{ij}^t}\right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z}^{\text{bin}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\text{bin}})} = \left(\prod_{i=1}^n x_i^{g_{ij}^t}\right) \frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}}, \mathbf{z}^{\text{bin}})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y}, \mathbf{z}^{\text{bin}})} = x_j^t. \end{aligned} $$

□

Example 4.5. As an illustration of Corollaries 4.1 and 4.2 as well as Theorem 4.4 we now present the C-matrices, G-matrices, and F-polynomials for the left companion cluster algebra ${}^L\mathcal{A}$ in Table 3 from which we invite the reader to directly verify these results.

To state a relationship between a generalized cluster algebra and its right-companion we need the following analogue of Proposition 4.3.

Proposition 4.6. *Let* $(\mathbf{x}, \mathbf{y}, B, \mathbf{Z})$ *be a generalized seed over* $\mathbb{P}$. *For any* $t \in \mathbb{T}_n$ *and any* $1 \le j \le n$ *we have the following equalities in* $\mathbb{Q}_{\text{sf}}(\mathbf{y})$ *and* $\mathbb{Q}_{\text{sf}}(\mathbf{x}, \mathbf{y})$ *respectively:*

$$ F_j^t(\mathbf{y}, \mathbf{0}) = {}^R F_j^t({}^R\mathbf{y}) \quad \text{and} \quad F_j^t(\hat{\mathbf{y}}, \mathbf{0}) = {}^R F_j^t({}^R\hat{\mathbf{y}}), \tag{4.2} $$

where ${}^R y_i = y_i^{d_i}$ *and* ${}^R\hat{y}_i = \hat{y}_i^{d_i}$.

$$
{}^LC(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^LG(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(1) = 1 \\ {}^LF_2(1) = 1 \end{cases}
$$

$$
{}^LC(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^LG(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(2) = 1 + {}^Ly_1 \\ {}^LF_2(2) = 1 \end{cases}
$$

$$
{}^LC(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad {}^LG(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(3) = 1 + {}^Ly_1 \\ {}^LF_2(3) = 1 + {}^Ly_2 + 3{}^Ly_1{}^Ly_2 + 3{}^Ly_1^2{}^Ly_2 + {}^Ly_1^3{}^Ly_2 \end{cases}
$$

$$
{}^LC(4) = \begin{pmatrix} 1 & -3 \\ 0 & -1 \end{pmatrix}, \quad {}^LG(4) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(4) = 1 + {}^Ly_2 + 2{}^Ly_1{}^Ly_2 + {}^Ly_1^2{}^Ly_2 \\ {}^LF_2(4) = 1 + {}^Ly_2 + 3{}^Ly_1{}^Ly_2 + 3{}^Ly_1^2{}^Ly_2 + {}^Ly_1^3{}^Ly_2 \end{cases}
$$

$$
{}^LC(5) = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}, \quad {}^LG(5) = \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}, \quad \begin{cases} {}^LF_1(5) = 1 + {}^Ly_2 + 2{}^Ly_1{}^Ly_2 + {}^Ly_1^2{}^Ly_2 \\ {}^LF_2(5) = 1 + 2{}^Ly_2 + {}^Ly_2^2 + 3{}^Ly_1{}^Ly_2 \\ \qquad\qquad +3{}^Ly_1{}^Ly_2^2 + 3{}^Ly_1^2{}^Ly_2^2 + {}^Ly_1^3{}^Ly_2^2 \end{cases}
$$

$$
{}^LC(6) = \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}, \quad {}^LG(6) = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}, \quad \begin{cases} {}^LF_1(6) = 1 + {}^Ly_2 + {}^Ly_1{}^Ly_2 \\ {}^LF_2(6) = 1 + 2{}^Ly_2 + {}^Ly_2^2 + 3{}^Ly_1{}^Ly_2 \\ \qquad\qquad +3{}^Ly_1{}^Ly_2^2 + 3{}^Ly_1^2{}^Ly_2^2 + {}^Ly_1^3{}^Ly_2^2 \end{cases}
$$

$$
{}^LC(7) = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}, \quad {}^LG(7) = \begin{pmatrix} 2 & 3 \\ -1 & -1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(7) = 1 + {}^Ly_2 + {}^Ly_1{}^Ly_2 \\ {}^LF_2(7) = 1 + {}^Ly_2 \end{cases}
$$

$$
{}^LC(8) = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad {}^LG(8) = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(8) = 1 \\ {}^LF_2(8) = 1 + {}^Ly_2 \end{cases}
$$

$$
{}^LC(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^LG(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^LF_1(9) = 1 \\ {}^LF_2(9) = 1 \end{cases}
$$

Table 3: C-matrices, G-matrices, and F-polynomials for the same mutation sequence (3.7) applied to the seeds of ${}^L\mathcal{A}$.

Proof. We will proceed by induction on the distance from t_0 to t in $\mathbb{T}_n$. To begin, note that by definition we have $F_j^{t_0} = 1 = {}^R\!F_j^{t_0}$. Consider $t \overset{k}{—} t'$ with t' further from t_0 than t and suppose $F_j^t = {}^R\!F_j^t$ for all j. Then by Proposition 2.8 and Proposition 3.13 we see for $j \neq k$ that $F_j^{t'} = F_j^t = {}^R\!F_j^t = {}^R\!F_j^{t'}$ while taking $j = k$ we have

$$\begin{aligned}
{}^R\!F_k^{t'}\big({}^R\mathbf{y}\big) &= \big({}^R\!F_k^t\big)^{-1}\Bigg(\prod_{i=1}^n {}^R y_i^{[-{}^Rc_{ik}^t]_+}\big({}^R\!F_i^t\big)^{[-b_{ik}^t d_k]_+}\Bigg)\Bigg(1+\prod_{i=1}^n {}^R y_i^{{}^Rc_{ik}^t}\big({}^R\!F_i^t\big)^{b_{ik}^t d_k}\Bigg)\\
&= \big(F_k^t\big)^{-1}\Bigg(\prod_{i=1}^n y_i^{[-c_{ik}^t d_k]_+}\big(F_i^t\big)^{[-b_{ik}^t d_k]_+}\Bigg)\Bigg(1+\prod_{i=1}^n y_i^{c_{ik}^t d_k}\big(F_i^t\big)^{b_{ik}^t d_k}\Bigg)\\
&= \big(F_k^t\big)^{-1}\Bigg(\prod_{i=1}^n y_i^{[-c_{ik}^t]_+}\big(F_i^t\big)^{[-b_{ik}^t]_+}\Bigg)^{d_k}\Bigg(1+\Bigg(\prod_{i=1}^n y_i^{c_{ik}^t}\big(F_i^t\big)^{b_{ik}^t}\Bigg)^{d_k}\Bigg)\\
&= F_k^{t'}(\mathbf{y},\mathbf{0}).
\end{aligned}$$

It follows by induction that $F_j^t(\mathbf{y},\mathbf{0}) = {}^R\!F_j^t\big({}^R\mathbf{y}\big)$ for all $t \in \mathbb{T}_n$ and $1 \leq j \leq n$. Finally notice that ${}^R\hat{y}_j = {}^R y_j \prod_{i=1}^n {}^R x_i^{b_{ij}d_j} = y_j^{d_j} \prod_{i=1}^n x_i^{b_{ij}d_j} = \hat{y}_j^{d_j}$ so that substituting the variables $\hat{y}_j$ into this identity gives $F_j^t(\hat{\mathbf{y}},\mathbf{0}) = {}^R\!F_j^t\big({}^R\hat{\mathbf{y}}\big)$ for all $t \in \mathbb{T}_n$ and $1 \leq j \leq n$. □

Write $x_i^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathcal{F}$ and $y_j^t\big|_{\mathbf{z}=\mathbf{0}} \in \mathbb{P}$ for the variables obtained by applying equations (3.15) and (3.14) respectively using the specialized F-polynomials $F_j^t(\mathbf{y},\mathbf{0})$ in place of the generic F-polynomials $F_j^t(\mathbf{y},\mathbf{z})$.

Theorem 4.7. *We have* $x_i^t\big|_{\mathbf{z}=\mathbf{0}} = {}^R x_i^t$ *and* $\big(y_j^t\big|_{\mathbf{z}=\mathbf{0}}\big)^{d_j} = {}^R y_j^t$.

Proof. To see the claim for coefficients we apply Theorem 2.9 and Theorem 3.14 along with Corollary 4.1 and Proposition 4.6 to get

$${}^R y_j^t = \Bigg(\prod_{i=1}^n {}^R y_i^{{}^Rc_{ij}^t}\Bigg)\prod_{i=1}^n {}^R\!F_i^t\big|_{\mathbb{P}}({}^R\mathbf{y})^{b_{ij}^t d_j} = \Bigg(\prod_{i=1}^n y_i^{c_{ij}^t d_j}\Bigg)\prod_{i=1}^n F_i^t\big|_{\mathbb{P}}(\mathbf{y},\mathbf{0})^{b_{ij}^t d_j} = \big(y_j^t\big)^{d_j}.$$

Finally to see the claim for cluster variables we apply Theorem 2.10 and Theorem 3.15 along with Corollary 4.2 and Proposition 4.6 to get

$${}^R x_j^t = \Bigg(\prod_{i=1}^n {}^R x_i^{{}^Rg_{ij}^t}\Bigg)\frac{{}^R\!F_j^t\big|_{\mathcal{F}}({}^R\hat{\mathbf{y}})}{{}^R\!F_j^t\big|_{\mathbb{P}}({}^R\mathbf{y})} = \Bigg(\prod_{i=1}^n x_i^{g_{ij}^t}\Bigg)\frac{F_j^t\big|_{\mathcal{F}}(\hat{\mathbf{y}},\mathbf{0})}{F_j^t\big|_{\mathbb{P}}(\mathbf{y},\mathbf{0})} = x_j^t.$$

□

Example 4.8. As an illustration of Corollaries 4.1 and 4.2 as well as Theorem 4.7 we now present the C-matrices, G-matrices, and F-polynomials for the right companion cluster algebra ${}^R\mathcal{A}$ in Table 4 from which we invite the reader to directly verify these results.

$$
{}^R C(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^R G(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(1) = 1 \\ {}^R F_2(1) = 1 \end{cases}
$$

$$
{}^R C(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^R G(2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(2) = 1 + {}^R y_1 \\ {}^R F_2(2) = 1 \end{cases}
$$

$$
{}^R C(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad {}^R G(3) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(3) = 1 + {}^R y_1 \\ {}^R F_2(3) = 1 + {}^R y_2 + {}^R y_1 {}^R y_2 \end{cases}
$$

$$
{}^R C(4) = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad {}^R G(4) = \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(4) = 1 + 3{}^R y_2 + 3{}^R y_2^2 + {}^R y_2^3 \\ \qquad\qquad + 3{}^R y_1 {}^R y_2^2 + 2{}^R y_1 {}^R y_2^3 + {}^R y_1^2 {}^R y_2^3 \\ {}^R F_2(4) = 1 + {}^R y_2 + {}^R y_1 {}^R y_2 \end{cases}
$$

$$
{}^R C(5) = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad {}^R G(5) = \begin{pmatrix} 1 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} {}^R F_1(5) = 1 + 3{}^R y_2 + 3{}^R y_2^2 + {}^R y_2^3 \\ \qquad\qquad + 3{}^R y_1 {}^R y_2^2 + 2{}^R y_1 {}^R y_2^3 + {}^R y_1^2 {}^R y_2^3 \\ {}^R F_2(5) = 1 + 2{}^R y_2 + {}^R y_2^2 + {}^R y_1 {}^R y_2^2 \end{cases}
$$

$$
{}^R C(6) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}, \quad {}^R G(6) = \begin{pmatrix} 2 & 1 \\ -3 & -2 \end{pmatrix}, \quad \begin{cases} {}^R F_1(6) = 1 + 3{}^R y_2 + 3{}^R y_2^2 + {}^R y_2^3 + {}^R y_1 {}^R y_2^3 \\ {}^R F_2(6) = 1 + 2{}^R y_2 + {}^R y_2^2 + {}^R y_1 {}^R y_2^2 \end{cases}
$$

$$
{}^R C(7) = \begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}, \quad {}^R G(7) = \begin{pmatrix} 2 & 1 \\ -3 & -1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(7) = 1 + 3{}^R y_2 + 3{}^R y_2^2 + {}^R y_2^3 + {}^R y_1 {}^R y_2^3 \\ {}^R F_2(7) = 1 + {}^R y_2 \end{cases}
$$

$$
{}^R C(8) = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \quad {}^R G(8) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(8) = 1 \\ {}^R F_2(8) = 1 + {}^R y_2 \end{cases}
$$

$$
{}^R C(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad {}^R G(9) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{cases} {}^R F_1(9) = 1 \\ {}^R F_2(9) = 1 \end{cases}
$$

Table 4: C-matrices, G-matrices, and F-polynomials for the same mutation sequence (3.7) applied to the seeds of ${}^R\mathcal{A}$.

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On the Torelli theorem for Deligne-Hitchin moduli spaces

by David Alfaya and Tomas L. Gómez

Abstract

We prove a Torelli theorem for the parabolic Deligne-Hitchin moduli space, and compare it with previous Torelli theorems for non-parabolic Deligne-Hitchin moduli spaces.

1 Introduction

Let X be a smooth projective complex curve. The classical Torelli theorem says that we can recover the isomorphism class of the curve from the isomorphism class of the Jacobian $J(X)$ (which we think of as the moduli space of degree 0 line bundles on X) with the standard polarization $\Theta(X)$. In other words, if $(J(X), \Theta(X))$ is isomorphic to $(J(X'), \Theta(X'))$ as polarized varieties, then there is an isomorphism between X and X'. This has been generalized for other moduli spaces. A theorem which asserts that we can recover the curve X from the isomorphism class of some moduli space is called a Torelli theorem.

For instance, a Torelli theorem is known for the moduli space of semistable vector bundles with fixed determinant on X [Tj, NR, MN, KP]. For the moduli space of Higgs bundles, the Torelli theorem was proved in [BG]. In the case of the moduli space of parabolic bundles and parabolic Higgs bundles we recover the curve with the parabolic points [BBB, BHK, GL, BGL].

Deligne [De] has given a glueing construction of the twistor space of the moduli space of Higgs bundles [Hi2, §9], and this complex analytic space is called the Deligne-Hitchin moduli space (see [Si3] for the description).

In [BGHL] a Torelli theorem for the Deligne-Hitchin moduli space is proved (it should be noted that in [BGHL] we fix the determinant of the underlying vector bundle to be trivial, so the structure group is $\mathrm{SL}(r, \mathbb{C})$, but in [Si3] only the degree is fixed, so the structure group is $\mathrm{GL}(r, \mathbb{C})$).

In [BGH], Deligne glueing is used to construct a Deligne-Hitchin moduli space for a semisimple structure group and a Torelli type theorem is proved for it.

In this talk we report on the work [AG], where we prove a Torelli theorem for the Deligne-Hitchin moduli space for parabolic vector bundles, which is constructed using Deligne glueing. The detailed constructions and proofs (sometimes with more generality) can be found in the original paper [AG]. The aim here is to highlight the main ideas and to explain the similarities and differences with [BGHL] and [BGH].

The theorem we are going to prove is (see next section for the definition of the Deligne-Hitchin moduli space)

Theorem 1.1. *Fix a genus $g \geq 3$, a number of points $n \geq 1$, and a rank $r \geq 2$. Let X and X' be smooth complex projective curves of genus g, and $D \subset X$, $D' \subset X'$ divisors consisting of n distinct points. Let α and α' be systems of weights satisfying the conditions in Remark 2.1. Assume $\sum_{x \in D} \beta(x)$ and $\sum_{x' \in D'} \beta'(x')$ are coprime to r.*

If $r = 2$ and the parabolic Deligne-Hitchin moduli spaces are isomorphic as analytic varieties

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \cong \mathcal{M}_{\mathrm{DH}}(X', r, \alpha')$$

then (X', D') is isomorphic to either (X, D) or $(\overline{X}, D)$.

We remark that the condition $r = 2$ is imposed because we use [BBB]. The situation is the following: the moduli spaces of parabolic Higgs bundles

$$\mathcal{M}(X, r, \alpha, \xi) \quad \text{and} \quad \mathcal{M}(\overline{X}, r, -\alpha, \overline{\xi})$$

with $\xi = \mathcal{O}_X(-\sum_{x \in D} \beta(x) \cdot x)$ embed into the parabolic Deligne-Hitchin moduli space. We show (for arbitrary r) that we can recover the images of these embeddings just from the isomorphism class of the parabolic Deligne-Hitchin moduli space. Then we apply the Torelli theorem for parabolic vector bundles [BBB] (which requires $r = 2$) to obtain our Torelli theorem. Therefore, if the result of [BBB] were generalized to arbitrary r, we would automatically get a Torelli theorem for parabolic Deligne-Hitchin moduli spaces for arbitrary rank r.

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2 Parabolic Deligne-Hitchin moduli space

In this section we recall some basic definitions and the constructions of the moduli spaces involved. Let X be a smooth projective complex curve, and let D be a divisor consisting of $n \geq 1$ distinct points (these are called the parabolic points). A parabolic vector bundle on X is a holomorphic vector bundle E or rank r endowed with a weighted filtration of the fiber E_x over each parabolic point $x \in D$:

$$E_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x} \supsetneq E_{x,l_x+1} = 0$$

$$0 \leq \alpha_1(x) < \alpha_2(x) < \cdots < \alpha_{l_x}(x) < 1 \ .$$

We say that $\alpha_i(x)$ is the weight associated to $E_{x,i}$. In this article we will assume that the filtrations are full flags, i.e., $l_x = r$ for all parabolic points x.

Let $(E, E_\bullet)$ be a parabolic vector bundle. We define its parabolic degree to be

$$\operatorname{pardeg}(E, E_\bullet) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{r} \alpha_i(x) \ . \tag{2.1}$$

Let $F \subset E$ be a subbundle. There is a parabolic structure induced on F as follows. For each parabolic point $x \in X$ we obtain a filtration $F_{x,i}$ of F_x by looking at the intersections $F_x \cap E_{x,j}$ for all j. The weight $\beta_i(x)$ of $F_{x,i}$ is defined as

$$\beta_i(x) = \max_j \{\alpha_j(x) : F_x \cap E_{x,j} = F_{x,i}\} \ .$$

We say that a parabolic bundle $(E, E_\bullet)$ is stable (respectively, semistable) if for all proper parabolic subbundles $F \subsetneq E$, with the induced parabolic structure,

$$\operatorname{pardeg}(F, F_\bullet) < \operatorname{pardeg}(E, E_\bullet) \qquad \text{(respectively, } \leq) \ .$$

Let $\mathcal{M}(X, r, \alpha, \xi)$ (or $\mathcal{M}(r, \alpha, \xi)$, or just $\mathcal{M}$, if the rest of the data is clear from the context) be the moduli space of semistable parabolic vector bundles on X of rank r, weights α and fixed determinant isomorphic to the line bundle ξ. It is projective of dimension

$$\dim \mathcal{M}(X, r, \alpha, \xi) = (r^2 - 1)(g - 1) + \frac{n(r^2 - r)}{2} \ .$$

Remark 2.1. We will assume the following conditions throughout the article

1. The weights are generic (see [AG] for the explicit meaning of generic in our situation), and the filtrations are full flags.

2. The weights are "concentrated", meaning that $\alpha_r(x) - \alpha_1(x) < \frac{4}{nr^2}$ for all parabolic points $x \in D$.

3. For each parabolic point the sum of the weights is an integer, i.e., for all parabolic points $x \in D$

$$\beta(x) := \sum_{i=1}^{r} \alpha_i(x) \in \mathbb{Z} \ .$$

4. The degree of ξ (the determinant of the vector bundles E) and the rank r are coprime.

Proposition 2.2. *Under the above assumptions, there are no strictly semistable parabolic vector bundles with respect the the weights α, the moduli space*

$$\mathcal{M}(X, r, \alpha, \xi)$$

is smooth, projective and parameterizes stable parabolic bundles. Furthermore, if $(E, E_\bullet)$ is a parabolic vector bundle, then it is stable as a vector bundle if and only if it is stable as a parabolic vector bundle.

For the proof, see [AG]. Recall that, as the rank and degree are coprime, no vector bundle is strictly semistable as a vector bundle.

A strongly parabolic Higgs bundle is a parabolic vector bundle $(E, E_\bullet)$ together with a homomorphism, called Higgs field,

$$\Phi : E \longrightarrow E \otimes K(D)$$

such that, for all parabolic points $x \in D$, the homomorphism induced in the fiber satisfies

$$\Phi(E_{x,i}) \subset E_{x,i+1} \otimes K(D)|_x$$

where K is the canonical line bundle of the curve X. A weakly parabolic Higgs bundle is defined analogously, but requiring the weaker condition

$$\Phi(E_{x,i}) \subset E_{x,i} \otimes K(D)|_x \ .$$

Unless otherwise stated, all Higgs fields will be strongly parabolic.

A subbundle $F \subset E$ is called Φ-invariant if $\Phi(F) \subset F \otimes K(D)$. We say that a parabolic Higgs bundle $(E, E_\bullet, \Phi)$ is stable (respectively, semistable) if the inequality (2.1) holds for all proper Φ-invariant subbundles.

Let $\mathcal{M}_{\mathrm{Higgs}}(X, r, \alpha, \xi)$ (or just $\mathcal{M}_{\mathrm{Higgs}}$ if the rest of the data is clear from the context) be the moduli space of semistable parabolic Higgs bundles on X of rank r, weights α, fixed determinant isomorphic to the line bundle ξ, and $\operatorname{tr}\Phi = 0$. Recall that we are assuming that the weights are generic. This implies that there are no strictly semistable parabolic Higgs bundles, and that this moduli space is smooth. Its dimension is

$$\dim \mathcal{M}_{\mathrm{Higgs}}(X, r, \alpha, \xi) = 2(r^2-1)(g-1) + n(r^2 - r) \ .$$

The cotangent space of $\mathcal{M}(r, \alpha, \xi)$ sits inside this moduli space as an open subset

$$T^*\mathcal{M}(X, r, \alpha, \xi) \subset \mathcal{M}_{\mathrm{Higgs}}(X, r, \alpha, \xi) \ . \tag{2.2}$$

This open subset consists of the parabolic Higgs bundles whose underlying parabolic bundle is stable. The complement consists of parabolic Higgs bundles which are stable as a parabolic Higgs bundle, but whose underlying parabolic bundle is unstable.

To see (2.2), note that the tangent space to a point $(E, E_\bullet)$ is

$$H^1(X, \mathrm{ParEnd}^0(E, E_\bullet)) ,$$

the first cohomology of the sheaf of traceless weakly parabolic endomorphisms (the traceless condition comes from the fact that the determinant is fixed). By Serre duality

$$H^1(X, \mathrm{ParEnd}^0(E, E_\bullet))^* \cong H^0(X, \mathrm{SParEnd}^0(E, E_\bullet) \otimes K(D)) ,$$

the space of strongly parabolic Higgs fields with $\mathrm{tr}\,\Phi = 0$, and any such Higgs bundle is stable because the underlying parabolic vector bundle is stable. This explains why it is natural to require that the Higgs field is traceless and strongly parabolic (as opposed to weakly parabolic).

The moduli space of parabolic Higgs bundles is endowed with a $\mathbb{C}^*$-action which, for each $t \in \mathbb{C}^*$, sends $(E, E_\bullet, \Phi)$ to $(E, E_\bullet, t\Phi)$. This is compatible, using (2.2), with scalar multiplication on the cotangent bundle of $\mathcal{M}(X, r, \alpha, \xi)$.

We now recall the definition of the Hitchin map and spectral curves in this setting. Denote by S the total space of the line bundle $K(D)$, let

$$p : S = \underline{\mathrm{Spec}}\,\mathrm{Sym}^\bullet(K(D))^* \longrightarrow X$$

be the projection and let $x \in H^0(S, p^*K(D))$ be the tautological section. Taking the characteristic polynomial of a Higgs field

$$\det(x \cdot \mathrm{id} - p^*\Phi) = x^r + \widetilde{s}_1 x^{r-1} + \widetilde{s}_2 x^{r-2} + \cdots + \widetilde{s}_r$$

we obtain sections $\widetilde{s}_i \in H^0(S, p^*K^iD^i)$ and it can be shown that these come from sections on X, i.e., there are sections $s_i \in H^0(X, K^iD^i)$ such that $p^*s_i = \widetilde{s}_i$.

Recall that we are assuming that Φ is strongly parabolic. Then the residue at each parabolic point is nilpotent and hence the eigenvalues of Φ vanish at the divisor D. Therefore, for each i the section s_i belongs to the subspace $H^0(X, K^iD^{i-1})$, where we use the shorthand $K^iD^j = K^i \otimes \mathcal{O}_X(D)^j$.

Furthermore, we are assuming that the trace of Φ is identically zero, so $s_1 = 0$. We then define the Hitchin space as

$$\mathcal{H}_0 = H^0(X, K^2D) \oplus \cdots \oplus H^0(X, K^rD^{r-1}) . \tag{2.3}$$

The Hitchin map is defined by taking the characteristic polynomial of the Higgs field, i.e., to each Higgs field Φ we associate the point in the Hitchin space defined by the elements s_i, $2 \le i \le r$ defined above

$$H : \mathcal{M}_{\mathrm{Higgs}}(r, \alpha, \xi) \longrightarrow \mathcal{H}_0 .$$

This map is projective with connected fibers.

From now on we will assume

$$\xi = \mathcal{O}_X\Big(- \sum_{x\in D} \beta(x)\cdot x\Big)$$

so that the parabolic degree of our vector bundles will be zero (recall that $\beta(x)$ is an integer because of the assumptions in Remark 2.1).

Deligne introduced the notion of λ-connection ($\lambda \in \mathbb{C}$) which interpolates between Higgs bundles and usual connections: if $\lambda = 0$ we get a Higgs bundle, and if $\lambda = 1$ we get a usual connection.

Definition 2.3. Let $(E, E_\bullet)$ be a parabolic bundle with an isomorphism $\det(E) \cong \xi$. A λ-connection, for the group $\mathrm{SL}(r,\mathbb{C})$, on X is a triple $(E, E_\bullet, \nabla)$ where ∇ is a $\mathbb{C}$-linear homomorphism

$$\nabla : E \longrightarrow E \otimes K(D)$$

such that

1. (Leibniz) If f is a holomorphic function and s is a holomorphic section of E (both over some open set of X),
$$\nabla(fs) = f\nabla(s) + \lambda s \otimes df \ .$$
2. For each parabolic point $x \in D$ the residue satisfies
$$\mathrm{Res}_x(\nabla)(E_{x,i}) \subseteq E_{x,i} \ .$$
3. For each parabolic point $x \in D$, the action of the residue of ∇ on $E_{x,i}/E_{x,i+1}$ is multiplication by $\lambda\alpha_i(x)$ (this is well defined because the residue preserves the filtration by the previous property).
4. The operator $\mathrm{tr}(\nabla) : \det(E) \longrightarrow \det(E) \otimes K(D)$ induced by ∇ coincides with $\lambda\nabla_{\xi,\beta}$ (defined below).

A parabolic vector bundle $(E, E_\bullet)$ induces a parabolic structure on the determinant $\xi = \det(E)$, with weights $\beta(x) = \sum_{i=1}^r \alpha_i(x)$, and using the correspondence between parabolic bundles of parabolic degree 0 and connections, this gives a connection on ξ (with poles along D) which we denote $\nabla_{\xi,\beta}$.

A subbundle $F \subset E$ is called ∇-invariant if $\nabla(F) \subset F \otimes K(D)$. We say that a λ-connection is stable (respectively, semistable) if the inequality (2.1) holds for all proper ∇-invariant parabolic subbundles.

Let $\mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$ (or just $\mathcal{M}_{\mathrm{Hodge}}$ if the rest of the data is clear from the context) be the moduli space of tuples $(\lambda, E, E_\bullet, \nabla)$ where ∇ is a semistable λ-connection (these objects are called parabolic Hodge bundles). It has a projection

$$\mathrm{pr}_\lambda : \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \longrightarrow \mathbb{C} \ . \tag{2.4}$$

Note that, if $\lambda = 0$, then a λ-connection is just a Higgs bundle, so we have

$$\mathrm{pr}_\lambda^{-1}(0) = \mathcal{M}_{\mathrm{Higgs}}(X, r, \alpha, \xi) \ .$$

The Hodge moduli space is endowed with a $\mathbb{C}^*$-action which, for each $t \in \mathbb{C}^*$, sends

$$(\lambda, E, E_\bullet, \nabla) \mapsto (t\lambda, E, E_\bullet, t\nabla) \ . \tag{2.5}$$

This extends the standard $\mathbb{C}^*$-action on the moduli space of parabolic Higgs bundles, and the projection pr_λ is equivariant, using the standard scalar multiplication on $\mathbb{C}$.

On the other hand, if $\lambda = 1$, then a λ-connection is a holomorphic flat connection on a parabolic vector bundle, i.e., a logarithmic connection such that the residue over each parabolic point x acts on $E_{x,i}/E_{x,i+1}$ as $\alpha_i(x)$. Therefore, the fiber over $\lambda = 1$ is the moduli space of parabolic $\mathrm{SL}(r, \mathbb{C})$ connections

$$\mathrm{pr}_\lambda^{-1}(1) = \mathcal{M}_{\mathrm{conn}}(X, r, \alpha, \xi) \ .$$

If ∇ is a λ-connection with $\lambda \neq 0$, then $(1/\lambda)\nabla$ is a usual connection, so the fiber of pr_λ over any $\lambda \neq 0$ is isomorphic to the moduli space of parabolic connections (i.e., meromorphic connections which respect the parabolic filtration). Therefore, the Hodge moduli space shows that the moduli space of Higgs bundles is a degeneration of the moduli space of connections (or, equivalently, the moduli space of connections is a deformation of the moduli space of Higgs bundles).

Now we are going to describe Deligne's glueing in the parabolic setting. Let $X_\mathbb{R}$ be the real manifold underlying X with the orientation induced by the complex structure. Fix a base point x_0 and a positively oriented simple loop

$$\gamma_x \in \pi_1(X_\mathbb{R} \setminus D, x_0)$$

around each parabolic point $x \in D$. Let $\mathcal{M}_{\mathrm{rep}}(X_\mathbb{R}, r, \alpha)$ be the set of irreducible representations

$$\rho : \pi_1(X_\mathbb{R} \setminus D, x_0) \longrightarrow \mathrm{SL}(r, \mathbb{C})$$

of the fundamental group, up to conjugation by $\mathrm{SL}(r, \mathbb{C})$, such that, for each parabolic point $x \in D$, the eigenvalues of $\rho(\gamma_x)$ are $\{e^{-2\pi i \alpha_i(x)}\}$. Note that, since the sum of the weights for each parabolic point is an integer, the determinant of $\rho(\gamma_x)$ is 1.

Simpson [Si1] has given a Riemann-Hilbert correspondence between stable parabolic connections of parabolic degree 0 and stable filtered local systems of degree 0 for the general linear group $\mathrm{GL}(n, \mathbb{C})$. It has a version for our setting (the group is $\mathrm{SL}(r, \mathbb{C})$ and the eigenvalues of the monodromies around the parabolic points are fixed by the weights α as above) which gives a biholomorphism

$$\mathrm{RH}_X : \mathcal{M}_{\mathrm{rep}}(X_\mathbb{R}, r, \alpha) \xrightarrow{\cong} \mathcal{M}_{\mathrm{conn}}(X, r, \alpha, \xi) \ . \tag{2.6}$$

Using this biholomorphism and the action (2.5) we obtain a holomorphic open embedding

$$\begin{array}{rcl} \mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) & \hookrightarrow & \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \\ (t, \rho) & \mapsto & (t, t \cdot \mathrm{RH}_X(\rho)) \end{array} \tag{2.7}$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*) \subset \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$.

Let $J_{\mathbb{R}}$ be the almost complex structure on $X_{\mathbb{R}}$ coming from X. Then $-J_{\mathbb{R}}$ is also an almost complex structure on $X_{\mathbb{R}}$ whose corresponding Riemann surface will be denoted by $\overline{X}$. Let $\overline{\xi}$ be the line bundle on $\overline{X}$ given by the complex structure conjugate to ξ. As a topological line bundle, it is isomorphic to ξ^{-1}. Note that the underlying manifold $\overline{X}_{\mathbb{R}}$ to $\overline{X}$ is the same as $X_{\mathbb{R}}$ but the induced orientation is the opposite. Therefore, if we consider the same loops γ_x defined above, we can identify

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) = \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha) \ .$$

We will also use the Riemann-Hilbert isomorphism for $\overline{X}$

$$\mathrm{RH}_{\overline{X}} : \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha) \xrightarrow{\cong} \mathcal{M}_{\mathrm{conn}}(\overline{X}, r, -\alpha, \overline{\xi}) \ .$$

The parabolic Deligne-Hitchin moduli space is defined by glueing

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) := \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi) \cup \mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi})$$

along the image of $\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}, r, \alpha) = \mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(\overline{X}_{\mathbb{R}}, r, -\alpha)$ using $\mathrm{RH}_{\overline{X}}$ and RH_X, identifying

$$(\lambda, \lambda \cdot \mathrm{RH}_X(\rho)) \in \mathcal{M}_{\mathrm{Hodge}}(X, r, \alpha, \xi)$$

with

$$(\lambda^{-1}, \lambda^{-1} \cdot \mathrm{RH}_{\overline{X}}(\rho)) \in \mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi}) \ .$$

The projection pr_λ in (2.4) extends to a morphism to $\mathbb{P}^1$ which we denote by the same letter

$$\mathrm{pr}_\lambda : \mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \longrightarrow \mathbb{P}^1 \ .$$

This moduli space will be denoted $\mathcal{M}_{\mathrm{DH}}$ if the rest of the data is clear from the context. It is clear, staring at the definition, that there is a holomorphic isomorphism,

$$\mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \cong \mathcal{M}_{\mathrm{DH}}(\overline{X}, r, -\alpha) \tag{2.8}$$

covering the antipodal map on $\mathbb{P}^1$. Because of this isomorphism, we cannot expect to recover the isomorphism class of (X, D), but only the unordered pair $\{(X, D), (\overline{X}, D)\}$.

3 Torelli theorem

Taking the Higgs field to be zero, we have an embedding of the moduli space of parabolic vector bundles in the moduli space of parabolic Higgs bundles. Furthermore, a Higgs bundle is the same thing as a 0-connection, so we also have an embedding into the Hodge moduli space. Finally, the Hodge moduli space is an open subset of the Deligne-Hitchin moduli space, so we have an embedding

$$\mathcal{M}(X, r, \alpha, \xi) \subset \mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \tag{3.1}$$

Note that the image of this embedding sits on the fiber over $\lambda = 0$. By definition, $\mathcal{M}_{\mathrm{Hodge}}(\overline{X}, r, -\alpha, \overline{\xi})$ is also an open subset of $\mathcal{M}_{\mathrm{DH}}(X, r, \alpha)$, so we also have an embedding

$$\mathcal{M}(\overline{X}, r, -\alpha, \overline{\xi}) \subset \mathcal{M}_{\mathrm{DH}}(X, r, \alpha) \tag{3.2}$$

and the image of this embedding sits over $\lambda = \infty \in \mathbb{P}^1$.

The strategy of the proof is the following. Given the isomorphism class of $\mathcal{M}_{\mathrm{DH}}(X, r, \alpha)$ as an analytic variety, we want to identify the image of $\mathcal{M}(X, r, \alpha, \xi)$ under the first embedding. Then we have the isomorphism class of $\mathcal{M}(X, r, \alpha, \xi)$, and we can apply the Torelli theorem [BBB] for the moduli space of parabolic bundles, in order to recover the isomorphism class of the pointed curve. Unfortunately, the Torelli theorem for parabolic bundles is only known for $r = 2$, and this is why we have to restrict our attention to rank 2.

Because of the isomorphism (2.8), we will not be able to distinguish between the images of (3.1) and (3.2).

The idea of the proof is to look at all vector fields on $\mathcal{M}_{\mathrm{DH}}(X, r, \alpha)$, i.e., sections of its tangent bundle. We look at the analytic subset of this moduli space defined as the simultaneous zeroes of all vector fields. More precisely, we will prove the following

Proposition 3.1. *Let Z be an irreducible component of*

$$\{z \in \mathcal{M}_{\mathrm{DH}}(X, r, \alpha) : \eta(z) = 0 \quad \text{for all } \eta \in H^0(\mathcal{M}_{\mathrm{DH}}, T\mathcal{M}_{\mathrm{DH}})\}$$

such that

$$\dim Z = \dim \mathcal{M}(X, r, \alpha, \xi)\ .$$

Then Z is the image of one of the embeddings (3.1) *and* (3.2)*.*

We start by showing that, at most, these two subsets satisfy the condition:

Proposition 3.2. *Let Z be an irreducible component of $\mathcal{M}_{\mathrm{DH}}^{\mathbb{C}^*}$, the fixed locus of the standard $\mathbb{C}^*$-action. Then*

$$\dim Z \le \dim \mathcal{M}(X, r, \alpha, \xi)$$

with equality only for Z the image of the embedding (3.1) *or* (3.2)*.*

Proof. Here we use the standard $\mathbb{C}^*$-action. This produces a vector field which vanishes at the fixed point locus. The projection pr_λ is equivariant with respect to the standard $\mathbb{C}^*$-action, so the irreducible components of the fixed point locus of the $\mathbb{C}^*$-action have to sit in the fibers of λ equal to 0 or ∞. We will first consider the case $\lambda = 0$, so the fixed locus sits in the moduli space of parabolic Higgs bundles. The Hitchin map is equivariant with respect to the $\mathbb{C}^*$-action on the parabolic Higgs moduli, so the fixed locus has to sit in $H^{-1}(0)$, which is called the nilpotent cone.

The nilpotent cone is equidimensional of dimension $\dim \mathcal{M}(X, r, \alpha, \xi)$. Exactly one of its components is the moduli space of parabolic bundles $\mathcal{M}(X, r, \alpha, \xi)$, and this is a fixed locus of the standard $\mathbb{C}^*$-action. The standard $\mathbb{C}^*$-action is nontrivial on the other components (by an argument similar to [Si2, Lemma 11.9]). We remark that the other fixed locus subvarieties correspond to variations of Hodge structures.

The case $\lambda = \infty$ is the same, but looking at the moduli space of parabolic Higgs bundles on $\overline{X}$. □

In [BGHL] the same proof was used. In [BGH] we do not have at our disposal an analogue of [Si2, Lemma 11.9], and the proof is based on a study of the infinitesimal deformations of the points fixed by the standard $\mathbb{C}^*$-action.

Now we show the opposite direction, i.e., we show that the images of these embeddings do satisfy the condition. We would like to show that all global vector fields vanish at $\mathcal{M}$, and we will do this by showing the stronger condition

$$H^0(\mathcal{M}(X, r, \alpha, \xi), T\mathcal{M}_{\mathrm{DH}}(X, r, \alpha)|_{\mathcal{M}(X,r,\alpha,\xi)}) = 0 \ .$$

This will be proved in several steps.

Lemma 3.3. *The holomorphic cotangent bundle*

$$T^*\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X, r, \alpha, \xi)$$

does not admit any nonzero holomorphic section.

Proof. Since we are assuming full flags, it is known that $\mathcal{M}(X, r, \alpha, \xi)$ is rational [BY, Theorem 6.1], and it is well-known that a smooth rational projective variety does not admit nonzero holomorphic 1-forms. □

In [BGHL] this is Lemma 2.1, and it was proved using the Hecke transform. The point is that, in the setting of [BGHL], we are looking at differentials (1-forms) on the moduli space of vector bundles with trivial determinant, which is not smooth. By doing a Hecke transform, we relate it to the moduli space of vector bundles with fixed determinant $\mathcal{O}_X(x_0)$ for a fixed point x_0. This is a smooth unirational projective variety [Se, p. 53], so it does not admit any nonzero holomorphic 1-form. An argument using the Hecke correspondence shows that the same holds for the moduli space of vector bundles with trivial determinant.

In [BGH] this is Proposition 4.2, and it was proved using "abelianization". By fixing a spectral curve, we obtained a dominant rational map from an abelian variety to the moduli space of bundles. Using this map, a holomorphic 1-form ω on the moduli space would give a rational 1-form on the abelian variety which, by a codimension argument, is shown to be defined on the whole abelian variety. Any 1-form on an abelian variety is closed, so ω is closed, but the first cohomology of the moduli space is zero [AB, Ch. 10], so $\omega = df$ for a holomorphic function on the moduli space, but since this is projective, f is constant, so $\omega = 0$.

Lemma 3.4. *The holomorphic tangent bundle*

$$T\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathcal{M}(X, r, \alpha, \xi)$$

does not admit any nonzero holomorphic section

Proof. We adapt an argument of Hitchin [Hi2, Theorem 6.2]. A section s of the tangent bundle gives, by contraction, a function on the total space of the cotangent bundle

$$s^\sharp : T^*\mathcal{M}(X, r, \alpha, \xi) \longrightarrow \mathbb{C}$$

which is linear on the fibers. The total space of the cotangent bundle is an open subset of the moduli of parabolic Higgs bundles, and the codimension of the complement is greater than two (here we use a calculation of Faltings [F, Lemma II.6 and V.(iii), p. 561]), so it extends to a function on the moduli of parabolic Higgs bundles. This descends, using the Hitchin map, to a function on the Hitchin space, which is homogeneous of degree 1 with respect to the standard $\mathbb{C}^*$-action (because $s^\sharp$ is linear on the fibers). But the Hitchin space (2.3) has no linear part, so this function has to be zero, and hence $s = 0$. □

In [BGHL] this is Lemma 2.2, and the proof is the same. In [BGH] this is Proposition 4.1, and we use a result of Faltings [F, Corollary III.3].

Corollary 3.5. *The tangent bundle to the parabolic Higgs moduli space, restricted to the moduli space of parabolic bundles, has no nonzero section, i.e.*

$$H^0(\mathcal{M}, T\mathcal{M}_{\text{Higgs}}|_{\mathcal{M}}) = 0 \ .$$

Proof. There is a short exact sequence on $\mathcal{M}$

$$0 \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{M}_{\text{Higgs}}|_{\mathcal{M}} \longrightarrow N \longrightarrow 0$$

where N is the normal of the embedding of $\mathcal{M}$ in $\mathcal{M}_{\text{Higgs}}$, but $N \cong T^*\mathcal{M}$ because we have a closed embedding (the zero section) and an open embedding

$$\mathcal{M} \subset T^*\mathcal{M} \subset \mathcal{M}_{\text{Higgs}} \ .$$

Now we use Lemmas 3.3 and 3.4.

□

Let $\mathcal{M}_{\mathrm{Hodge}}^{\mathrm{s-par}}$ and $\mathcal{M}_{\mathrm{conn}}^{\mathrm{s-par}}$ be the subsets of $\mathcal{M}_{\mathrm{Hodge}}$ and $\mathcal{M}_{\mathrm{conn}}$ where the underlying parabolic bundle is stable as a parabolic bundle. Recall that we are assuming generic weights, so that $\mathcal{M}^{\mathrm{s-par}} = \mathcal{M}$.

Proposition 3.6. *The projection*

$$\mathrm{pr}_E : \mathcal{M}_{\mathrm{conn}}^{\mathrm{s-par}} \longrightarrow \mathcal{M}$$

sending a connection to the underlying parabolic bundle, admits no holomorphic section.

Proof. Let s be such a section. The parabolic version of the Riemann-Hilbert correspondence (2.6) gives a biholomorphism between the moduli space of stable parabolic connections and the moduli space of irreducible representations of the fundamental group of the open curve, and this moduli space is an affine variety. Then the section s would give a holomorphic map from the projective variety $\mathcal{M}$ to an affine variety, but this map has to be constant, so the map s does not exist. □

In [BGHL] this is Proposition 3.2. There we remark that the projection pr_E is a torsor under the cotangent bundle. A section of pr_E then gives a nonzero cohomology class in the first cohomology class of the cotangent bundle, but in the setting of [BGHL] it is proved that this cohomology group is zero.

In [BGH] this is Proposition 4.4. We remark that the projection pr_E is isomorphic to the torsor of holomorphic connections on certain determinant line bundle, but this line bundle is ample, so it does not admit holomorphic connections.

Corollary 3.7. *Let*

$$\mathrm{pr}_E : \mathcal{M}_{\mathrm{Hodge}}^{\mathrm{s-par}} \longrightarrow \mathcal{M}$$

be the projection which sends each Hodge bundle to the underlying parabolic bundle. The only section s of this projection is the standard embedding of $\mathcal{M}$ in $\mathcal{M}_{\mathrm{Hodge}}$.

Proof. The composition of the section s with the projection pr_λ is a regular function on the projective variety $\mathcal{M}$, hence it is constant which, after scaling with the $\mathbb{C}^*$-action, we may assume it is 1 or 0.

It cannot be 1, because the fiber of pr_λ over 1 is the moduli space of connections, and it would contradict Proposition 3.6.

If it is 0 then, since the fiber of pr_λ over 0 is the moduli space of parabolic Higgs bundles, the section s factors through $\mathcal{M}_{\mathrm{Higgs}}^{\mathrm{s-par}}$, but this is isomorphic to the total space of $T^*\mathcal{M}$. Then s is a section of this vector bundle and, by Lemma 3.3, it has to be the zero section. □

Corollary 3.8.

$$H^0(\mathcal{M}(X,r,\alpha,\xi), T\mathcal{M}_{\mathrm{Hodge}}(X,r,\alpha,\xi)|_{\mathcal{M}(X,r,\alpha,\xi)}) = 0$$

Proof. Using the short exact sequence on $\mathcal{M}$

$$0 \longrightarrow T\mathcal{M} \longrightarrow T\mathcal{M}^{\text{s-par}}_{\text{Hodge}}|_{\mathcal{M}} \longrightarrow N \longrightarrow 0$$

and Lemma 3.4, it suffices to show that N has no nonzero sections. Note that N is isomorphic to $\mathcal{M}^{\text{s-par}}_{\text{Hodge}}$ as varieties over $\mathcal{M}$, sending $(\lambda, E, E_\bullet, \nabla)$ to the derivative at $t = 0$ of the map $\mathbb{C} \longrightarrow \mathcal{M}^{\text{s-par}}_{\text{Hodge}}$ given by

$$t \mapsto (t\lambda, E, E_\bullet, t\nabla)$$

so by Corollary 3.7 we conclude that N has no nonzero section. □

We are now ready to prove the main Theorem 1.1.

Corollary 3.8 implies that the images of the embeddings (3.1) and (3.2) are in the fixed point locus of any $\mathbb{C}^*$-action on the parabolic Deligne-Hitchin moduli space (if they were not, the derivative of the action would give vector field whose restriction to these images is nonzero). This together with Proposition 3.2, shows that these are the only irreducible components Z with this property having

$$\dim Z = \dim \mathcal{M}$$

hence proving Proposition 3.1.

Finally, using Proposition 3.1, from the isomorphism class of the parabolic Deligne-Hitchin moduli space we recover the isomorphism class of $\mathcal{M}(X, r, \alpha, \xi)$ or $\mathcal{M}(\overline{X}, r, -\alpha, \overline{\xi})$. Therefore, assuming $r = 2$ we can apply [BBB] to finish the proof of the main Theorem.

The same ideas also give Torelli theorems for $\mathcal{M}_{\text{Higgs}}$ and $\mathcal{M}_{\text{Hodge}}$. Indeed, in these cases we again have an embedding of $\mathcal{M}$, and the image is characterized as the only irreducible component, with dimension $\dim \mathcal{M}$, of the simultaneous zeroes of the sections of the tangent bundle. See [AG] for details.

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Yang–Mills–Higgs connections on Calabi–Yau manifolds, II

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Abstract

In this paper we study Higgs and co–Higgs G–bundles on compact Kähler manifolds X. Our main results are:

1. If X is Calabi–Yau (i.e., it has vanishing first Chern class), and (E, θ) is a semistable Higgs or co–Higgs G–bundle on X, then the principal G–bundle E is semistable. In particular, there is a deformation retract of $\mathcal{M}_H(G)$ onto $\mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli space of semistable principal G–bundles with vanishing rational Chern classes on X, and analogously, $\mathcal{M}_H(G)$ is the moduli space of semistable principal Higgs G–bundles with vanishing rational Chern classes.
2. Calabi–Yau manifolds are characterized as those compact Kähler manifolds whose tangent bundle is semistable for every Kähler class, and have the following property: if (E, θ) is a semistable Higgs or co–Higgs vector bundle, then E is semistable.

1 Introduction

In our previous paper [BBGL] we showed that the existence of semistable Higgs bundles with a nontrivial Higgs field on a compact Kähler manifold X constrains the geometry of X. In particular, it was shown that if X is Kähler-Einstein with $c_1(TX) \geq 0$, then it is necessarily Calabi-Yau, i.e., $c_1(TX) = 0$. In this paper we extend the analysis of the interplay between the existence of semistable Higgs bundles and the geometry of the underlying manifold (actually, we shall

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also consider co-Higgs bundles, and allow the structure group of the bundle to be any reductive linear algebraic group). Thus, if X is Calabi–Yau and (E, θ) is a semistable Higgs or co–Higgs G–bundle on X, it is proved that the underlying principal G–bundle E is semistable (Lemma 5.1). This has a consequence on the topology of the moduli spaces of principal (Higgs) G-bundles having vanishing rational Chern classes. We can indeed prove that there is a deformation retract of $\mathcal{M}_H(G)$ onto $\mathcal{M}(G)$, where $\mathcal{M}(G)$ is the moduli space of semistable principal G–bundles with vanishing rational Chern classes, and analogously, $\mathcal{M}_H(G)$ is the moduli space of semistable principal Higgs G–bundles with vanishing rational Chern classes (cf. [BF, FL] for similar deformation retract results).

As a further application, we can prove a characterization of Calabi–Yau manifolds in terms of Higgs and co-Higgs bundles; the characterization in question says that if X is a compact Kähler manifold with semistable tangent bundle with respect to every Kähler class, having the following property: for any semistable Higgs or co–Higgs vector bundle (E, θ) on X, the vector bundle E is semistable, then X is Calabi-Yau (Theorem 5.2).

In Section 4 We give a result about the behavior of semistable Higgs bundles under pullback by finite morphisms of Kähler manifolds. Let (X, ω) be a Ricci–flat compact Kähler manifold, M a compact connected Kähler manifold, and

$$f : M \longrightarrow X$$

a surjective holomorphic map such that each fiber of f is a finite subset of M. Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is semistable (respectively, stable). Then the principal G–bundle f^*E_G is semistable (respectively, polystable).

2 Preliminaries

Let X be a compact connected Kähler manifold equipped with a Kähler form ω. Using ω, the degree of torsion-free coherent analytic sheaves on X is defined as follows:

$$\text{degree}(F) := \int_X c_1(F) \bigwedge \omega^{d-1} \in \mathbb{R},$$

where $d = \dim_{\mathbb{C}} X$. The holomorphic cotangent bundle of X will be denoted by Ω_X.

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. The connected component of the center of G containing the identity element will be denoted by $Z_0(G)$. The Lie algebra of G will be denoted by $\mathfrak{g}$. A Zariski closed connected subgroup $P \subseteq G$ is called a parabolic subgroup of G if G/P is a projective variety. The unipotent radical of a parabolic subgroup P will be denoted by $R_u(P)$. A Levi subgroup of a parabolic subgroup P is a Zariski closed subgroup

$L(P) \subset P$ such that the composition

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism. Levi subgroups exist, and any two Levi subgroups of P differ by an inner automorphism of P [Bo, § 11.22, p. 158], [Hu2, § 30.2, p. 184]. The quotient map $G \longrightarrow G/P$ defines a principal P–bundle on G/P. The holomorphic line bundle on G/P associated to this principal P–bundle for a character χ of P will be denoted by $G(\chi)$. A character χ of a parabolic subgroup P is called *strictly anti–dominant* if $\chi|_{Z_0(G)}$ is trivial, and the associated holomorphic line bundle on $G(\chi) \longrightarrow G/P$ is ample.

For a principal G–bundle E_G on X, the vector bundle

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow X$$

associated to E_G for the adjoint action of G on its Lie algebra $\mathfrak{g}$ is called the *adjoint bundle* for E_G. So the fibers of $\mathrm{ad}(E_G)$ are Lie algebras identified with $\mathfrak{g}$ up to inner automorphisms. Using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ and the exterior multiplication of differential forms we have a symmetric bilinear pairing

$$(\mathrm{ad}(E_G) \otimes \Omega_X) \times (\mathrm{ad}(E_G) \otimes \Omega_X) \longrightarrow \mathrm{ad}(E_G) \otimes \Omega^2_X$$

which will be denoted by $\bigwedge$.

A *Higgs field* on a holomorphic principal G–bundle E_G on X is a holomorphic section θ of $\mathrm{ad}(E_G) \otimes \Omega_X$ such that

$$\theta \bigwedge \theta = 0 . \tag{2.1}$$

A *Higgs G–bundle* on X is a pair of the form (E_G, θ), where E_G is holomorphic principal G–bundle on X and θ is a Higgs field on E_G. A Higgs G–bundle (E_G, θ) is called *stable* (respectively, *semistable*) if for every quadruple of the form (U, P, χ, E_P), where

- $U \subset X$ is a dense open subset such that the complement $X \setminus U$ is a complex analytic subset of X of complex codimension at least two,
- $P \subset G$ is a proper parabolic subgroup,
- χ is a strictly anti–dominant character of P, and
- $E_P \subset E_G|_U$ is a holomorphic reduction of structure group to P over U such that $\theta|_U$ is a section of $\mathrm{ad}(E_P) \otimes \Omega_U$,

the following holds:

$$\text{degree}(E_P \times^\chi \mathbb{C}) > 0$$

(respectively, $\text{degree}(E_P \times^\chi \mathbb{C}) \geq 0$); note that since $X \setminus U$ is a complex analytic subset of X of complex codimension at least two, the line bundle $E_P \times^\chi \mathbb{C}$ on U extends uniquely to a holomorphic line bundle on X.

A semistable Higgs G–bundle (E_G, θ) is called *polystable* if there is a Levi subgroup $L(Q)$ of a parabolic subgroup $Q \subset G$ and a Higgs $L(Q)$–bundle (E', θ') on X such that

- the Higgs G–bundle obtained by extending the structure group of (E', θ') using the inclusion $L(Q) \hookrightarrow G$ is isomorphic to (E_G, θ), and
- the Higgs $L(Q)$–bundle (E', θ') is stable.

Fix a maximal compact subgroup $K \subset G$. Given a holomorphic principal G–bundle E_G and a C^∞ reduction of structure group $E_K \subset E_G$ to the subgroup K, there is a unique connection on the principal K–bundle E_K that is compatible with the holomorphic structure of E_G [At, pp. 191–192, Proposition 5]; it is known as the *Chern connection.* A C^∞ reduction of structure group of E_G to K is called a *Hermitian structure* on E_G.

Let Λ_ω denote the adjoint of multiplication of differential forms on X by ω.

Given a Higgs G–bundle (E_G, θ) on X, a Hermitian structure $E_K \subset E_G$ is said to satisfy the Yang–Mills–Higgs equation if

$$\Lambda_\omega(\mathcal{K} + \theta \bigwedge \theta^*) = \mathfrak{z}\,, \tag{2.2}$$

where $\mathcal{K}$ is the curvature of the Chern connection associated to E_K and $\mathfrak{z}$ is some element of the Lie algebra of $Z_0(G)$; the adjoint θ^* is constructed using the Hermitian structure E_K. A Higgs G–bundle admits a Hermitian structure satisfying the Yang–Mills–Higgs equation if and only if it is polystable [Si], [BiSc, p. 554, Theorem 4.6].

Given a polystable Higgs G–bundle (E, θ), any two Hermitian structures on E_G satisfying the Yang–Mills–Higgs equation differ by a holomorphic automorphism of E_G that preserves θ; however, the associated Chern connection is unique [BiSc, p. 554, Theorem 4.6].

3 Higgs G–bundles on Calabi–Yau manifolds

Henceforth, till the end of Section 4, we assume that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero. From this assumption it follows that every Kähler class on X contains a Ricci–flat Kähler metric [Ya, p. 364, Theorem 2]. We will assume that the Kähler form ω on X is Ricci–flat.

3.1 Higgs G–bundles on Calabi-Yau manifolds

Let (E_G, θ) be a polystable Higgs G–bundle on X. For any holomorphic tangent vector $v \in T_xX$, note that $\theta(x)$ is an element of the fiber $\mathrm{ad}(E_G)_x$. For any point $x \in X$, consider the complex subspace

$$\widehat{\Theta}_x := \{\theta(x)(v) \mid v \in T_xX\} \subset \mathrm{ad}(E_G)_x\,. \tag{3.1}$$

Form (2.1) it follows immediately that $\widehat{\Theta}_x$ is an abelian subalgebra of the Lie algebra $\mathrm{ad}(E_G)_x$.

Let ∇ be the connection on $\mathrm{ad}(E_G)$ induced by the unique connection on E_G given by the solutions of the Yang–Mills–Higgs equation.

Lemma 3.1.

1. *The abelian subalgebra* $\widehat{\Theta}_x \subset \mathrm{ad}(E_G)_x$ *is semisimple.*

2. $\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$ *is preserved by the connection* ∇ *on* $\mathrm{ad}(E_G)$*. In particular,*
$$\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$$
is a holomorphic subbundle.

Proof. First take $G = \mathrm{GL}(n,\mathbb{C})$, so that (E_G, θ) defines a Higgs vector bundle (F, φ) of rank n. Let
$$\widehat{\Theta}'_x \subset \mathrm{End}(F_x)$$
be the subalgebra constructed as in (3.1) for the Higgs vector bundle (F, φ). From [BBGL, Proposition 2.5] it follows immediately that there is a basis of the vector space F_x such that
$$\varphi(x)(v) \in \mathrm{End}(F_x)$$
is diagonal with respect to it for all $v \in T_x$. This implies that the subalgebra $\widehat{\Theta}'_x$ is semisimple (this uses the Jordan–Chevalley decomposition, see e.g. [Hu1, Ch. 2]).

Consider the $\mathcal{O}_X$–linear homomorphism
$$\eta : TX \longrightarrow \mathrm{End}(F)$$
that sends any $w \in T_yX$ to $\varphi(y)(w) \in \mathrm{End}(F_y)$, where φ as before is the Higgs field on the holomorphic vector bundle F. Proposition 2.2 of [BBGL] says that φ is flat with respect to the connection on $\mathrm{End}(F) \otimes \Omega_X$ induced by the connection ∇ on $\mathrm{End}(F) = \mathrm{ad}(E_G)$ and the Levi–Civita connection on Ω_X for ω. Therefore, the above homomorphism η intertwines the Levi–Civita connection on TX and the connection on $\mathrm{End}(F)$. Consequently, the image $\eta(TX) \subset \mathrm{End}(F)$ is preserved by the connection on $\mathrm{End}(F)$. On the other hand, $\eta(TX)$ coincides with $\{\widehat{\Theta}'_x\}_{x\in X} \subset \mathrm{End}(F)$.

Therefore, the lemma is proved when $G = \mathrm{GL}(n, \mathbb{C})$.

For a general G, take any homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(N, \mathbb{C})$$

such that $\rho(Z_0(G))$ lies inside the center of $\mathrm{GL}(N, \mathbb{C})$. Let (E_ρ, θ_ρ) be the Higgs vector bundle of rank N given by (E_G, θ) using ρ. For any Hermitian structure on E_G solving the Yang–Mills–Higgs equation for (E_G, θ), the induced Hermitian structure on E_ρ solves the Yang–Mills–Higgs equation for (E_ρ, θ_ρ). We have shown above that the lemma holds for (E_ρ, θ_ρ).

Since the lemma holds for (E_ρ, θ_ρ) for every ρ of the above type, we conclude that the lemma holds for (E_G, θ). □

As before, (E_G, θ) is a polystable Higgs G–bundle on X. Fix a Hermitian structure

$$E_K \subset E_G \tag{3.2}$$

that satisfies the Yang–Mills–Higgs equation for (E_G, θ).

Take another Higgs field β on E_G. Let

$$\widetilde{\beta} : TX \longrightarrow \mathrm{ad}(E_G)$$

be the $\mathcal{O}_X$–linear homomorphism that sends any tangent vector $w \in T_yX$ to

$$\beta(y)(w) \in \mathrm{ad}(E_G)_y\,.$$

Theorem 3.2. *Assume that the image $\widetilde{\beta}(TX)$ is contained in the subbundle*

$$\{\widehat{\Theta}_x\}_{x\in X} \subset \mathrm{ad}(E_G)$$

in Lemma 3.1. Then E_K in (3.2) *also satisfies the Yang–Mills–Higgs equation for (E_G, β). In particular, (E_G, β) is polystable.*

Proof. From Theorem 4.2 of [BBGL] we know that E_K in (3.2) satisfies the Yang–Mills–Higgs equation for $(E_G, 0)$. Therefore, it suffices to show that $\beta \bigwedge \beta^* = 0$ (see (2.2)).

Let

$$\gamma : TX \longrightarrow \mathrm{ad}(E_G)$$

be the $C^\infty(X)$–linear homomorphism that sends any $w \in T_yX$ to $\theta^*(y)(w) \in \mathrm{ad}(E_G)_y$. Clearly, we have

$$\gamma(TX)^* = \{\widehat{\Theta}_x\}_{x\in X}\,; \tag{3.3}$$

as before, the superscript "$*$" denotes adjoint with respect to the reduction E_K. Since the subbundle $\{\widehat{\Theta}_x\}_{x\in X}$ is preserved by the connection on $\mathrm{ad}(E_G)$, from (3.3) it follows that

$$\{\widehat{\Theta}_x\}_{x\in X} + \gamma(TX) \subset \mathrm{ad}(E_G) \tag{3.4}$$

is a subbundle preserved by the connection; it should be clarified that the above need not be a direct sum.

We know that $\theta \bigwedge \theta^* = 0$ [BBGL, Lemma 4.1]. This and (2.1) together imply that the subbundle in (3.4) is an abelian subalgebra bundle. We have

$$\widetilde{\beta}(TX) \subset \{\widehat{\Theta}_x\}_{x\in X},$$

and hence β^* is a section of $\gamma(TX)\otimes\Omega_X \subset \mathrm{ad}(E_G)\otimes\Omega_X$. Since the subbundle in (3.4) is an abelian subalgebra bundle, we now conclude that $\beta \bigwedge \beta^* = 0$. □

The proof of Theorem 3.2 gives the following:

Corollary 3.3. *Let ϕ be a Higgs field on E_G such that $\phi \bigwedge \phi^* = 0$. Then E_K in* (3.2) *also satisfies the Yang–Mills–Higgs equation for* (E_G, ϕ). *In particular,* (E_G, ϕ) *is polystable.*

Proof. As noted in the proof of Theorem 3.2, the reduction E_K satisfies the Yang–Mills–Higgs equation for $(E_G, 0)$. Since $\phi \bigwedge \phi^* = 0$, it follows that E_K in (3.2) satisfies the Yang–Mills–Higgs equation for (E_G, ϕ). □

Remark 3.4. The condition in Theorem 3.2 that $\widetilde{\beta}(TX) \subset \{\widehat{\Theta}_x\}_{x\in X}$ does not depend on the Hermitian structure E_K; it depends only on the Higgs G–bundle (E_G, θ). In contrast, the condition $\phi \bigwedge \phi^* = 0$ in Corollary 3.3 depends also on E_K.

3.2 A deformation retraction

Let $\mathcal{M}_H(G)$ denote the moduli space of semistable Higgs G–bundles (E_G, θ) on X such that all rational characteristic classes of E_G of positive degree vanish. It is known (it is a straightforward consequence of Theorem 2 in [Si]) that if the following three conditions hold:

1. (E_G, θ) is semistable,
2. for all characters χ of G, the line bundle on X associated to E_G for χ is of degree zero, and
3. the second Chern class $c_2(\mathrm{ad}(E_G)) \in H^4(X, \mathbb{Q})$ vanishes,

then all characteristic classes of E_G of positive degree vanish. Let $\mathcal{M}(G)$ denote the moduli space of semistable principal G–bundles E_G on X such that all rational characteristic classes of E_G of positive degree vanish.

We have an inclusion

$$\xi : \mathcal{M}(G) \longrightarrow \mathcal{M}_H(G), \quad E_G \longmapsto (E_G, 0). \tag{3.5}$$

Proposition 3.5. *There is a natural holomorphic deformation retraction of* $\mathcal{M}_H(G)$ *to the image of* ξ *in* (3.5)*.*

Proof. Points of $\mathcal{M}_H(G)$ parametrize the polystable Higgs G–bundles (E_G, θ) on X such that all rational characteristic classes of E_G of positive degree vanish. Given such a Higgs G–bundle (E_G, θ), from Theorem 3.2 we know that $(E_G, t\cdot\theta)$ is polystable for all $t \in \mathbb{C}$. Therefore, we have a holomorphic map

$$F : \mathbb{C} \times \mathcal{M}_H(G) \longrightarrow \mathcal{M}_H(G), \quad (t, (E_G, \theta)) \longmapsto (E_G, t\cdot\theta).$$

The restriction of F to $\{1\} \times \mathcal{M}_H(G)$ is the identity map of $\mathcal{M}_H(G)$, while the image of the restriction of F to $\{0\} \times \mathcal{M}_H(G)$ is the image of ξ. Moreover, the restriction of F to $\{0\} \times \xi(\mathcal{M}(G))$ is the identity map. □

Fix a point $x_0 \in X$. Since G is an affine variety and $\pi_1(X, x_0)$ is finitely presented, the geometric invariant theoretic quotient

$$\mathcal{M}_R(G) := \mathrm{Hom}(\pi_1(X, x_0), G)/\!\!/G$$

for the adjoint action of G on $\mathrm{Hom}(\pi_1(X, x_0), G)$ is an affine variety. The points of $\mathcal{M}_R(G)$ parameterize the equivalence classes of homomorphisms from $\pi_1(X, x_0)$ to G such that the Zariski closure of the image is a reductive subgroup of G. Consider the quotient space

$$\mathcal{M}_R(K) := \mathrm{Hom}(\pi_1(X, x_0), K)/K,$$

where K as before is a maximal compact subgroup of G. The inclusion of K in G produces an inclusion

$$\xi' : \mathcal{M}_R(K) \longrightarrow \mathcal{M}_R(G). \tag{3.6}$$

Corollary 3.6. *There is a natural deformation retraction of* $\mathcal{M}_R(G)$ *to the subset* $\mathcal{M}_R(K)$ *in* (3.6)*.*

Proof. The nonabelian Hodge theory gives a homeomorphism of $\mathcal{M}_R(G)$ with $\mathcal{M}_H(G)$. On the other hand, $\mathcal{M}_R(K)$ is identified with $\mathcal{M}(G)$, and the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}(G) & \xrightarrow{\xi} & \mathcal{M}_H(G) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{M}_R(K) & \xrightarrow{\xi'} & \mathcal{M}_R(G) \end{array}$$

Hence Proposition 3.5 produces the deformation retraction in question. □

4 Pullback of Higgs bundles by finite morphisms

Take (X, ω) to be as before. Let M be compact connected Kähler manifold, and let

$$f : M \longrightarrow X$$

be a surjective holomorphic map such that each fiber of f is a finite subset of M. In particular, we have $\dim M = \dim X$. It is known that the form $f^*\omega$ represents a Kähler class on the Kähler manifold M [BiSu, p. 438, Lemma 2.1]. The degree of torsion-free coherent analytic sheaves on M will be defined using the Kähler class given by $f^*\omega$.

Proposition 4.1. *Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is semistable. Then the principal G–bundle f^*E_G is semistable.*

Proof. Since the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ is semistable, it follows that (E_G, θ) is semistable. Indeed, the pullback of any reduction of structure group of (E_G, θ) that contradicts the semistability condition also contradicts the semistability condition for $(f^*E_G, f^*\theta)$. Since the Higgs G–bundle (E_G, θ) is semistable, we conclude that the principal G–bundle E_G is semistable [Bi, p. 305, Lemma 6.2]. This, in turn, implies that f^*E_G is semistable (see [BiSu, p. 441, Theorem 2.4] and [BiSu, p. 442, Remark 2.5]). □

Proposition 4.2. *Let (E_G, θ) be a Higgs G–bundle on X such that the pulled back Higgs G–bundle $(f^*E_G, f^*\theta)$ on M is stable. Then the principal G–bundle f^*E_G is polystable.*

Proof. The principal G–Higgs bundle (E_G, θ) is stable, because any reduction of it contradicting the stability condition pulls back to a reduction that contradicts the stability condition for $(f^*E_G, f^*\theta)$. Since (E_G, θ) is stable, we know that E_G is polystable [Bi, p. 306, Lemma 6.4]. Now f^*E_G is polystable because E_G is so [BiSc, p. 439, Proposition 2.3], [BiSc, p. 442, Remark 2.6]. □

5 Co–Higgs bundles

We recall the definition of a co–Higgs vector bundle [Ra1, Ra2, Hi].

Let (X, ω) be a compact connected Kähler manifold and E a holomorphic vector bundle on X. A *co–Higgs field* on E is a holomorphic section

$$\theta \in H^0(X, \mathrm{End}(E) \otimes TX)$$

such that the section $\theta \bigwedge \theta$ of $\mathrm{End}(E) \otimes \bigwedge^2 TX$ vanishes identically. A co–Higgs bundle on X is a pair (E, θ), where E is a holomorphic vector bundle on X and θ is a co–Higgs field on E [Ra1, Ra2, Hi].

A co–Higgs bundle (E, θ) is called *semistable* if for all nonzero coherent analytic subsheaves $F \subset E$ with $\theta(F) \subset F \otimes TX$, the inequality

$$\mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \leq \frac{\text{degree}(E)}{\text{rank}(E)} := \mu(E)$$

holds.

5.1 Co–Higgs bundles on Calabi–Yau manifolds

In this subsection we assume that $c_1(TX) \in H^2(X, \mathbb{Q})$ is zero, and the Kähler form ω on X is Ricci–flat. Take a holomorphic vector bundle E on X.

Lemma 5.1. *Let θ be a Higgs field or a co–Higgs field on E such that (E, θ) is semistable. Then the vector bundle E is semistable.*

Proof. Let θ be a co–Higgs field on E such that the co–Higgs bundle (E, θ) is semistable. Assume that E is not semistable. Let F be the maximal semistable subsheaf of E, in other words, F is the first term in the Harder–Narasimhan filtration of E. The maximal semistable subsheaf of E/F will be denoted by F_1, so $\mu_{\max}(E/F) = \mu(F_1)$. Note that we have

$$\mu(F) > \mu(F_1) = \mu_{\max}(E/F)\,. \tag{5.1}$$

Since ω is Ricci–flat we know that TX is polystable. The tensor product of a semistable sheaf and a semistable vector bundle is semistable [AB, p. 212, Lemma 2.7]. Therefore, the maximal semistable subsheaf of $(E/F) \otimes TX$ is

$$F_1 \otimes TX \subset (E/F) \otimes TX\,.$$

Now,

$$\mu(F_1 \otimes TX) = \mu(F_1)$$

because $c_1(TX) = 0$. Hence from (5.1) it follows that

$$\mu(F) > \mu(F_1 \otimes TX) = \mu_{\max}((E/F) \otimes TX)\,. \tag{5.2}$$

Let

$$q : E \longrightarrow E/F$$

be the quotient homomorphism. From (5.2) it follows that there is no nonzero homomorphism from E to $(E/F) \otimes TX$. In particular, the composition

$$F \hookrightarrow E \xrightarrow{\theta} E \otimes TX \xrightarrow{q \otimes \mathrm{Id}} (E/F) \otimes TX$$

vanishes identically. This immediately implies that $\theta(F) \subset F \otimes TX$. Therefore, the co–Higgs subsheaf $(F, \theta|_F)$ of (E, θ) violates the inequality in the definition of semistability. But this contradicts the given condition that (E, θ) is semistable. Hence we conclude that E is semistable.

Note that Ω_X is polystable because TX is polystable. Hence the above proof also works when the co-Higgs field θ is replaced by a Higgs field. □

A particular case of this result was shown in [Ra2] for X a K3 surface. Moreover, a result implying this Lemma was proved in [BH].

5.2 A characterization of Calabi–Yau manifolds

Theorem 5.2. *Let X be a compact connected Kähler manifold such that for every Kähler class $[\omega] \in H^2(X, \mathbb{R})$ on it the following two hold:*

1. *the tangent bundle TX is semistable, and*

2. *for every semistable Higgs or co–Higgs bundle (E, θ) on X, the underlying holomorphic vector bundle E is semistable.*

Then $c_1(TX) = 0$.

Proof. We will show that degree$(TX) = 0$ for every Kähler class on X. For this, take any Kähler class $[\omega]$.

First assume that degree$(TX) > 0$ with respect to $[\omega]$. We will construct a co–Higgs field on the holomorphic vector bundle

$$E := \mathcal{O}_X \oplus TX \,. \tag{5.3}$$

Since the vector bundle $\mathrm{Hom}(TX, \mathcal{O}_X) = \Omega_X$ is a direct summand End(E), we have

$$\mathrm{End}(TX) = \Omega_X \otimes TX = \mathrm{Hom}(TX, \mathcal{O}_X) \otimes TX \subset \mathrm{End}(E) \otimes TX \,.$$

Hence $\mathrm{Id}_{TX} \in H^0(X, \mathrm{End}(TX))$ is a co–Higgs field on E; this co–Higgs field on E will be denoted by θ.

We will show that the co–Higgs bundle (E, θ) is semistable.

For show that, take any coherent analytic subsheaf $F \subset E$ such that $\theta(F) \subset F \otimes TX$. First consider the case where

$$F \bigcap (0, TX) = 0 \,.$$

Then the composition

$$F \hookrightarrow E = \mathcal{O}_X \oplus TX \longrightarrow \mathcal{O}_X$$

is injective. Hence

$$\mu(F) \leq \mu(\mathcal{O}_X) = 0 < \mu(E) \,.$$

Hence the co–Higgs subsheaf $(F, \theta|_F)$ of (E, θ) does not violate the inequality condition for semistability.

Next assume that

$$F \bigcap (0, TX) \neq 0 \,.$$

Now in view of the given condition that $\theta(F) \subset F \otimes TX$, from the construction of the co–Higgs field θ is follows immediately that

$$F \bigcap (\mathcal{O}_X,\, 0) \neq 0\,.$$

Hence we have

$$F = (F \bigcap (0,\, TX)) \oplus (F \bigcap (\mathcal{O}_X,\, 0))\,. \tag{5.4}$$

Note that

$$\mu(F \bigcap (0,\, TX)) \leq \mu(TX)$$

because TX is semistable, and also we have $\mu(F \bigcap (\mathcal{O}_X,\, 0)) \leq \mu(\mathcal{O}_X)$. Therefore, from (5.4) it follows that

$$\mu(F) \leq \mu(E)\,.$$

Hence again the co–Higgs subsheaf $(F,\, \theta|_F)$ of $(E,\, \theta)$ does not violate the inequality condition for semistability. So $(E,\, \theta)$ is semistable.

Hence by the given condition, the holomorphic vector bundle E is semistable. But this implies that $\text{degree}(TX) = 0$. This contradicts the assumption that $\text{degree}(TX) > 0$.

Now assume that $\text{degree}(TX) < 0$. We will construct a Higgs field on the vector bundle E in (5.3).

The vector bundle $\text{Hom}(\mathcal{O}_X,\, TX) = TX$ is a direct summand $\text{End}(E)$. Hence we have

$$\text{End}(TX) = TX \otimes \Omega_X = \text{Hom}(\mathcal{O}_X,\, TX) \otimes \Omega_X \subset \text{End}(E) \otimes \Omega_X\,.$$

Consequently, $\text{Id}_{TX} \in H^0(X,\, \text{End}(TX))$ is a Higgs field on E; this Higgs field on E will be denoted by θ'.

We will show that the above Higgs vector bundle $(E,\, \theta)$ is semistable.

Take any coherent analytic subsheaf

$$F \subset E$$

such that $\theta(F) \subset F \otimes \Omega_X$ and $\text{rank}(F) < \text{rank}(E)$. First consider the case where

$$F \bigcap (\mathcal{O}_X,\, 0) = 0\,.$$

Then we have $F \subset (0,\, TX) \subset E$. Since TX is semistable, we have

$$\mu(F) \leq \mu(TX)\,.$$

On the other hand, $\mu(TX) < \mu(E)$, because $\text{degree}(TX) < 0 = \mu(\mathcal{O}_X)$. Combining these we get

$$\mu(F) < \mu(E)\,,$$

and consequently, the Higgs subsheaf $(F, \theta|_F)$ of (E, θ) does not violate the inequality condition for semistability.

Now assume that

$$F \bigcap (\mathcal{O}_X, 0) \neq 0 .$$

Hence

$$\operatorname{rank}(F \bigcap (\mathcal{O}_X, 0)) = 1 , \tag{5.5}$$

because $F \bigcap (\mathcal{O}_X, 0)$ is a nonzero subsheaf of $\mathcal{O}_X$. Now from the construction of the Higgs field θ it follows that

$$\operatorname{rank}(F \bigcap (0, TX)) = \operatorname{rank}(TX) .$$

Combining this with (5.5) we conclude that $\operatorname{rank}(F) = \operatorname{rank}(E)$. This contradicts the assumption that $\operatorname{rank}(F) < \operatorname{rank}(E)$. Hence we conclude that the Higgs vector bundle (E, θ) is semistable.

Now the given condition says that E is semistable, which in turn implies that

$$\operatorname{degree}(TX) = 0 .$$

This contradicts the assumption that $\operatorname{degree}(TX) < 0$.

Therefore, we conclude that $\operatorname{degree}(TX) = 0$ for all Kähler classes $[\omega]$ on X. In other words,

$$c_1(TX) \cup ([\omega])^{d-1} = 0 \tag{5.6}$$

for every Kähler class $[\omega]$ on X, where d as before is the complex dimension of d. But the $\mathbb{R}$–linear span of

$$\{[\omega]^{d-1} \in H^{2d-2}(X, \mathbb{R}) \mid [\omega] \quad \text{Kähler class}\}$$

is the full $H^{2d-2}(X, \mathbb{R})$. Therefore, from (5.6) it follows that

$$c_1(TX) \cup \delta = 0$$

for all $\delta \in H^{2d-2}(X, \mathbb{R})$. Now from the Poincaré duality it follows that $c_1(TX) \in H^2(X, \mathbb{R})$ vanishes. □

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Hitchin Pairs for non-compact real Lie groups

by Peter B. Gothen[1]

Abstract

Hitchin pairs on Riemann surfaces are generalizations of Higgs bundles, allowing the Higgs field to be twisted by an arbitrary line bundle. We consider this generalization in the context of G-Higgs bundles for a real reductive Lie group G. We outline the basic theory and review some selected results, including recent results by Nozad and the author [32] on Hitchin pairs for the unitary group of indefinite signature $\mathrm{U}(p,q)$.

1 Introduction

Let X be a closed Riemann surface with holomorphic cotangent bundle $K = \Omega^1_X$. A rank n *Higgs bundle* on X is a pair (E,ϕ), where $E \to X$ is a rank n holomorphic vector bundle and $\phi\colon E \to E \otimes K$ is an endomorphism valued holomorphic 1-form on X. Higgs bundles are fundamental objects in the non-abelian Hodge theorem [20, 22, 37, 60]. In the simplest (abelian) case of $n = 1$ this can be expressed as the isomorphism

$$\mathrm{Hom}(\pi_1 X, \mathbb{C}^*) \simeq T^* \operatorname{Jac}(X),$$

whose infinitesimal version gives the Hodge decomposition $H^1(X,\mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X)$. Thus, for $n = 1$, a flat line bundle on X corresponds to a pair (E,ϕ) consisting of a holomorphic line bundle $E \to X$ and a holomorphic 1-form ϕ on X. For general n, non-abelian Hodge theory produces an isomorphism

$$\mathrm{Hom}(\pi_1 X, \mathrm{GL}(n,\mathbb{C})) /\!\!/ \mathrm{GL}(n,\mathbb{C}) \simeq \mathcal{M}(\mathrm{GL}(n,\mathbb{C})).$$

Here the space on the right hand side is the moduli space of isomorphism classes of Higgs bundles (of degree 0) and the space on the left hand side is the space of representations of $\pi_1 X$ modulo the action of $\mathrm{GL}(n,\mathbb{C})$ by overall conjugation.

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Note that, viewed in this way, the non-abelian Hodge theorem generalizes the Narasimhan–Seshadri theorem [54] to non-compact groups.

For many purposes, rather than considering ϕ as a 1-form, one might as well consider pairs (E, ϕ), where $\phi\colon E \to E \otimes L$ is twisted by an arbitrary line bundle $L \to X$. Such a pair is known as a *Hitchin pair* or a *twisted Higgs bundle*. This point of view was probably first explored systematically by Nitsure [55]. The non-abelian Hodge theorem generalizes to this context and involves, on the one side, meromorphic Higgs bundles and on the other side meromorphic connections. This generalization has been carried out by Simpson [61], for Higgs fields with simple poles, and Biquard–Boalch [5], for more general polar parts (see Boalch [8] for a survey).

Another generalization of the non-abelian Hodge theorem has to do with representations of $\pi_1 X$ in groups G other than the general linear group. This already goes back to Hitchin's seminal papers [37, 38, 39] and indeed was also treated by Simpson [62]. Here we shall focus on the theory for real G, which has quite a different flavour from the theory for complex G. A systematic approach to non-abelian Hodge theory for real reductive groups G and applications to the study of character varieties has been explored in a number of papers; see, for example, [30, 12, 24, 25]. The focus of the present paper are the objects which are obtained by allowing for an arbitrary twisting line bundle L in G-Higgs bundles rather than just the canonical bundle K. These objects are known as *G-Hitchin pairs*.

There are many other important aspects of Higgs bundle theory and without any pretense of completeness, we mention here a few. One of the important features of the Higgs bundle moduli space for complex G is that it is an algebraically completely integrable Hamiltonian system (see Hitchin[38]), known as the Hitchin system. This is closely related to the fact that this moduli space is a holomorphic symplectic manifold admitting a hyper-Kähler metric. This aspect of the theory can be generalized to Hitchin pairs using Poisson geometry, as pioneered by Bottacin [9] and Markman [47]; see Biquard–Boalch [5] for the existence of hyper-Kähler metrics on the symplectic leaves. Closely related is the theory of parabolic Higgs bundles (see, for example, Konno [45] and Yokogawa [67]). Parabolic G-Higgs bundles for real G have been considered by, among others, Logares [46], García-Prada–Logares–Muñoz [28] and Biquard–García-Prada–Mundet [6]. Higgs bundles also play an important role in mirror symmetry (see, for example, Hausel–Thaddeus [34]) and in the geometric Langlands correspondence (see, for example, Kapustin–Witten [42]). Also a number of results on G-Higgs bundles for real groups can be obtained via the study of the Hitchin fibration; for this we refer the reader to Baraglia–Schaposnik [4], García-Prada–Peón-Nieto–Ramanan [29], Hitchin–Schaposnik [40], Peón-Nieto [57] and Schaposnik [58], as well as further references found therein.

In this paper we describe the basics of the theory of G-Hitchin pairs and give a few examples (Section 2). We explain the Hitchin–Kobayashi correspondence which relates the (parameter dependent) stability condition for G-Hitchin pairs to solutions to Hitchin's gauge theoretic equations (Section 3). We then describe recent work of Nozad and the author [32] on $\mathrm{U}(p,q)$-Hitchin pairs (introduced in Section 4), the Milnor–Wood inequality for such pairs (Section 5) and how wall-crossing arguments can be used to study their moduli (Section 6).

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2 Hitchin pairs for real groups

Let G be a connected real reductive Lie group. Following Knapp [44], we shall take this to mean that the following data has been fixed:

- a maximal compact subgroup $H \subset G$;
- a Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$;
- a non-degenerate $\mathrm{Ad}(G)$-invariant quadratic form, negative definite on $\mathfrak{h}$ and positive definite on $\mathfrak{m}$, which restricts to the Killing form on the semisimple part $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$.

Note that the above data complexify (with the possible exception of G) and that there is an isotropy representation

$$\iota\colon H^{\mathbb{C}} \to \mathrm{Aut}(\mathfrak{m}^{\mathbb{C}})$$

coming from restricting and complexifying the adjoint representation of G.

Let X be Riemann surface and let $K = \Omega^1_X$ be its holomorphic cotangent bundle. Fix a line bundle $L \to X$. For a principal $H^{\mathbb{C}}$-bundle $E \to X$ and a representation $\rho\colon H^{\mathbb{C}} \to \mathrm{GL}(V)$ of $H^{\mathbb{C}}$, we denote the associated vector bundle by $E(V) = E \times_\rho V$.

Definition 2.1. A *G-Hitchin pair* (twisted by L) on X is a pair (E,ϕ), where $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle and $\phi \in H^0(X, L \otimes E(\mathfrak{m}^{\mathbb{C}}))$ is a holomorphic 1-form with values in the vector bundle defined by the isotropy representation of $H^{\mathbb{C}}$. If $L = K$, the pair (E,ϕ) is called a *G-Higgs bundle.*

Example 2.2. If G is compact, a G-Hitchin pair is nothing but a holomorphic principal $G^{\mathbb{C}}$-bundle.

Example 2.3. If $G = \mathrm{GL}(n, \mathbb{C})$, a G-Hitchin pair is a pair (E, ϕ), where $E \to X$ is a rank n holomorphic vector bundle and $\phi \in H^0(X, L \otimes \mathrm{End}(E))$ is an L-twisted endomorphism of E. A $\mathrm{SL}(n, \mathbb{C})$-Higgs bundle is given by the same data, with the additional requirements that $\det(E) = \mathcal{O}_X$ and $\phi \in H^0(X, L \otimes \mathrm{End}_0(E))$, where $\mathrm{End}_0(E) \subset \mathrm{End}(E)$ is the subbundle of ϕ with $\phi = 0$.

Example 2.4. Let $G = \mathrm{SL}(n, \mathbb{R})$. A maximal compact subgroup is $\mathrm{SO}(n)$ defined by the standard inner product $\langle x, y \rangle = \sum x_i y_i$ and the isotropy representation is the subspace of $A \in \mathfrak{sl}(n, R)$ which are symmetric with respect to the inner product:

$$\langle Ax, y \rangle = \langle x, Ay \rangle.$$

Hence a $\mathrm{SL}(n, \mathbb{R})$-Hitchin pair can be viewed as a pair $((U, Q), \phi)$, where (U, Q) is a holomorphic orthogonal bundle, i.e., $U \to X$ is a rank n vector bundle with a non-degenerate holomorphic quadratic form Q, and $\phi \in H^0(X, L \otimes S^2_Q U)$. Here $S^2_Q U \subset \mathrm{End}(U)$ denotes the subbundle of endomorphism of U, which are symmetric with respect to Q.

Example 2.5. Let $G = \mathrm{U}(p, q)$, the group of linear transformations of $\mathbb{C}^{p+q}$ which preserves an indefinite hermitian form of signature (p, q) on $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$. Taking the obvious $\mathrm{U}(p) \times \mathrm{U}(q)$ as the maximal compact subgroup, we have $H^{\mathbb{C}} = \mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C})$ and the isotropy representation is

$$\mathrm{GL}(p, \mathbb{C}) \times \mathrm{GL}(q, \mathbb{C}) \to \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$$

acting by restricting the adjoint representation of $\mathrm{GL}(p+q, \mathbb{C})$. Hence a $\mathrm{U}(p, q)$-Hitchin pair can be identified with a quadruple (V, W, β, γ), where

$$\beta \in H^0(L \otimes \mathrm{Hom}(W, V)) \quad \text{and} \quad \gamma \in H^0(L \otimes \mathrm{Hom}(V, W)).$$

The $\mathrm{GL}(p+q, \mathbb{C})$-Hitchin pair associated via the inclusion $\mathrm{U}(p, q) \subset \mathrm{GL}(p+q, \mathbb{C})$ is (E, ϕ), where $E = V \oplus W$ and $\phi = \left(\begin{smallmatrix} 0 & \beta \\ \gamma & 0 \end{smallmatrix}\right)$. Of course a $\mathrm{SU}(p, q)$-Hitchin pair is given by the same data, with the additional requirement that $\det(V) \otimes \det(W) = \mathcal{O}_X$.

Example 2.6. Let $G = \mathrm{Sp}(2n, \mathbb{R})$, the real symplectic group in dimension $2n$, defined as the subgroup of $\mathrm{SL}(2n, \mathbb{R})$ of transformations of $\mathbb{R}^{2n}$ preserving the standard symplectic form, which can be written in coordinates $(x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ as

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n.$$

Then a $\mathrm{Sp}(2n, \mathbb{R})$-Hitchin pair can identified with a triple (V, β, γ), where $V \to X$ is a rank n vector bundle and

$$\beta \in H^0(L \otimes S^2 V) \quad \text{and} \quad \gamma \in H^0(L \otimes S^2 V^*).$$

Note how the inclusions $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SL}(2n, \mathbb{R})$ and $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{SU}(n, n)$ are reflected in the associated vector bundle data. In the former case, the rank $2n$ orthogonal bundle (U, Q) is given by $U = V \oplus V^*$ with the quadratic form $Q = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

3 The Hitchin–Kobayashi correspondence

We now move on to the central notion of stability for G-Hitchin pairs. The stability condition depends on a parameter $c \in i\mathfrak{z}$, where $\mathfrak{z}$ denotes the centre of $\mathfrak{h}$.

From the point of view of construction of moduli spaces, stability allows for a GIT construction of the moduli space $\mathcal{M}_d^c(X, G)$ of c-semistable G-Higgs bundles for a fixed topological invariant $d \in \pi_1(H)$; this construction has been carried out by Schmitt (see [59]).

On the other hand, there is a Hitchin–Kobayashi correspondence for G-Higgs bundles, which gives necessary and sufficient conditions in terms of stability for the existence of solutions to the so-called Hitchin's equations. To state these equations, we need some notation. By a hermitian metric on the $H^{\mathbb{C}}$-bundle E we mean a reduction of structure group to $H \subset H^{\mathbb{C}}$, i.e., a smooth section $h\colon X \to E(H^{\mathbb{C}}/H)$. We denote the corresponding principal H-bundle by E_h. Note that h defines a compact real structure, denoted by σ_h, on the bundle of Lie algebras $E(\mathfrak{g}^{\mathbb{C}})$, compatible with the decomposition $E(\mathfrak{g}^{\mathbb{C}}) = E(\mathfrak{h}^{\mathbb{C}}) \oplus E(\mathfrak{m}^{\mathbb{C}})$. If we combine σ_h with the conjugation on complex 1-forms on X, we obtain a complex antilinear involution $A^1(E(\mathfrak{g}^{\mathbb{C}})) \to A^1(E(\mathfrak{g}^{\mathbb{C}}))$. This restricts to an antilinear map which, by a slight abuse of notation, we denote by the same symbol:

$$\sigma_h\colon A^{1,0}(E(\mathfrak{m}^{\mathbb{C}})) \to A^{0,1}(E(\mathfrak{m}^{\mathbb{C}})).$$

Fix a hermitian metric h_L on L and let ω_X denote the Kähler form of a metric on X compatible with its complex structure, normalized so that $\int_X \omega_X = 2\pi$. Then, for $c \in i\mathfrak{z}$, Hitchin's equation for a metric h on E is the following

$$F(A_h) + [\phi, \sigma_h(\phi)]\omega_X = -ic\omega_X. \tag{3.1}$$

Here A_h denotes the Chern connection on E_h (i.e., the unique H-connection compatible with the holomorphic structure on E) and $F(A_h)$ its curvature. Moreover, the bracket $[\phi, \sigma_h(\phi)]$ is defined by combining the Lie bracket on $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ with the contraction $L \otimes \overline{L} \to \mathcal{O}_X$ given by the metric h_L. Note also that in the case when $L = K$, the second term on the left hand side can be written simply as $[\phi, \sigma_h(\phi)]$ where the bracket on the Lie algebra is now combined with the wedge product on forms.

In order to state the Hitchin–Kobayashi correspondence for G-Hitchin pairs, giving necessary and sufficient conditions for the existence of solutions to the Hitchin equation, one needs an appropriate stability condition. The general condition needed can be found in [24] (based, in turn, on Bradlow–García-Prada-Mundet [14] and Mundet [41]). It is fairly involved to state in general, so we shall refer the reader to loc. cit. for the full statement and here just give a couple of examples which cover our present needs. Note that, just as the Hitchin equation, the stability condition will depend on a parameter $c \in i\mathfrak{z}$.

Example 3.1. (Cf. Hitchin[37], Simpson [60, 62].) Consider $\mathrm{GL}(n,\mathbb{C})$-Hitchin pairs (E,ϕ), where $E \to X$ is a rank n vector bundle and $\phi \in H^0(X, L\otimes \mathrm{End}(E))$. Recall that the *slope* of a vector bundle E on X is the ratio between its degree and its rank: $\mu(E) = \deg(E)/\operatorname{rk}(E)$. A $\mathrm{GL}(n,\mathbb{C})$-Hitchin pair (E,ϕ) is *semistable* if

$$\mu(F) \leq \mu(E) \tag{3.2}$$

for all non-zero subbundles $F \subset E$ which are preserved by ϕ, i.e., such that $\phi(F) \subset F \otimes L$. Moreover, (E,ϕ) is *stable* if additionally strict inequality holds in (3.2) whenever $F \neq E$. Finally, (E,ϕ) is polystable if it is the direct sum of stable Higgs bundles, all of the same slope. In this case $i\mathfrak{z} \simeq \mathbb{R}$ and the stability parameter is fixed to be the real constant $c = \mu(E)$. Note that this constraint is of a topological nature and can be obtained from Chern–Weil theory by integrating the trace of the Hitchin equation, which in this case is:

$$F(A_h) + [\phi, \phi^{*_h}]\omega_X = -ic\,\mathrm{Id}\,\omega_X.$$

Example 3.2. (Cf. [10].) Consider $\mathrm{U}(p,q)$-Hitchin pairs (V,W,β,γ). In this case, $i\mathfrak{z} \simeq \mathbb{R}\times\mathbb{R}$ and the Hitchin equation becomes

$$\begin{aligned} F(A_h(V)) + (\beta\beta^{*_h} - \gamma^{*_h}\gamma)\omega_X &= -ic_1\,\mathrm{Id}_V\,\omega_X, \\ F(A_h(W)) + (\gamma\gamma^{*_h} - \beta^{*_h}\beta)\omega_X &= -ic_2\,\mathrm{Id}_W\,\omega_X. \end{aligned} \tag{3.3}$$

Here $A_h(V)$ and $A_h(W)$ denote the Chern connections on V and W, respectively, and the parameter $(c_1,c_2) \in \mathbb{R}\times\mathbb{R}$ is constrained by Chern–Weil theory by

$$\frac{p}{p+q}c_1 + \frac{q}{p+q}c_2 = \mu(V\oplus W).$$

The stability condition is most conveniently described by introducing the *α-slope* of (V,W,β,γ) by

$$\mu_\alpha(V,W,\beta,\gamma) = \mu(V\oplus W) + \alpha\frac{p}{p+q}$$

for a real parameter α, related to (c_1,c_2) by $\alpha = c_2 - c_1$. The α-stability conditions are completely analogous to the ones of Example 3.1, but applied to $\mathrm{U}(p',q')$-subbundles, defined in the obvious way by $V' \subset V$ and $W' \subset W$ such that $\beta(W') \subset V'\otimes L$ and $\gamma(V') \subset W' \otimes L$.

The Hitchin–Kobayashi correspondence for G-Hitchin pairs [37, 62, 14, 24] can now be stated as follows.

Theorem 3.3. *Let (E,ϕ) be a G-Hitchin pair. There exists a hermitian metric h in E solving Hitchin's equation (3.1) if and only if (E,ϕ) is c-polystable. Moreover, the solution h is unique up to H-gauge transformations of E_h.*

Next we explain how to give an interpretation in terms of moduli spaces. Fix a C^∞ principal H-bundle $\mathcal{E}$ of topological class $d \in \pi_1 H$ and consider the configuration space of G-Higgs pairs on $\mathcal{E}$:

$$\mathcal{C}(\mathcal{E}) = \{(\bar\partial_A, \phi) \mid \bar\partial_A \phi = 0\}.$$

Here $\bar\partial_A$ is a $\bar\partial$-operator on $\mathcal{E}$ defining a structure of holomorphic principal $H^{\mathbb{C}}$-bundle $E_A \to X$ and the C^∞-Higgs field $\phi \in A^{1,0}(\mathcal{E}(\mathfrak{m}^{\mathbb{C}}))$. Let $\mathcal{C}^{c-\mathrm{ps}}(\mathcal{E}) \subset \mathcal{C}(\mathcal{E})$ be the subset of c-polystable G-Higgs pairs. The *complex gauge group* $\mathcal{G}^{\mathbb{C}}$ is the group of C^∞ automorphisms of the principal $H^{\mathbb{C}}$-bundle $\mathcal{E}_{\mathbb{C}}$ obtained by extending the structure group to the complexification $H^{\mathbb{C}}$ of H. It acts on $\mathcal{C}^{c-\mathrm{ps}}(\mathcal{E})$ and we can identify, as sets[2],

$$\mathcal{M}^c_d(X, G) = \mathcal{C}^{c-\mathrm{ps}}(\mathcal{E})/\mathcal{G}^{\mathbb{C}}.$$

Now consider Hitchin's equation (3.1) as an equation for a pair (A, ϕ) of a (metric) connection A on $\mathcal{E}$ and a Higgs field $\phi \in A^{1,0}(\mathcal{E}(\mathfrak{m}^{\mathbb{C}}))$. The complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts transitively on the space of metrics on $\mathcal{E}$ with stabilizer the *unitary gauge group* $\mathcal{G}$, by which we understand the C^∞ automorphism group of the H-bundle $\mathcal{E}$. Thus the Hitchin–Kobayashi correspondence of Theorem 3.3 says that there is a complex gauge transformation taking (A, ϕ) to a solution to Hitchin's equation if and only if (E_A, ϕ) is c-polystable, and this solution is unique up to unitary gauge transformation. In other words, we have a bijection

$$\mathcal{M}^c_d(X, G) \simeq \{(A, \phi) \mid (A, \phi) \text{ satisfies (3.1)}\}/\mathcal{G}. \tag{3.4}$$

When G is compact, there is no Higgs field and the Hitchin equation simply says that the Chern connection is (projectively) flat. Hence (3.4) identifies the moduli space of semistable $G^{\mathbb{C}}$-bundles with the moduli space of (projectively) flat G-connections. This latter space can in turn be identified with the *character variety* of representations of (a central extension of) the fundamental group of X in G.

For non-compact G, assume that $L = K$ and that the parameter $c \in iZ(\mathfrak{g})$. Then the Hitchin equation can be interpreted as a (projective) flatness condition for the G-connection B defined by

$$B = A_h + \phi - \sigma_h(\phi). \tag{3.5}$$

It is a fundamental theorem of Donaldson [22] and (more generally) Corlette [20] that for any flat reductive[3] connection B on a principal G-bundle $\mathcal{E}_G$, there exists a so-called harmonic metric on $\mathcal{E}_G$. A consequence of harmonicity is that when

[2] Indeed a construction of the moduli space using complex analytic methods in the style of Kuranishi should be possible, though we are not aware of the existence of such a construction in the literature.

[3] When G is linear this simply means that the holonomy representation is completely reducible.

the metric is used to decompose B as in (3.5), then (A, ϕ) satisfies the Hitchin equation. Combining this with the Hitchin–Kobayashi correspondence gives the non-abelian Hodge theorem[4]: an identification between the moduli space of G-Higgs bundles and the character variety for representations of (a central extension of) $\pi_1 X$ in G.

Example 3.4. If we want to apply the non-abelian Hodge theorem to $\mathrm{U}(p,q)$-Higgs bundles, we need to fix the parameter in Hitchin's equation to be in the centre of $\mathrm{U}(p,q)$, i.e., in the notation of Example 3.2, we must take $c_1 = c_2 = c = \mu(V \oplus W)$. Of course this corresponds to the value for $\mathrm{GL}(n,\mathbb{C})$-Higgs bundles under the inclusion $\mathrm{U}(p,q) \subset \mathrm{GL}(p+q,\mathbb{C})$ (cf. Examples 2.5 and 3.1).

4 Hitchin pairs for $\mathrm{U}(p,q)$ and quiver bundles

We saw in Example 3.2, that there is a degree of freedom in the choice of stability parameter for $\mathrm{U}(p,q)$-Hitchin pairs. There is another way of viewing this parameter dependence for the stability condition, which is to notice that a $\mathrm{U}(p,q)$-Hitchin pair can be viewed as a *quiver bundle* (see, e.g., King [43], Álvarez-Cónsul–García-Prada [1, 2], and also [31]). To explain this, recall that a quiver Q is an oriented graph (which we shall assume to be finite), given by a set of vertices Q_0, a set of arrows Q_1 and head and tail maps

$$h, t\colon Q_1 \to Q_0.$$

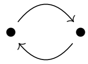
For each $a \in Q_1$, let $M_a \to X$ be a holomorphic vector bundle on X and let $M = \{M_a\}$ be the collection of these *twisting bundles*.

Definition 4.1. A *Q-bundle twisted by M* on X is a collection of holomorphic vector bundles $E_i \to X$ indexed by the vertices $i \in Q_0$ of Q and a collection of holomorphic maps $\phi_a\colon M_a \otimes E_{ta} \to E_{ha}$ indexed by the arrows $a \in Q_1$ of Q.

Remark 4.2. It is easy to see that Q-bundles on X form a category which can be made into an abelian category by considering coherent Q-sheaves, in a way analogous to what happens for vector bundles.

It should now be clear that L-twisted $\mathrm{U}(p,q)$-Hitchin pairs can be viewed as Q-bundles for the quiver

(4.1)



where both arrows are twisted by L^*.

[4] See the references cited in the Introduction for the generalization to the meromorphic situation.

There is a natural stability condition for quiver bundles which, just as for Hitchin pairs, gives necessary and sufficient conditions for the existence of solutions to certain natural gauge theoretic equations (cf. King [43] and Álvarez-Cónsul–García-Prada [1, 2]). This condition depends on a parameter vector

$$\alpha = (\alpha_i)_{i \in Q_0} \in \mathbb{R}^{Q_0}$$

and it is defined using the *α-slope* of a Q-bundle E:

$$\mu_\alpha(E) = \frac{\sum_i (\deg(E_i) + \alpha_i \operatorname{rk}(E_i))}{\sum_i \operatorname{rk}(E_i)}.$$

Thus E is α-stable if for any proper non-zero sub-Q-bundle E' of E, we have

$$\mu_\alpha(E') < \mu_\alpha(E),$$

and α-semi- and polystability are defined just as for vector bundles.

Note that the stability condition is unchanged under an overall translation of the stability parameter

$$(\alpha_i) \mapsto (\alpha_i + a)$$

for any constant $a \in \mathbb{R}$. Thus we may as well take $\alpha_0 = 0$ and we see that the number of effective stability parameters is $|Q_0| - 1$. In the case of Q-bundles for the quiver (4.1), i.e., $\mathrm{U}(p, q)$-Hitchin pairs, we then have one real parameter $\alpha = \alpha_1$ and the general Q-bundle stability condition reproduces the stability for $\mathrm{U}(p, q)$-Hitchin pairs of Example 3.2.

5 The Milnor–Wood inequality for $\mathrm{U}(p, q)$-Hitchin pairs

The Milnor–Wood inequality has its origins [48, 66] in the theory of flat bundles. From this point of view there is a long sequence of generalizations and important contributions (see, for example, Dupont [23], Toledo [64], Domic–Toledo [21], Turaev [65], Clerc–Ørsted [19], Burger–Iozzi-Wienhard [17, 18]). Here we shall, however, focus on its Higgs bundle incarnation, again first considered by Hitchin [37]. From this point of view it is a bound on the topological class of a $\mathrm{U}(p, q)$-Hitchin pair. In order to state it we need the following definition.

Definition 5.1. Let $E = (V, W, \beta, \gamma)$ be a $\mathrm{U}(p, q)$-Hitchin pair. The *Toledo invariant* of E is

$$\tau(E) = \frac{2pq}{p+q}\big(\mu(V) - \mu(W)\big).$$

Note that, if we set $a = \deg(V)$ and $b = \deg(W)$, then we can write $\tau(E) = 2(qa - pb)/(p + q)$.

The Milnor–Wood inequality for $\mathrm{U}(p, q)$-Hitchin pairs can now be stated as follows:

Proposition 5.2 (Gothen–Nozad [32, Proposition 3.3]). *Let* $E = (V, W, \beta, \gamma)$ *be an* α*-semistable* $\mathrm{U}(p,q)$*-Hitchin pair with twisting line bundle* L*. Then*

$$-\operatorname{rk}(\beta)\deg(L) + \alpha\big(\operatorname{rk}(\beta) - \frac{2pq}{p+q}\big) \le \tau(E) \le \operatorname{rk}(\gamma)\deg(L) + \alpha\big(\operatorname{rk}(\gamma) - \frac{2pq}{p+q}\big).$$

The proof is analogous to the one for $\mathrm{U}(p,q)$-Higgs bundles in [10]. It applies the α-semistability condition for $\mathrm{U}(p,q)$-Hitchin pairs to certain subobjects defined in a natural way using β and γ. We refer the reader to [32] for details.

Remark 5.3. The Toledo invariant has been defined for G-Higgs bundles for any non-compact simple reductive group G of hermitian type by Biquard–García-Prada–Rubio [7]. These authors also prove a very general Milnor–Wood inequality for such G-Higgs bundles. In the case when $L = K$ their theorem specializes to our Proposition 5.2.

The inequality of Proposition 5.2 has several interesting consequences, for example we get the following bounds on the Toledo invariant (cf. [32, Proposition 3.4]).

Proposition 5.4. *Let* $E = (V, W, \beta, \gamma)$ *be an* α*-semistable* $\mathrm{U}(p,q)$*-Hitchin pair with twisting line bundle* L *with* $\deg(L) \ge 0$*. Then the following hold:*

(*i*) *If* $\alpha \le -\deg(L)$ *then*

$$\min\{p,q\}\big(-\alpha\frac{|p-q|}{p+q} - \deg(L)\big) \le \tau(E) \le -\alpha\frac{2pq}{p+q}.$$

(*ii*) *If* $-\deg(L) \le \alpha \le \deg(L)$ *then*

$$\min\{p,q\}\big(-\alpha\frac{|p-q|}{p+q} - \deg(L)\big) \le \tau(E) \le \min\{p,q\}\big(\deg(L) - \alpha\frac{|p-q|}{p+q}\big).$$

(*iii*) *If* $\deg(L) \le \alpha$ *then*

$$-\alpha\frac{2pq}{p+q} \le \tau(E) \le \min\{p,q\}\big(\deg(L) - \alpha\frac{|p-q|}{p+q}\big).$$

Note, in particular, that for $\alpha = 0$ (the value relevant for the non-abelian Hodge theorem) we have by (*ii*) of the proposition that

$$|\tau(E)| \le \min\{p,q\}\deg(L). \tag{5.1}$$

In the case of $\mathrm{U}(p,q)$-Higgs bundles (i.e., $\alpha = 0$ and $L = K$) this is the usual Milnor–Wood inequality (cf. [10]).

The study of properties of Higgs bundles with extremal values for the Toledo invariant is an interesting question. This has been studied for various specific groups G of hermitian type by Hitchin [37] for $\mathrm{PSL}(2,\mathbb{R})$, Gothen [30] for $\mathrm{Sp}(4,\mathbb{R})$, García-Prada–Gothen–Mundet [25] for $\mathrm{Sp}(2n,\mathbb{R})$, Bradlow–García-Prada–Gothen [10, 11, 16] for $\mathrm{SO}^*(2n)$ and $\mathrm{U}(p,q)$. A general study for G-Higgs bundles for non-compact groups of hermitian type was carried out by Biquard–García-Prada–Rubio [7]. From the point of view of representations of surface groups much work has also been done and without being at all exhaustive, we mention here a few works: Toledo [64], Hernández [36] and Burger–Iozzi–Wienhard [17, 18]. From either point of view, one of the key properties of maximal objects (Higgs bundles or representations) is that they exhibit rigidity phenomena, of which we mention but two examples. Firstly, a classical theorem of Toledo, which states that a maximal representation of $\pi_1 X$ in $\mathrm{U}(p,1)$ factors through $\mathrm{U}(1,1)\times\mathrm{U}(p-1)$. Secondly we mention [10, Proposition 3.30], which says that the moduli space of maximal $\mathrm{U}(p,p)$-Higgs bundles is isomorphic to the moduli space of K^2-twisted Hitchin pairs of rank p — so here Hitchin pairs play an important role even in the theory of usual Higgs bundles. Toledo's theorem and its generalizations for surface group representations have clear parallels on the Higgs bundle side of the non-abelian Hodge theory correspondence. On the other hand, the surface group representation parallel of the second kind of rigidity phenomenon is perhaps less clear; see, however, Guichard–Wienhard [33] for the case of representations in $\mathrm{Sp}(2n,\mathbb{R})$.

6 Wall crossing for $\mathrm{U}(p,q)$-Hitchin pairs

We finish this paper by describing an application of wall-crossing techniques to moduli of $\mathrm{U}(p,q)$-Hitchin pairs, following [56, 32]. These techniques have a long history in the subject, going back at least to Thaddeus' proof [63] of the rank 2 Verlinde formula. The main results on connectedness of moduli spaces of $\mathrm{U}(p,q)$-Higgs bundles from [10] were based on the wall-crossing results for *triples* of [11]: triples are Q-bundles for a quiver with two vertices and one arrow between them, so they correspond to $\mathrm{U}(p,q)$-Hitchin pairs with one of the Higgs fields β or γ vanishing. Later some of these results have been generalized to *holomorphic chains*, i.e., Q-bundles for a quiver of type A_n, see Álvarez-Cónsul–García-Prada–Schmitt [3], García-Prada–Heinloth–Schmitt [27], García-Prada–Heinloth [26] and Heinloth [35]. Similar ideas have also been employed by other authors to study various properties of moduli spaces, including their Hodge numbers, such as the works of Bradlow–García-Prada– Muñoz–Newstead [15], Bradlow–García-Prada–Mercat–Muñoz–Newstead [13], Muñoz [49, 50, 51] and Muñoz–Ortega–Vázquez-Gallo [52, 53].

One common feature of all these results is that they deal with quivers without oriented cycles, corresponding to nilpotent Higgs fields. It is therefore interesting to investigate to what extend the aforementioned results can be generalized to quivers with oriented cycles. Since we need at least two vertices to have effective stability parameters, the simplest possible case is that of $\mathrm{U}(p,q)$-Hitchin pairs, corresponding to the quiver (4.1).

It turns out that a direct generalization of the arguments for triples of [11] runs into difficulties. To explain this, we first remark that the stability condition can only change for certain discrete values of the parameter α, called *critical values*. Fix topological invariants $t = (p,q,a,b)$ of $\mathrm{U}(p,q)$-Hitchin pairs, where $a = \deg(V)$ and $b = \deg(W)$. Then α is a critical value of the stability parameter for $\mathrm{U}(p,q)$-Hitchin pairs of type t if it is numerically possible to have a proper subobject $E' \subset E$ of a $\mathrm{U}(p,q)$-Hitchin pair $E = (V,W,\beta,\gamma)$ of type t such that

$$\mu_\alpha(E') = \mu_\alpha(E) \quad \text{and} \quad \frac{p'}{p'+q'} \neq \frac{p}{p+q} \tag{6.1}$$

(Here the type of E' is $t' = (p',q',a',b')$.) This means that α is critical if and only if it is possible for $\mathrm{U}(p,q)$-Hitchin pairs to exist which are α'-stable for $\alpha' < \alpha$ and α'-unstable for $\alpha' > \alpha$ (and vice-versa). Denote by $\mathcal{M}_{\alpha^\pm}$ the moduli space of $\alpha^\pm$-semistable $\mathrm{U}(p,q)$-Hitchin pairs of type t, where $\alpha^\pm = \alpha \pm \epsilon$ for $\epsilon > 0$ small. Then one is led to introduce "flip loci" $\mathcal{S}_{\alpha^\pm} \subset \mathcal{M}_{\alpha^\pm}$ corresponding to $\mathrm{U}(p,q)$-Hitchin pairs which change their stability properties as the critical value α is crossed. If one can estimate appropriately the codimension of these flip loci, it will follow that $\mathcal{M}_{\alpha^\pm}$ are birationally equivalent. The $\mathrm{U}(p,q)$-Hitchin pairs E in the flip loci have descriptions as extensions

$$0 \to E' \to E \to E'' \to 0$$

for α-semistable $\mathrm{U}(p,q)$-Hitchin pairs (of lower rank) E' and E'' satisfying (6.1). Such extensions are controlled by the first hypercohomology of a two-term complex of sheaves $\underline{Hom}^\bullet(E'',E')$ (see [32, Definition 2.14], cf. [31]). Thus, in order to control the number of extensions one needs vanishing results for the zeroth and second hypercohomology groups. This, together with an analysis of the moduli space for large α, was the strategy followed in [10] to prove irreducibility of moduli spaces of holomorphic triples.

The main difficulty in generalizing this approach to $\mathrm{U}(p,q)$-Hitchin pairs is that the vanishing results do not generalize without additional hypotheses (compare, for example, [11, Proposition 3.6] and [32, Proposition 3.22]). However, for a certain range of the parameter α and the Toledo invariant, things can be made to work. Thus we can obtain birationality of moduli spaces of $\mathrm{U}(p,q)$-Hitchin pairs under certain constraints (see [32, Theorem 5.3]). This combined with the results from [10] on connectedness of moduli of $\mathrm{U}(p,q)$-Higgs bundles finally gives the main result:

Theorem 6.1 ([32, Theorem 5.5]). *Denote by $\mathcal{M}_\alpha(p,q,a,b)$ the moduli space of semistable K-twisted $\mathrm{U}(p,q)$-Hitchin pairs. Suppose that $\tau = \frac{2pq}{p+q}(a/p - b/q)$*

satisfies $|\tau| \leq \min\{p,q\}(2g-2)$. *Suppose also that either one of the following conditions holds:*

(1) $a/p - b/q > -(2g-2)$, $q \leq p$ *and* $0 \leq \alpha < \frac{2pq}{pq-q^2+p+q}\big(b/q - a/p - (2g-2)\big) + 2g-2$,

(2) $a/p - b/q < 2g-2$, $p \leq q$ *and* $\frac{2pq}{pq-p^2+p+q}(b/q - a/p + 2g - 2) - (2g-2) < \alpha \leq 0$.

Then the closure of the stable locus in the moduli space $\mathcal{M}_\alpha(p,q,a,b)$ *is irreducible. In particular, if* $\gcd(p+q, a+b) = 1$, *then* $\mathcal{M}_\alpha(p,q,a,b)$ *is irreducible.*

Remark 6.2. Unless $p = q$, the conditions on $a/b - b/q$ in the theorem are guaranteed by the hypothesis $|\tau| \leq \min\{p,q\}(2g-2)$ (see [32, Remark 5.6]).

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Quadric bundles applied to non-maximal Higgs bundles

by André Oliveira

Abstract

We present a survey on the moduli spaces of rank 2 quadric bundles over a compact Riemann surface X. These are objects which generalise orthogonal bundles and which naturally occur through the study of the connected components of the moduli spaces of Higgs bundles over X for the real symplectic group $\mathrm{Sp}(4,\mathbb{R})$, with non-maximal Toledo invariant. Hence they are also related with the moduli space of representations of $\pi_1(X)$ in $\mathrm{Sp}(4,\mathbb{R})$. We explain this motivation in some detail.

1 Components of Higgs bundles moduli spaces

Higgs bundles over a compact Riemann surface X were introduced by Nigel Hitchin in [25] as a pair (V,φ) where V is a rank 2 and degree d holomorphic vector bundle on X, with fixed determinant, and φ a section of $\mathrm{End}(V)\otimes K$ with trace zero. K denotes the canonical line bundle of X — the cotangent bundle of X. Nowadays those are also known as $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles. In the same paper, Hitchin determined the Poincaré polynomial of the corresponding moduli space $\mathcal{M}_d(\mathrm{SL}(2,\mathbb{C}))$, for d odd. The method was based on Morse-Bott theory, so smoothness of the moduli was an essential feature. It was then clear that $\mathcal{M}_d(\mathrm{SL}(2,\mathbb{C}))$ has an extremely rich topological structure, so a natural question was to ask about the topology of the moduli spaces of Higgs bundles for other groups. For $\mathrm{SL}(3,\mathbb{C})$, this was achieved by P. Gothen in [20] and more recently, and using a new approach, O. García-Prada, J. Heinloth and A. Schmitt in [13, 12] obtained the same for $\mathrm{SL}(4,\mathbb{C})$ and recursive formulas for $\mathrm{SL}(n,\mathbb{C})$. Other recent developments were achieved in [34] on the study of $\mathcal{M}_d(\mathrm{SL}(n,\mathbb{C}))$, which seem to confirm some fascinating conjectures [24]. All these cases were done under the condition of coprimality between rank and degree, so that the moduli spaces are smooth.

However, for a general real, connected, semisimple Lie group G, the moduli spaces $\mathcal{M}_c(G)$ of G-Higgs bundles with fixed topological type $c\in\pi_1(G)$, are non-smooth. This is one of the reasons why the topology of $\mathcal{M}_c(G)$ is basically unknown. Still, their most basic topological invariant — the number of connected components — is a honourable exception in this unknown territory, and much is known about it. If G is compact then $\mathcal{M}_c(G)$ is non-empty and connected for any $c \in \pi_1(G)$ [33] and the same is true if G is complex [17, 28]. In both cases the same holds even if G is just reductive or even non-connected (the only difference is that for non-connected groups, the topological type is indexed not by $\pi_1(G)$, but by a different set [30]). When G is a real group, the situation can be drastically different. There are two cases where extra components are known to occur: when G is a split real form of $G^{\mathbb{C}}$ and when G is a non-compact group of hermitian type.

Suppose G is a split real form of $G^{\mathbb{C}}$. Intuitively this means that G is the "maximally non-compact" real form of $G^{\mathbb{C}}$; see for example [32] for the precise definition. For instance $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{Sp}(2n,\mathbb{R})$ are split real forms of $\mathrm{SL}(n,\mathbb{C})$ and of $\mathrm{Sp}(2n,\mathbb{C})$, respectively. For these groups, Hitchin proved in [27] that there always exists at least one topological type c for which $\mathcal{M}_c(G)$ is disconnected and that the "extra" component is contractible and indeed isomorphic to a vector space — this is the celebrated *Hitchin component* also known as *Teichmüller component*. We will not pursue in this direction here.

A non-compact semisimple Lie group G of *hermitian type* is characterised by the fact that G/H is a hermitian symmetric space, where $H \subset G$ is a maximal compact subgroup. Thus G/H admits a complex structure compatible with the Riemannian structure, making it a Kähler manifold. If G/H is irreducible, the centre of the Lie algebra of H is one-dimensional and this implies that the torsion-free part of $\pi_1(G) = \pi_1(H)$ is isomorphic to $\mathbb{Z}$, hence the topological type gives rise to an integer d (usually the degree of some vector bundle), called the *Toledo invariant*. This Toledo invariant is subject to a bound condition, called the *Milnor-Wood inequality*, beyond which the moduli spaces $\mathcal{M}_d(G)$ are empty. Moreover, when $|d|$ is maximal (and G is of tube type [5]) there is a so-called *Cayler partner phenomena* which implies the existence of extra components for $\mathcal{M}_d(G)$. This has been studied for many classes of hermitian type groups [5] and proved in an intrinsic and general way recently in [2].

On the other hand, the connected components of $\mathcal{M}_d(G)$ for non-maximal and non-zero Toledo invariant are not known in general. One exception is the case of $\mathrm{U}(p,q)$, which has been basically dealt in [3, 4]. Two other exceptions are the cases of $G = \mathrm{Sp}(4,\mathbb{R})$ and of $G = \mathrm{SO}_0(2,3)$ — the identity component of $\mathrm{SO}(2,3)$. In these two cases, it is known [16, 22] that all the non-maximal subspaces are connected for each fixed topological type. Note that in the case of $\mathrm{SO}_0(2,3)$, the topological type is given by an element $(d,w) \in \mathbb{Z} \times \mathbb{Z}/2 = \pi_1(\mathrm{SO}_0(2,3))$, with d being the Toledo invariant; so for each d there are two components, labeled by w. We expect that the same holds true in general, that is, $\mathcal{M}_d(G)$ is connected for non-maximal d and fixed topological type.

In this paper we give an overview of the proof given in [22] of the connectedness of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ and $\mathcal{M}_d(\mathrm{SO}_0(2,3))$ for non-maximal and non-zero d. In this study one is naturally lead to consider a certain type of pairs, which we call *quadric* *bundles*, and the corresponding moduli spaces, depending on a real parameter α. Denote them by $\mathcal{N}_\alpha(d)$. The relevant parameter for the study of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ and $\mathcal{M}_d(\mathrm{SO}_0(2,3))$ is $\alpha = 0$. The idea is to obtain a description of the connected components of $\mathcal{N}_{\alpha_m^-}(d)$, for a specific value α_m^- of the parameter α, and then vary α, analysing the wall-crossing in the spirit of [38, 4]. It turns out that a crucial step in that proof (namely in the description of of $\mathcal{N}_{\alpha_m^-}(d)$) is a detailed analysis of the Hitchin fibration for L-twisted $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles, taking into account *all* the fibres of the Hitchin map and not only the generic ones. This was done in [23], and we briefly describe this analysis.

In the last section of the paper we briefly mention some other results concerning the spaces $\mathcal{N}_\alpha(d)$, obtained in [31], that lead to the description of some geometric and topological properties of these moduli spaces. In particular, these results imply that, under some conditions on d and on the genus of X, a Torelli type theorem holds for $\mathcal{N}_\alpha(d)$.

2 From Higgs bundles to quadric bundles

2.1 Definitions and examples

Let X be a compact Riemann surface of genus $g \geq 2$, with canonical line bundle $K = T^*X^{1,0}$, the holomorphic cotangent bundle. Let G be a real semisimple, connected, Lie group. Fix a maximal compact subgroup $H \subseteq G$ with complexification $H^\mathbb{C} \subseteq G^\mathbb{C}$. If $\mathfrak{h}^\mathbb{C} \subseteq \mathfrak{g}^\mathbb{C}$ are the corresponding Lie algebras, then the Cartan decomposition is $\mathfrak{g}^\mathbb{C} = \mathfrak{h}^\mathbb{C} \oplus \mathfrak{m}^\mathbb{C}$, where $\mathfrak{m}^\mathbb{C}$ is the vector space defined as the orthogonal complement of $\mathfrak{h}^\mathbb{C}$ with respect to the Killing form. Since $[\mathfrak{m}^\mathbb{C}, \mathfrak{h}^\mathbb{C}] \subset \mathfrak{m}^\mathbb{C}$, then $\mathfrak{m}^\mathbb{C}$ is a representation of $H^\mathbb{C}$ via the *isotropy representation* $H^\mathbb{C} \to \mathrm{GL}(\mathfrak{m}^\mathbb{C})$ induced by the adjoint representation $\mathrm{Ad} : G^\mathbb{C} \to \mathrm{GL}(\mathfrak{g}^\mathbb{C})$. If $E_{H^\mathbb{C}}$ is a principal $H^\mathbb{C}$-bundle over X, denote by $E_{H^\mathbb{C}}(\mathfrak{m}^\mathbb{C}) = E_{H^\mathbb{C}} \times_{H^\mathbb{C}} \mathfrak{m}^\mathbb{C}$ the vector bundle associated to $E_{H^\mathbb{C}}$ via the isotropy representation.

Definition 2.1. A *G-Higgs bundle* over X is a pair $(E_{H^\mathbb{C}}, \varphi)$ where $E_{H^\mathbb{C}}$ is a principal holomorphic $H^\mathbb{C}$-bundle and φ is a global holomorphic section of $E_{H^\mathbb{C}}(\mathfrak{m}^\mathbb{C}) \otimes K$, called the *Higgs field*.

In practice we usually replace the principal $H^\mathbb{C}$-bundle $E_{H^\mathbb{C}}$ by the corresponding vector bundle associated to some standard representation of $H^\mathbb{C}$ in some $\mathbb{C}^n$. Let us give two examples.

If $G = \mathrm{SL}(n,\mathbb{C})$, then $H^\mathbb{C} = G$ gives rise to a rank n vector bundle V with trivial determinant and since $\mathfrak{m}^\mathbb{C} = \mathfrak{sl}(n,\mathbb{C})$, the Higgs field φ is a traceless K-twisted endomorphism of V. If we fix the determinant of V to be any line bundle and impose the same traceless condition to $\varphi : V \to V \otimes K$, then we also call the pair (V, φ) an $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle, although it is really a “twisted” $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle. All these are usually just called *Higgs bundles with fixed determinant.* These are the "original" Higgs bundles, introduced in [25].

If $G = \mathrm{Sp}(2n,\mathbb{R})$, we can take $H = \mathrm{U}(n)$ as a maximal compact subgroup. So $H^{\mathbb{C}} = \mathrm{GL}(n,\mathbb{C})$ gives rise to a rank n holomorphic vector bundle V. The Cartan decomposition is $\mathfrak{sp}(2n,\mathbb{C}) = \mathfrak{gl}(n,\mathbb{C}) \oplus \mathfrak{m}^{\mathbb{C}}$ where the inclusion $\mathfrak{gl}(n,\mathbb{C}) \hookrightarrow \mathfrak{sp}(2n,\mathbb{C})$ is given by $A \mapsto \mathrm{diag}(A, -A^T)$.
So $\mathfrak{m}^{\mathbb{C}} = \left\{(B,C) \in \mathfrak{gl}(n,\mathbb{C})^2 \mid B = B^T,\ C = C^T\right\}$. Hence we have that:

Definition 2.2. An $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle is a triple (V,β,γ) where V is a holomorphic rank n vector bundle, $\beta \in H^0(X, S^2V \otimes K)$ and $\gamma \in H^0(X, S^2V^* \otimes K)$.

In an $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle (V,β,γ), we can then think of γ as a map $\gamma : V \to V^* \otimes K$ such that $\gamma^t \otimes \mathrm{Id}_K = \gamma$ and likewise for $\beta : V^* \to V \otimes K$.

A G-Higgs bundle $(E_{H^{\mathbb{C}}},\varphi)$ is topologically classified by the topological invariant of the corresponding $H^{\mathbb{C}}$-bundle $E_{H^{\mathbb{C}}}$, given by an element $\pi_1(H) \cong \pi_1(G)$.

In [14], a general notion of (semi,poly)stability of G-Higgs bundles was developed, allowing for proving a Hitchin–Kobayashi correspondence between polystable G-Higgs bundles and solutions to certain gauge theoretic equations known as *Hitchin equations*. On the other hand, A. Schmitt introduced stability conditions for more general objects, which also apply for the G-Higgs bundles context, and used these in his general construction of moduli spaces; cf. [35]. In particular his stability conditions coincide with the ones relevant for the Hitchin–Kobayashi correspondence. It should be noted that the stability conditions depend on a parameter $\alpha \in \sqrt{-1}\mathfrak{h} \cap \mathfrak{z}$, where $\mathfrak{z}$ is the centre of $\mathfrak{h}^{\mathbb{C}}$. In most cases this parameter is fixed by the topological type, so it really does not play any relevant role. This happens for any compact or complex Lie group and most real groups. Indeed, the only case where the parameter is not fixed by the topology is when G is of hermitian type. This is the case of $\mathrm{Sp}(2n,\mathbb{R})$, so it is important for us to take into account the presence of α.

Denote by $\mathcal{M}_d^\alpha(G)$ the moduli space of S-equivalence classes of α-semistable G-Higgs bundles with topological invariant $d \in \pi_1(G)$. On each S-equivalence class there is a unique (up to isomorphism) α-polystable representative, so we can consider $\mathcal{M}_d^\alpha(G)$ as the moduli space isomorphism classes of α-polystable G-Higgs bundles.

Remark 2.3. Given any line bundle $L \to X$, of non-negative degree, everything we just said generalises to *L-twisted G-Higgs pairs.* The only difference to G-Higgs bundles is that the Higgs field is a section of $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes L$ instead of $E_{H^{\mathbb{C}}}(\mathfrak{m}^{\mathbb{C}}) \otimes K$.

2.2 Higgs bundles for $\mathrm{Sp}(4,\mathbb{R})$ and quadric bundles

2.2.1 Moduli of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles

We already know that an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle is a triple (V,β,γ) with $\mathrm{rk}(V)=2$ and

$$\beta \in H^0(X, S^2V\otimes K), \quad \gamma \in H^0(X, S^2V^*\otimes K).$$

The topological type is given by the degree of V: $d=\deg(V)\in\mathbb{Z}=\pi_1(\mathrm{Sp}(4,\mathbb{R}))$. In fact, $\mathrm{Sp}(4,\mathbb{R})$ is of hermitian type, and the invariant d is the Toledo invariant mentioned in Section 1.

Given a real parameter α, here is the α-(semi)stability condition for $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles; see [14, 15] for the deduction of these conditions.

Definition 2.4. Let (V,β,γ) be an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle with $\deg(V)=d$. It is α-semistable if the following hold:

1. if $\beta=0$ then $d-2\alpha\geq 0$;
2. if $\gamma=0$ then $d-2\alpha\leq 0$.
3. for any line subbundle $L\subset V$, we have:
   (a) $\deg(L)\leq\alpha$ if $\gamma(L)=0$;
   (b) $\deg(L)\leq d/2$ if $\beta(L^{\perp})\subset L\otimes K$ and $\gamma(L)\subset L^{\perp}\otimes K$;
   (c) $\deg(L)\leq d-\alpha$ if $\beta(L^{\perp})=0$.

Here $L^{\perp}$ stands for the kernel of the projection $V^*\to L^*$, so it is the annihilator of L under γ; note that we are not considering any metric on V whatsoever. As usual, there are also the notions of α-stability (by considering strict inequalities) and of α-polystability; cf. [22].

Remark 2.5. If we view a semistable rank two vector bundle V of degree d as an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle with $\beta=\gamma=0$, then it is α-semistable if and only if $\alpha=d/2$.

Our aim is to present an overview on the study of the connected components of the moduli space of 0-polystable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ for certain values of d. To keep the notation simpler, we will just write

$$\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))=\mathcal{M}_d^0(\mathrm{Sp}(4,\mathbb{R}))$$

for the case $\alpha=0$. In this case we will just say "polystable" instead of 0-polystable and likewise for stable and semistable.

Remark 2.6. (Relation with representations $\pi_1(X) \to \mathrm{Sp}(4,\mathbb{R})$) We consider $\alpha = 0$ because this is the appropriate value for which non-abelian Hodge theory applies. More precisely, the *non-abelian Hodge Theorem* for $\mathrm{Sp}(4,\mathbb{R})$ states that an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle is polystable if and only if it corresponds to a reductive representation of $\pi_1(X)$ in $\mathrm{Sp}(4,\mathbb{R})$. This implies that $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ is homeomorphic to the space of reductive representations of $\pi_1(X)$ in $\mathrm{Sp}(4,\mathbb{R})$, with topological invariant d, modulo the action of conjugation by $\mathrm{Sp}(4,\mathbb{R})$, that is to $\mathcal{R}_d(\mathrm{Sp}(4,\mathbb{R})) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(X), \mathrm{Sp}(4,\mathbb{R}))/\mathrm{Sp}(4,\mathbb{R})$. This theorem is in fact valid for any real semisimple Lie group and also for real reductive groups with some slight modifications. The proof in the classical $G = \mathrm{SL}(n,\mathbb{C})$ case follows from [8, 10, 25, 37]. The more general case follows from [8, 14]. See for instance [16, 15] for more information for the case of $\mathrm{Sp}(2n,\mathbb{R})$ and [5] for an overview on the approach for the general group case.

The *Milnor-Wood inequality* for $G = \mathrm{Sp}(4,\mathbb{R})$ states that if an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle of degree d is semistable, then [9, 21, 14, 2]

$$|d| \leq 2g - 2.$$

(A similar type of inequality was proved for the first time for $G = \mathrm{PSL}(2,\mathbb{R})$ by Milnor in [29], on the representations side; cf. Remark 2.6.)

So $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ is empty if $|d| > 2g-2$. If $|d| = 2g-2$ then we say that we are in the *maximal Toledo* case, which is in fact the case where more interesting phenomena occur. Indeed, it is known [21] that $\mathcal{M}_{\pm(2g-2)}(\mathrm{Sp}(4,\mathbb{R}))$ has $3 \times 2^{2g} + 2g - 4$ components and that it is isomorphic to the moduli space of K^2-twisted $\mathrm{GL}(2,\mathbb{R})$-Higgs bundles — this is an example of the Cayley partner phenomena mentioned in the introduction (see also [5, 2]). In subsection 3.2.2 below we will see this for a subvariety of $\mathcal{M}_{2g-2}(\mathrm{Sp}(4,\mathbb{R}))$. It is also known that $\mathcal{M}_0(\mathrm{Sp}(4,\mathbb{R}))$ is connected [21]. The corresponding results for these two extreme cases for $|d|$ in higher rank are also known; cf. [15].

Nevertheless, in this paper we are interested in the components of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ for *non-maximal* and *non-zero* Toledo invariant: $0 < |d| < 2g-2$. The duality $(V,\beta,\gamma) \mapsto (V^*,\gamma,\beta)$ gives an isomorphism $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R})) \cong \mathcal{M}_{-d}(\mathrm{Sp}(4,\mathbb{R}))$, thus we just consider $0 < d < 2g-2$.

2.2.2 The approach to count components

The general idea, introduced in [25, 27], to study the connected components of $\mathcal{M}_c(G)$ is to use the functional $f : \mathcal{M}_c(G) \to \mathbb{R}$ mapping a G-Higgs bundle to the (square of the) L^2-norm of the Higgs field. The fact that f is proper and bounded below implies that it attains a minimum on each connected component of $\mathcal{M}_c(G)$. Hence the number of connected components of $\mathcal{M}_c(G)$ is bounded above by the one of the subvariety $\mathcal{N}_c(G) \subset \mathcal{M}_c(G)$ of local minimum of f. The procedure is thus to identify $\mathcal{N}_c(G)$, study its connected components and then draw conclusions about the components of $\mathcal{M}_c(G)$. Of course if $\mathcal{N}_c(G)$ turns out to be connected, then it immediately follows that $\mathcal{M}_c(G)$ is connected as well.

Explicitly, for $\mathrm{Sp}(4,\mathbb{R})$, the Higgs field splits as β and γ, so we have

$$f(V,\beta,\gamma) = \|\beta\|^2_{L^2} + \|\gamma\|^2_{L^2} = \int_X \mathrm{tr}(\beta\beta^{*,h}) + \int_X \mathrm{tr}(\gamma^{*,h}\gamma), \tag{2.1}$$

where $h : V \to \bar{V}^*$ is the metric on V which provides the Hitchin-Kobayashi correspondence and hence we are taking in (2.1) the adjoint with respect to h.

The following result completely identifies the subvariety of local minima in the non-zero and non-maximal cases. For this identification it is important that, over the smooth locus of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$, the function f is a moment map of the hamiltonian circle action $(V,\beta,\gamma) \mapsto (V, e^{i\theta}\beta, e^{i\theta}\gamma)$. By work of Frankel [11], a smooth point of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ is a critical point of f exactly when it is a fixed point of this $\mathrm{U}(1)$-action. Then there is a cohomological criteria [3, Corollary 4.15] which identifies the local minima among this fixed point set. Finally one has to perform a subsequent analysis to identify the local minima over the singular locus of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$.

Proposition 2.7 ([21])**.** *Let (V,β,γ) represent a point of $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$, with $0 < d < 2g-2$. Then it is a local minimum of f if and only if $\beta = 0$.*

Thus, for $0 < d < 2g-2$, the subvariety of local minima $\mathcal{N}_d(\mathrm{Sp}(4,\mathbb{R})) \subset \mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ is given by pairs (V,γ) where V is a rank 2 bundle, of degree d and γ is a section of $S^2V^* \otimes K$. This is what we call a *quadric bundle*. Since d is positive, γ must indeed be a non-zero section, as we saw in Remark 2.5.

Definition 2.8. A *quadric bundle* on X is a pair (V,γ), where V is a holomorphic vector bundle over X and γ is a holomorphic non-zero section of $S^2V^* \otimes K$.

Quadric bundles are sometimes also called *conic bundles* or *quadratic pairs* in the literature. In particular, this happens in the papers [22, 31] by the author where they were named quadratic pairs. But the term "quadric bundles" used in [18] is indeed more adequate, since it is more specific and moreover reveals the fact that these can be seen as bundles of quadrics, since for each $p \in X$ the map γ restricted to the fibre V_p defines a bilinear symmetric form, hence a quadric in $\mathbb{P}^{\mathrm{rk}(V)-1}$. When $\mathrm{rk}(V) = 2$, the term conic bundle is then perfectly adequate also.

The rank and degree of a quadric bundle are of course the rank and degree of V. We will only consider the rank 2 case. The rank n case appears naturally by considering $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles.

Remark 2.9. More generally, one can define U-quadric bundles, for a fixed holomorphic line bundle U over X. The only difference for the preceding definition is that γ is a non-zero section of $S^2V^* \otimes U$. We will mostly be interested in (K-)quadric bundles, but more general U-quadric bundles will also appear, more precisely when $U = LK$, for some line bundle L, in relation with the group $\mathrm{SO}_0(2,3)$. All results below can be adapted to this more general setting [22].

Quadric bundles of rank up to 3 were studied in [19] by Gómez and Sols, where they introduced an appropriate α-semistability condition, depending on a real parameter α, and constructed moduli spaces of S-equivalence classes of α-semistable quadric bundles using GIT. The construction of the moduli spaces follows from the general methods of [35]. Denote the moduli space of S-equivalence classes of α-semistable U-quadric bundles on X of rank 2 and degree d by $\mathcal{N}_{X,\alpha}(2,d) = \mathcal{N}_\alpha(d)$.

A simplified δ-(semi)stability condition for quadric bundles of arbitrary rank has been obtained in [18]. In rank 2 our α-semistability condition reads as follows (see [22, Proposition 2.15]). It is equivalent to the corresponding one on [18] by taking $\alpha = d/2 - \delta$.

Definition 2.10. Let (V,γ) be a rank 2 quadric bundle of degree d.

- The pair (V,γ) is α-*semistable* if and only if $\alpha \leq d/2$ and, for any line bundle $L \subset V$, the following conditions hold:
  1. $\deg(L) \leq \alpha$, if $\gamma(L) = 0$;
  2. $\deg(L) \leq d/2$, if $\gamma(L) \subset L^\perp K$;
  3. $\deg(L) \leq d - \alpha$, if $\gamma(L) \not\subset L^\perp K$.
- The pair (V,γ) is α-*stable* if and only if it is α-semistable for any line bundle $L \subset V$, the conditions (1), (2) and (3) above hold with strict inequalities.

Clearly these conditions are compatible with the ones of Definition 2.4. There is also the notion of α-polystability, but we omit it (see again Proposition 2.15 of [22]). The important thing to note is that on each S-equivalence class of α-semistable quadric bundles there is a unique α-polystable representative. Thus the points of $\mathcal{N}_\alpha(d)$ parametrize the isomorphism classes of α-polystable quadric bundles of rank 2 and, furthermore, $\mathcal{N}_\alpha(d)$ is a subvariety of $\mathcal{M}_d^\alpha(\mathrm{Sp}(4,\mathbb{R}))$.

The next result follows from Proposition 2.7 and the discussion preceding it.

Proposition 2.11. *Let $0 < d < 2g-2$. The number of connected components of the moduli space $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ of semistable $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles of degree d is bounded above by the number of connected components of $\mathcal{N}_0(d)$, the moduli space of 0-polystable quadric bundles of degree d.*

3 Moduli of quadric bundles and wall-crossing

3.1 Non-emptiness conditions

The next result gives a Milnor-Wood type of inequality for quadric bundles.

Proposition 3.1. *If $\mathcal{N}_\alpha(d)$ is non-empty then $2\alpha \leq d \leq 2g-2$.*

Proof. The first statement is immediate from α-semistability, hence let us look to the second inequality.

Let (V,γ) be quadric bundle of rank 2 and degree d. If $\mathrm{rk}(\gamma)=2$ (generically), then $\det(\gamma)$ is a non-zero section of $\Lambda^2 V^{-2}K^2$ so $d \leq 2g-2$.

Suppose now that there exists an α-semistable quadric bundle (V,γ) of rank 2 and degree $d > 2g-2$, with $\mathrm{rk}(\gamma) < 2$. Since $\gamma \neq 0$, we must have $\mathrm{rk}(\gamma) = 1$. Let N be the line subbundle of V given by the kernel of γ and let $I \subset V^*$ be such that IK is the saturation of the image sheaf of γ. Hence γ induces a non-zero map of line bundles $V/N \to IK$, so

$$-d + \deg(N) + \deg(I) + 2g - 2 \geq 0. \tag{3.1}$$

But, from the α-semistability condition, we have $\deg(N) \leq \alpha$ and $\deg(I) \leq \alpha - d$, because $\gamma(I^{\perp}) = 0$. This implies $-d+\deg(N)+\deg(I)+2g-2 < 0$, contradicting (3.1). We conclude that there is no such (V,γ). □

In fact, the inequalities of this proposition are equivalent to the non-emptiness of the moduli. This follows from the results below. So from now on we assume

$$2\alpha \leq d \leq 2g-2.$$

Indeed most of the times we will consider $2\alpha < d < 2g-2$.

Remark 3.2. Recall that our motivation comes from $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles and there (see Proposition 2.7) we imposed $d > 0$. However, the moduli spaces of quadric bundles make perfect sense and can be non-empty also for $d \leq 0$. Hence we will *not* impose $d > 0$ for the quadric bundles moduli spaces, although when $d \leq 0$ we lose the relation with Higgs bundles.

3.2 Moduli for small parameter

3.2.1 Stabilization parameter

For a fixed $d \leq 2g-2$, we know that there are no moduli spaces $\mathcal{N}_\alpha(d)$ whenever α is "large", meaning $\alpha > d/2$. Here we prove that there is a different kind of phenomena when α is "small". Precisely, we show that all the moduli spaces $\mathcal{N}_{\alpha'}(d)$ are isomorphic for any $\alpha' < d-g+1$. Moreover, in all of them, the map γ is generically non-degenerate. Write $\alpha_m = d-g+1$.

Proposition 3.3. *If (V,γ) is an α-semistable pair with $\alpha < \alpha_m$, then γ is generically non-degenerate. Moreover, if $\alpha_2 \leq \alpha_1 < \alpha_m$, then $\mathcal{N}_{\alpha_1}(d)$ and $\mathcal{N}_{\alpha_2}(d)$ are isomorphic.*

Proof. Recall that we always have $\gamma \neq 0$. If $\mathrm{rk}(\gamma) = 1$, considering again the line bundles $N = \ker(\gamma) \subset V$ and $I \subset V^*$ as in the proof of Proposition 3.1, we see that $0 \leq -d + \deg(N) + \deg(I) + 2g - 2 \leq 2\alpha - 2d + 2g - 2$, i.e., $\alpha \geq \alpha_m$. This settles the first part of the proposition.

Let $(V, \gamma) \in \mathcal{N}_{\alpha_1}(d)$. The only way that (V, γ) may not belong to $\mathcal{N}_{\alpha_2}(d)$ is from the existence of an α_2-destabilizing subbundle which, since $\alpha_2 \leq \alpha_1$ and looking at Definition 2.10, must be a line subbundle $L \subset V$ such that $\gamma(L) = 0$ and $\deg(L) > \alpha_2$. This in turn implies that $\mathrm{rk}(\gamma) = 1$ generically, which is impossible due to the first part of the proposition.

Conversely, if $(V, \gamma) \in \mathcal{N}_{\alpha_2}(d)$, then $(V, \gamma) \in \mathcal{N}_{\alpha_1}(d)$ unless there is an α_1-destabilizing subbundle L of (V, γ) such that $d - \alpha_1 < \deg(L) \leq d - \alpha_2$, and $\gamma(L) \not\subset L^{\perp}K$. Therefore the composite $L \to V \xrightarrow{\gamma} V^* \otimes U \to L^{-1}K$ is non-zero so $-2\deg(L) + 2g - 2 \geq 0$. On the other hand, $d - \alpha_1 < \deg(L)$ together with $\alpha_1 < \alpha_m$, gives $-2\deg(L) + 2g - 2 < 0$, yielding again to a contradiction. $\square$

We now aim to study the connectedness of the spaces $\mathcal{N}_\alpha(d)$, for $\alpha < \alpha_m$. Although our main motivation comes from the study of $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles with non-maximal Toledo invariant (cf. Proposition 2.7), let us say a few words about $\mathcal{N}_\alpha(2g-2)$, which really has a different behaviour from all the other cases.

3.2.2 Maximal Toledo invariant

Take $d = 2g - 2$. In this case $\alpha_m = g - 1 = d/2$, so the stabilisation parameter of the previous results is really the largest value for which non-emptiness holds. This means that, whenever non-empty, *all the moduli spaces* $\mathcal{N}_\alpha(2g-2)$ *are isomorphic*, independently of α. Accordingly, in this maximal case, we drop the α from the notation and just write $\mathcal{N}(2g-2)$.

The other special feature about this case is that if $(V, \gamma) \in \mathcal{N}(2g-2)$, then $\gamma : V \to V^* \otimes K$ is an isomorphism, since we already know that it must be injective and now the degrees match. By choosing a square root $K^{1/2}$ of K, γ gives rise to a symmetric isomorphism $q = \gamma \otimes \mathrm{Id}_{K^{-1/2}} : V \otimes K^{-1/2} \cong V^* \otimes K^{1/2}$, i.e. to a nowhere degenerate quadratic form on $V \otimes K^{-1/2}$. In other words, $(V \otimes K^{-1/2}, q)$ is an orthogonal vector bundle. Now, there is a semistability condition for orthogonal bundles (namely that any isotropic subbundle must have non-positive degree; [33]), and it can be seen that the orthogonal bundle $(V \otimes K^{-1/2}, q)$ is semistable if and only if (V, γ) is α-semistable for any $\alpha < \alpha_m$. So:

Proposition 3.4. *The moduli space* $\mathcal{N}(2g-2)$ *is isomorphic to the moduli space of rank* 2 *orthogonal vector bundles (without fixed topological type).*

The existence of this isomorphism justifies the disconnectedness of $\mathcal{N}(2g-2)$. This is an example of the Cayley correspondence mentioned in the introduction. All this goes through higher rank, telling us that quadric bundles are the natural generalisation of orthogonal vector bundles, when we remove the non-degeneracy condition, providing another motivation for the consideration of these objects.

3.2.3 Quadric bundles, twisted Higgs pairs and the fibres of the Hitchin map

Write α_m^- for any value of α less than $\alpha_m = d - g + 1$. We shall now deal with the spaces $\mathcal{N}_{\alpha_m^-}(d)$ for any $d < 2g - 2$. We will do it by relating pairs (V, γ) with certain twisted rank 2 Higgs bundles and using the Hitchin map on the corresponding moduli space.

Consider a quadric bundle $(V, \gamma) \in \mathcal{N}_{\alpha_m^-}(d)$. By Proposition 3.3, $\det(\gamma)$ is a non-zero holomorphic section of $\Lambda^2 V^{-2} K^2$. Since now $d < 2g - 2$, the section $\det(\gamma)$ has zeros, so we consider the corresponding effective divisor $\operatorname{div}(\det(\gamma)) \in \operatorname{Sym}^{4g-4-2d}(X)$.

Write $\operatorname{Jac}^d(X)$ for the "Jacobian variety" of degree d holomorphic line bundles over X. Let $\mathcal{P}_X$ be the 2^{2g}-cover of $\operatorname{Sym}^{4g-4-2d}(X)$ obtained by pulling back the cover $\operatorname{Jac}^{2g-2-d}(X) \to \operatorname{Jac}^{4g-4-2d}(X)$, $L \mapsto L^2$, under the Abel-Jacobi map $\operatorname{Sym}^{4g-4-2d}(X) \to \operatorname{Jac}^{4g-4-2d}(X)$. The elements of $\mathcal{P}_X$ are pairs (D, L) in the product $\operatorname{Sym}^{4g-4-2d}(X) \times \operatorname{Jac}^{2g-2-d}(X)$ such that $\mathcal{O}(D) \cong L^2$.

In order to describe $\mathcal{N}_{\alpha_m^-}(d)$, we shall use the following map, which is analogue to the so-called Hitchin map defined by Hitchin in [25], and which will recall below in (3.3):

$$
\begin{array}{rccl}
h: & \mathcal{N}_{\alpha_m^-}(d) & \longrightarrow & \mathcal{P}_X \\
 & (V, \gamma) & \longmapsto & (\operatorname{div}(\det(\gamma)), \Lambda^2 V^{-1} K).
\end{array} \tag{3.2}
$$

Our goal is to be able to say something about the fibres of this map.

To relate h with the Hitchin map, recall first that, given any line bundle L of non-negative degree, an *L-twisted Higgs pair* of type is a pair (V, φ), where V is a holomorphic vector bundle over X and $\varphi \in H^0(X, \operatorname{End}(V) \otimes L)$. So, we are just twisting the Higgs field by L instead of K.

Definition 3.5. A rank 2 and degree d, L-twisted Higgs pair (V, φ) is *semistable* if $\deg(F) \leq d/2$ for any line subbundle $F \subset V$ such that $\varphi(F) \subset FL$.

Let $\mathcal{M}_L^\Lambda$ denote the moduli space of L-twisted Higgs pairs of rank 2 and degree d, with fixed determinant $\Lambda \in \operatorname{Jac}^d(X)$ and with traceless Higgs field. In this particular case, the *Hitchin map* in $\mathcal{M}_L^\Lambda$ is defined by:

$$
\begin{array}{rccl}
\mathcal{H}: & \mathcal{M}_L^\Lambda & \longrightarrow & H^0(X, L^2) \\
 & (V, \varphi) & \longmapsto & \det(\varphi).
\end{array} \tag{3.3}
$$

We can naturally associate a ξ-twisted Higgs pair to a given quadric bundle (V, γ), of rank 2, where $\xi = \Lambda^2 V^{-1} K$. This is done by taking advantage of the fact that for a 2-dimensional vector space $\mathbb{V}$, there is an isomorphism $\mathbb{V} \otimes \Lambda^2 \mathbb{V}^* \cong \mathbb{V}^*$ given by $v \otimes \phi \mapsto \phi(v \wedge -)$. Then, from such quadric bundle, simply associate the ξ-twisted Higgs pair $(V, g^{-1}\gamma)$, where g is the isomorphism

$$
g : V \otimes \xi \xrightarrow{\cong} V^* \otimes K \tag{3.4}
$$

given by $g(v \otimes \phi \otimes u) = \phi(v \wedge -) \otimes u$; so indeed $g^{-1}\gamma : V \to V \otimes \xi$. Choosing appropriate local frames, g is locally given by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ so

$$\det(g^{-1}\gamma) = \det(\gamma) \ \text{ and } \ \mathrm{tr}(g^{-1}\gamma) = 0, \tag{3.5}$$

due to the symmetry of γ. Moreover, it is easy to see that (V, γ) is α_m^--semistable if and only if the corresponding $(V, g^{-1}\gamma)$ is semistable as in Definition 3.5. So if $(V, \gamma) \in \mathcal{N}_{\alpha_m^-}$, then $(V, g^{-1}\gamma)$ represents a point in $\mathcal{M}_\xi^{\xi^{-1}K}$.

Let us now go back to the map h in (3.2). Let (D, ξ) be any pair in $\mathcal{P}_X$. We want to describe the fibre of h over (D, ξ), i.e., the space of isomorphism classes of α_m^--polystable quadric bundles (V, γ) such that $\mathrm{div}(\det(\gamma)) = D$ and $\Lambda^2 V \cong \xi^{-1}K$. The following result gives the fibre $h^{-1}(D, \xi)$ in terms of the fibre $\mathcal{H}^{-1}(s)$, for a certain section s of ξ^2.

Proposition 3.6 ([22]). *Let $(D, \xi) \in \mathcal{P}_X$ and choose some $s \in H^0(X, \xi^2)$ such that* $\mathrm{div}(s) = D$. *Then $h^{-1}(D, \xi) \in \mathcal{N}_{\alpha_m^-}(d)$ is isomorphic to $\mathcal{H}^{-1}(s) \in \mathcal{M}_\xi^{\xi^{-1}K}$.*

The isomorphism of this proposition is of course given by the above correspondence between quadric bundles and ξ-twisted Higgs pairs. Notice that everything makes sense because of (3.5). A word of caution is however required here since there is a choice of a section s associated to the divisor D in Proposition 3.6. However, the given description of $h^{-1}(D, \xi)$ does not depend of this choice, due to Lemma 4.6 of [22]; see also Remark 4.10 in loc. cit. for more details.

Using this we can prove the following.

Theorem 3.7 ([22]). *Let $d < 2g - 2$. The moduli space $\mathcal{N}_{\alpha_m^-}(d)$ is connected and has dimension $7g - 7 - 3d$.*

The basic idea to prove connectedness is to prove that any fibre of h is connected. For that we use Proposition 3.6 and want to prove that $\mathcal{H}^{-1}(s)$ is connected for *every* $0 \neq s \in H^0(X, \xi^2)$. This is done using the theory of spectral covers and their Jacobians and Prym varieties, as developed in [1, 25, 26]. Besides these classical references, the reader may also check the details of the following definitions for instance in [23].

For every $s \neq 0$, there is a naturally associated curve X_s — the *spectral curve* of s — inside the total space of $\pi : \xi \to X$. The projection $\pi|_{X_s} : X_s \to X$ is a $2:1$ cover of X, with the branch locus being given by the divisor of s.

For generic $s \in H^0(X, \xi^2)$ the curve X_s is smooth. It is well-known that $\mathcal{H}^{-1}(s)$ is indeed (a torsor for) the *Prym variety* of X_s. This Prym variety is, in particular, a complex torus, so connected. If $\deg(\xi) \geq 2g-2$ then the connectedness of every fibre of $\mathcal{H}$ follows from the connectedness of the generic fibre (see Proposition 3.7 of [23]). It is nevertheless important to notice that $\deg(\xi) = -d+2g-2$, so $\deg(\xi)$ can be any positive integer. Moreover, it is precisely the case $\deg(\xi) < 2g - 2$ that is of most interest to us, since that is the case relevant to $\mathrm{Sp}(4, \mathbb{R})$-Higgs bundles. So, for these cases, our knowledge of the generic fibre is not enough to draw conclusions on the connectedness of the singular fibres, that is, the ones where the spectral curve X_s acquires singularities. However, this was achieved by P. Gothen and the author in [23] as follows.

When the spectral case is irreducible, we use the correspondence between Higgs pairs on X and rank one torsion free sheaves on X_s [1] to show that the fibre of the Hitchin map is essentially the compactification by rank 1 torsion free sheaves of the Prym of the double cover $X_s \to X$. In order to prove the connectedness of the fibre, we made use of the the compactification of the Jacobian of X_s by the *parabolic modules* of Cook [6, 7]. One advantage of this compactification is that it fibres over the Jacobian of the normalisation of X_s, as opposed to the compactification by rank one torsion free sheaves. In the case of reducible spectral curve X_s, we gave a direct description of the fibre as a stratified space. All together, the statement of our result, adapted to the situation under consideration in Proposition 3.6, is the following.

Theorem 3.8 ([23]). *Consider the Hitchin map $\mathcal{H} : \mathcal{M}_\xi^{\xi^{-1}K} \to H^0(X, \xi^2)$. For any s, $\mathcal{H}^{-1}(s)$ is connected. Moreover, for $s \neq 0$, the dimension of the fibre is* $\dim(\mathcal{H}^{-1}(s)) = -d + 3g - 3$.

As $\mathcal{P}_X$ is connected and of dimension $4g - 4 - 2d$, this settles Theorem 3.7.

The following corollary of Theorem 3.7 is immediate.

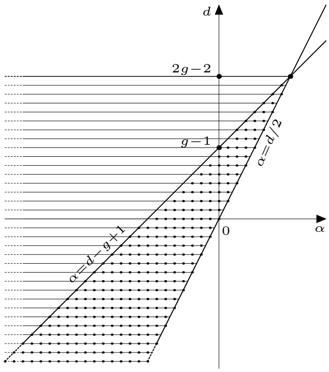
Corollary 3.9. *If $g - 1 < d < 2g - 2$, then the moduli space $\mathcal{N}_0(d)$ is connected.*

For the cases $0 < d < g - 1$, we must take into account other values of the parameter and not just α_m^-.

3.3 Critical values and wall-crossing

Having established the connectedness of the space $\mathcal{N}_{\alpha_m^-}(d)$ in Theorem 3.7, the purpose of this section is to study the variation of the moduli spaces $\mathcal{N}_\alpha(d)$ with the stability parameter α. Recall that the goal is to be able to say something about the connectedness of $\mathcal{N}_0(d)$. As in several other cases [38, 4], we have *critical values* for the parameter. These are special values α_k, for which the α-semistability condition changes. One proves that indeed there are a finite number of these critical values and, more precisely, that α is a critical value if and only if it is equal to $d/2$ or to $\alpha_k = [d/2] - k$, with $k = 0, \dots, -d + g - 1 + [d/2]$. By definition, on each open interval between consecutive critical values, the α-semistability condition does not vary, hence the corresponding moduli spaces are isomorphic. If α_k^+ denotes the value of any parameter between the critical values α_k and α_{k+1}, we can write without ambiguity $\mathcal{N}_{\alpha_k^+}(d)$ for the moduli space of α_k^+-semistable quadric bundles of for any α between α_k and α_{k+1}. Likewise, define $\mathcal{N}_{\alpha_k^-}(d)$, with α_k^- denoting any value between the critical values α_{k-1} and α_k. With this notation we have $\mathcal{N}_{\alpha_k^+}(d) = \mathcal{N}_{\alpha_{k+1}^-}(d)$.

The information obtained so far on the variation of $\mathcal{N}_\alpha(d)$ with α and d is summarised in the next graphic.



Above the line $d = 2g-2$, $\mathcal{N}_\alpha(d)$ is empty as well as on the right of the line $\alpha = d/2$. For a fixed $d < 2g-2$, the region on the left of the line $\alpha = \alpha_m = d-g+1$, is the region $\mathcal{N}_{\alpha_m^-}(d)$ described in the previous section, where there are no critical values. The critical values are represented by the dots between the lines $\alpha = \alpha_m$ and $\alpha = d/2$.

Given a critical value α_k we have the corresponding subvariety $\mathcal{S}_{\alpha_k^+}(d) \subset \mathcal{N}_{\alpha_k^+}(d)$ consisting of those pairs which are α_k^+-semistable but α_k^--unstable. In the same manner, define the subvariety $\mathcal{S}_{\alpha_k^-}(d) \subset \mathcal{N}_{\alpha_k^-}(d)$. Consequently,

$$\mathcal{N}_{\alpha_k^-}(d) \setminus \mathcal{S}_{\alpha_k^-}(d) \cong \mathcal{N}_{\alpha_k^+}(d) \setminus \mathcal{S}_{\alpha_k^+}(d). \tag{3.6}$$

So the spaces $\mathcal{S}_{\alpha_k^\pm}(d)$ encode the difference between the spaces $\mathcal{N}_{\alpha_k^-}(d)$ and $\mathcal{N}_{\alpha_k^+}(d)$ on opposite sides of the critical value α_k. This difference is usually known as the *wall-crossing* phenomena through α_k. In order to study this wall-crossing we need a description of the spaces $\mathcal{S}_{\alpha_k^\pm}(d)$.

In Section 3 of [22] we studied these $\mathcal{S}_{\alpha_k^\pm}(d)$ for any critical value. In particular, it was enough to conclude that they have high codimension in $\mathcal{N}_{\alpha_k^\pm}(d)$. For technical reasons, we had to impose the condition $d < g-1$. More precisely, we have the following.

Proposition 3.10. *Suppose that $d < g-1$. Then* $\dim \mathcal{N}_\alpha(d) = 7g-7-3d$, *for any $\alpha \leq d/2$. Moreover, for any k, the codimensions of $\mathcal{S}_{\alpha_k^\pm}(d) \subset \mathcal{N}_{\alpha_k^\pm}(d)$ are strictly positive.*

3.4 Connectedness of the moduli spaces of quadric bundles

Since we already know from Theorem 3.7 that $\mathcal{N}_{\alpha_m^-}(d)$ is connected for any $d < 2g-2$, Proposition 3.10 yields the following (cf. [22, Theorem 5.3]):

Theorem 3.11. *The moduli spaces $\mathcal{N}_\alpha(d)$ are connected for any $d < g-1$ and any $\alpha < d/2$.*

Corollary 3.12. *If $0 < d < g-1$, then the moduli space $\mathcal{N}_0(d)$ is connected.*

Recall that we want to study the connected components of the moduli space $\mathcal{N}_0(d)$ of 0-polystable quadric bundles, for any $0 < d < 2g-2$. From Corollaries 3.9 and 3.12 we see that the only remaining case to understand is when $d = g-1$. Notice that the space $\mathcal{N}_0(g-1)$ is really $\mathcal{N}_{\alpha_m}(g-1)$. Now, although the codimensions of every $\mathcal{S}_{\alpha_k^\pm}(d)$ are only known under the condition $d < g-1$, it follows from Corollaries 3.11 and 3.15 of [22] that the codimensions of both $\mathcal{S}_{\alpha_m^\pm}(d)$ are known to be positive also when $d = g-1$. From this, arguing as in the last paragraph of the proof of [22, Theorem 5.3], we prove that also $\mathcal{N}_0(g-1)$ is connected. So we conclude that:

Theorem 3.13. *The moduli space of 0-polystable quadric bundles of degree d is connected for every $0 < d < 2g-2$.*

4 Conclusion and further remarks

4.1 Non-maximal components for $\mathrm{Sp}(4,\mathbb{R})$ and $\mathrm{SO}_0(2,3)$

Proposition 2.11 and Theorem 3.13 imply then that we have achieved our objective of calculating the number of connected components of the moduli space of $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundles over X, with non-maximal and non-zero Toledo invariant:

Theorem 4.1. *If $0 < |d| < 2g-2$ then $\mathcal{M}_d(\mathrm{Sp}(4,\mathbb{R}))$ is connected.*

This theorem has in fact been proved before [22], by O. García-Prada and I. Mundet i Riera in [16], using different techniques. More precisely, they do consider quadric bundles, but prove the connectedness directly, i.e., fixing $\alpha = 0$ and not implementing the variation of the parameter.

Our method easily generalises for U-quadric bundles — see Remark 2.9 — for any line bundle U. In particular, if we consider LK-quadric bundles, with L some line bundle of degree 1, then these are related with $\mathrm{SO}_0(2,3)$-Higgs bundles with non-maximal (and non-zero) Toledo invariant, in the same way K-quadric bundles arise in the $\mathrm{Sp}(4,\mathbb{R})$ case. Applying Definition 2.1, it is easy to check that an $\mathrm{SO}_0(2,3)$-Higgs bundle is defined by a tuple $(L, W, Q_W, \beta, \gamma)$ where L is a line bundle, (W, Q_W) is an orthogonal rank 3 bundle and the Higgs field is defined by maps $\beta : W \to LK$ and $\gamma : W \to L^{-1}K$. These are topologically classified by the degree d of L (this is the Toledo invariant) and by the second Stiefel-Whitney class w_2 of W. Note that $\mathrm{SO}_0(2,3)$ is isomorphic to $\mathrm{PSp}(4,\mathbb{R})$. Moreover, an $\mathrm{SO}_0(2,3)$-Higgs bundle lifts to an $\mathrm{Sp}(4,\mathbb{R})$-Higgs bundle if and only if $d \equiv w_2 \bmod 2$.

The precise same methods that we described for $\mathrm{Sp}(4,\mathbb{R})$, yield then the following (see [22, Theorem 6.26]):

Theorem 4.2. *If* $0 < |d| < 2g-2$ *and* $w_2 \in \mathbb{Z}/2$ *then* $\mathcal{M}_{d,w_2}(\mathrm{SO}_0(2,3))$ *is connected.*

Recalling that non-abelian Hodge theory implies that the moduli space of G-Higgs bundles over X is homeomorphic to the space of conjugacy classes of reductive representations of $\pi_1(X)$ in G (cf. Remark 2.6), we conclude the both Theorems 4.1 and 4.2 have their counterparts on the representations side (see [16, 22] for the detailed statements).

4.2 Some different directions

4.2.1 Torelli theorem

Our method to analyse the components of the moduli spaces $\mathcal{N}_\alpha(d)$ of α-semistable quadric bundles of degree d was to start with the study in the lowest extreme of α, that is the study of $\mathcal{N}_{\alpha_m^-}(d)$. One can ask what happens in the highest possible extreme, namely $\alpha = \alpha_M = d/2$. Since this is a critical value, take a slight lower value, $\alpha_M^- = d/2 - \epsilon$, for a small $\epsilon > 0$. Here, different phenomena arise.

Briefly, it is easy to check that if (V, γ) is α_M^--semistable, then V is itself semistable as a rank 2 vector bundle. If $M(d)$ denotes the moduli space of polystable rank 2 degree d vector bundles on X, this yields a forgetful map $\pi : \mathcal{N}_{\alpha_M^-}(d) \to M(d)$. If $d = 2g-2$, this map is an embedding because $\mathcal{N}_{\alpha_M^-}(d)$ is, as we saw, the moduli of orthogonal vector bundles and thus follows from [36]. If $g-1 \leq d < 2g-2$, the determination of the image of π is a Brill-Noether problem. If $d < g-1$, π is surjective and if, further, $d < 0$, the map π, suitably restricted, is a projective bundle over the stable locus $M^s(d) \subset M(d)$. This is explained in Proposition 3.13 of [22].

So assume $d < 0$, and from now on let us just consider quadric bundles (V, γ) where the determinant of V is fixed to be some line bundle Λ of degree d. Let $\mathcal{N}_\alpha(\Lambda) \subset \mathcal{N}_\alpha(d)$ and $M(\Lambda) \subset M(d)$ denote the corresponding obvious moduli spaces. Using the projective bundle π onto the stable locus of $M(\Lambda)$ and through a detailed analysis of the the smooth locus $\mathcal{N}_\alpha^{sm}(\Lambda) \subset \mathcal{N}_\alpha(\Lambda)$, we were able to obtain some geometric and topological results on $\mathcal{N}_\alpha(d)$. This procedure is taken in detail in [31] again in the more general setting of U-quadric bundles.

For instance we proved that $\mathcal{N}_\alpha(\Lambda)$ is irreducible and $\mathcal{N}_\alpha^{sm}(\Lambda)$ is simply-connected — see Corollaries 4.3 and 4.4 of [31]. The irreducibility was already known from [19], using different methods.

Under some slight conditions on the genus of X, we calculated the torsion-free part of the first three integral cohomology groups of the smooth locus $\mathcal{N}_\alpha^{sm}(\Lambda) \subset \mathcal{N}_\alpha(\Lambda)$ for any α. In particular [31, Proposition 5.6] says that $H^3(\mathcal{N}_\alpha^{sm}(\Lambda), \mathbb{Z})$ is isomorphic to $H^1(X, \mathbb{Z})$. This fact, together with the assumption that the genus of X is at least 5, and after properly defining a polarisation on $H^3(\mathcal{N}_\alpha^{sm}(\Lambda), \mathbb{Z})$ compatible with the one on $H^1(X, \mathbb{Z})$, allowed us to prove that a Torelli type theorem holds for $\mathcal{N}_\alpha(\Lambda)$. From this it follows that the same is also true for the non-fixed determinant moduli. To emphasise now the base curve, write $\mathcal{N}_{X,\alpha}(\Lambda)$ for the moduli space of α-polystable quadric bundles of rank two with fixed determinant Λ on X. Let $\mathcal{N}_{X,\alpha}(d)$ be the same thing but just fixing the degree and not the determinant.

Theorem 4.3 ([31]). *Let X and X' be smooth projective curves of genus $g, g' \geq 5$, Λ and Λ' line bundles of degree $d < 0$ and $d' < 0$ on X and X', respectively. If $\mathcal{N}_{X,\alpha}(\Lambda) \cong \mathcal{N}_{X',\alpha}(\Lambda')$ then $X \cong X'$. The same holds for $\mathcal{N}_{X,\alpha}(d)$ and $\mathcal{N}_{X',\alpha}(d')$.*

In other words, the isomorphism class of the curve X is determined by the one of the projective variety $\mathcal{N}_{X,\alpha}(\Lambda)$.

4.2.2 Higher ranks

One natural question is to wonder if the procedure we described here can be generalised to ranks higher than 2. First, Proposition 2.11 is true for any rank (it is even true for any real reductive Lie group). Proposition 2.7 also generalises in a straightforward way for $\mathrm{Sp}(2n, \mathbb{R})$ for $n > 2$, so we are again lead to the study of higher rank quadric bundles. The technical problems start here because the α-semistability condition can be much more complicated in higher rank, involving not only subbundles but filtrations (see [19] and [18]). One consequence is that the study of $\mathcal{N}_{\alpha_m^-}(d)$ and mainly of $\mathcal{S}_{\alpha_k^\pm}(d)$ should become much more complicated.

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Program of the School

- Mario Garcia-Fernandes (ICMAT, Madrid):
  An introduction to the Strominger system of partial differential equations
- Jochen Heinloth (University Duisburg-Essen):
  Geometry of moduli spaces of Higgs bundles on curves
- Semyon Klevtsov (University of Cologne - Humboldt fellow):
  Quantum Hall, Bergman kernel and Kähler metrics
- Tomoki Nakanishi (Nagoya University, Japan):
  Cluster algebras and applications

Program of the Conference

The following invited talks were presented in the conference week:

1. Luis Alvarez-Consul (ICMAT, Madrid, Spain): *Gravitating vortices, cosmic strings and the Kähler-Yang-Mills equations*
2. Artur de Araujo (Universidade do Porto, Portugal): *Generalized orthogonal quiver bundles*
3. Jorgen Ellegard Andersen (QGM, Aarhus University, Denmark) : *Geometric quantisation of moduli spaces of Higgs bundles*
4. Indranil Biswas (TIFR, India): *Bohr-Sommerfeld Lagrangians of moduli spaces of Higgs bundles*
5. Elena Bunkova (Steklov Institute of Mathematics, Moscow, Russia):*Formal group for elliptic function of level 3*
6. Mirek Englis (Czech Republic): *Orthogonal Polynomials, Laguerre Fock Space and Quasi-classical Asymptotics*
7. Anton Fonarev (Steklov Mathematical Institute, Moscow, Russia): *Derived categories of curves as components of Fano varieties*
8. Hajime Fujita (Japan): *Danilov type formula for toric origami manifolds via localization of index*
9. Tomas Gomez (ICMAT, Madrid, Spain): *Torelli theorem for the parabolic Deligne-Hitchin moduli space*
10. Peter Gothen (Centro de Matemtica da Universidade do Porto, Portugal) : *Birationality of moduli of U(p,q)-Hitchin pairs*
11. Tomoyuki Hisamoto (Nagoya University, Japan): *On uniform K-stability*
12. Alexander Karabegov (Abilene Christian University, USA): *On the phase form of a deformation quantization with separation of variables*
13. Akishi Kato (Tokyo University, Japan): *Quiver mutation loops and partition q-series*
14. Sergei Lando (HSE Moscow, Russia): *Computation of universal polynomials for characteristic classes of singularities*
15. Jan Manschot (Trinity College, Dublin, Ireland): *Sheaves on surfaces and generalized Appell functions*
16. Andrey E. Mironov (Sobolev Institute of Mathematics, Novosibirsk, Russia): *Commuting ordinary differential operators with polynomial coefficients and automorphisms of the first Weyl algebra*

17. Motohiko Mulase (UCD, Davis, USA) : *Topological recursion, 2D TQFT, and quantization of Hitchin spectral curves.*
18. Andre Oliveira (UTAD and University of Porto, Portugal) : *Geometry of quadratic pairs moduli spaces*
19. Yuji Sano (Kumamoto University, Japan) : *Quantization of extremal metrics and the modified K-energy*
20. Andrei I. Shafarevich (Moscow State University, Russia) : *Lagrangian manifolds and Maslov indices corresponding to the spectral series of Schrödinger operators with delta-potentials on symmetric manifolds.*
21. Oleg K. Sheinman (Steklov Mathematical Institute, Russia) : *Lax operator algebras, Hitchin systems, spinning tops, and more.*
22. Evgeny Smirnov (Independent University, Moscow, Russia): *Spherical double flag varieties*
23. Siye Wu (Taiwan): *Non-orientable surfaces and electric-magnetic duality*

In addition there were 4 poster presentations given by Victor Castellanos (Mexico), Oscar Chavez Molina (Mexico), Oleksandr Iena (Luxembourg), and Vladimir Salnikov (Luxembourg),

Table of Content

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