The Asymptotic Expansion of Bergman Kernel

Xianzhe Dai, Kefeng Liu, Xiaonan Ma

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Polarized manifolds

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- X is a compact complex manifold of complex dimension n;
- $L \longrightarrow X$ a positive line bundle.

Let h be an hermitian metric on L whose curvature is $-2\pi i\omega$, where ω defines a Kähler metric on X.

Together with h, ω defines an L^2 inner product on

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Let $s_1(x), \cdots, s_{d_p}(x)$ be an orthonormal basis of $H^0(X, L^p)$, with $d_p = \dim H^0(X, L^p)$. Then the (diagonal) Bergman kernel (or density of the states) is

$$B_p(x) = \sum_{j=1}^{d_p} |s_j(x)|^2.$$

Note that $B_p(x)$ is the restriction to the diagonal of the full Bergman kernel

$$B_p(x,y) = \sum_{j=1}^{d_p} s_j(x) \otimes s_j^*(y),$$

which is the kernel function of the orthogonal projection onto $H^0(X, L^p)$.

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In other words, it is an invariant of the Kähler metric ω.

Theorem (Tian, Zelditch, Bouche, Catlin, Lu): (1) For fixed ω, there is an asymptotic expansion as $p \to \infty$:

$$B_p(x) \sim A_0(x)p^n + A_1(x)p^{n-1} + \cdots ,$$

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where $A_j(x)$ are smooth functions on X defined locally by ω.
(2) In particular ,

$$A_0(x) = 1, \quad A_1(x) = \frac{1}{8\pi} S(x),$$

where $S(x)$ is the scalar curvature of ω.

(3) The expansion holds in C^∞ in that for any $r, N \geq 0$,

$$\|B_p(x) - \sum_{j=0}^{N} A_j(x) p^{n-j}\|_{C^r(X)} \leq K_{r,N,\omega} p^{n-N-1}.$$

Moreover the expansion is uniform in that for any r, N there is an integer s such that if ω runs over a set of metrics which are bounded in C^s, and with ω bounded below, the constants $K_{r,N,\omega}$ are bounded by some $K_{r,N}$ independent of ω.

Note that

$$\int_X B_p(x) = \dim H^0(X, L^p) = a_0 p^n + a_1 p^{n-1} + \cdots ,$$

from Kodaira vanishing theorem and Riemann-Roch. Moreover,

$$a_0 = \int_X 1, \quad a_1 = \frac{1}{8\pi} \int_X S(x).$$

Thus, in a sense, the asymptotic expansion of Bergman kernel above can be thought as a local version of the Riemann-Roch.

The symplectic versions were studied by **Borthwick-Uribe**, **Shiffman-Zleditch** using the Szegö kernels of **Boutet de Monvel-Guillemin**. In the holomorphic case, the Szegö kernels are exactly (modulo smooth operators) the Szegö kernel associated to the holomorphic sections by **Boutet de Monvel-Sjöstrand**.

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Let (X, ω) be a compact symplectic manifold of real dimension $2n$ which satisfies the prequantization condition.

That is, there exists a Hermitian line bundle L over X endowed with a Hermitian connection ∇^L with the property that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega$$

, where $R^L = (\nabla^L)^2$ is the curvature of (L, ∇^L).

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- the Levi-Civita connection ∇^{TX} on (TX, g^{TX}) with curvature R^{TX}, and ∇^{TX} induces a natural connection $\nabla^{\det}$ on $\det(T^{(1,0)}X)$ with curvature $R^{\det}$.

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With these data, we have the *spin^c* Dirac operator D_p:

$$D_p : \Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^{n} \Omega^{0,q}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E).$$

Let $\{S_i^p\}_{i=1}^{d_p}$ ($d_p = \dim \ker D_p$) be any orthonormal basis of $\ker D_p$. We define the diagonal of the Bergman kernel of D_p (or the distortion function) by

$$B_p(x) = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x))^* \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x \quad (1)$$

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The full Bergman kernel is

$$B_p(x,y) = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(y))^*. \qquad (2)$$

If $X,\omega)$ is actually Kähler, and L is a holomorphic line bundle satisfying the prequantization condition, then

$$D_p = \overline{\partial}_p + \overline{\partial}_p^*,$$

where $\overline{\partial}_p$ is the $\overline{\partial}$ operator on X with values in L^p. Hence this definition agrees with the standard one. (Here we take $E = \mathbb{C}$ to be trivial.)

Theorem

There exist smooth coefficients $b_r(x) \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$ which are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r-1$ (resp. $2r$) and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, with $b_0 = (\det \mathbf{J})^{1/2} I_{\mathbb{C}\otimes E}$, such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\left| B_p(x) - \sum_{r=0}^{k} b_r(x) p^{n-r} \right|_{C^l} \leq C_{k,l} p^{n-k-1}. \qquad (3)$$

Moreover, the expansion is uniform in that for any $k, l \in \mathbb{N}$, there is an integer s such that if all data $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E)$ run over a set which are bounded in C^s and with g^{TX} bounded below, there exists the constant $C_{k,\,l}$ independent of g^{TX}, and the C^l-norm includes also the derivatives on the parameters.

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Let $\{w_i\}$ be an orthonormal frame of $(T^{(1,0)}X, g^{TX})$. Set

$$\omega_d = -\sum_{l,m} R^L(w_l, \overline{w}_m)\,\overline{w}^m \wedge\, i_{\overline{w}_l}. \tag{4}$$

Theorem

There exist smooth sections $b_{r,u}$ of $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)$ on X which are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r - 1$ (resp. $2r$) and functions on the eigenvalues of $\mathbf{J}$ at x, and $b_{0,u} = \left(\det\left(\frac{|\mathbf{J}|}{1-e^{-4\pi u|\mathbf{J}|}}\right)\right)^{1/2} e^{2u\omega_d}$, such that for each $u > 0$ fixed, we have the asymptotic expansion as $p \to \infty$,

$$\exp(-\frac{u}{p}D_p^2)(x,x) = \sum_{r=0}^{k} b_{r,u}(x)p^{n-r} + \mathcal{O}(p^{n-k-1}).$$

Moreover, there exists $c > 0$ such that as $u \to +\infty$,

$$b_{r,u}(x) = b_r(x) + \mathcal{O}(e^{-cu}).$$

These results give us a way to compute the coefficient $b_r(x)$, as it is relatively easy to compute $b_{r,u}(x)$. As an example, we compute b_1 which plays an important role in Donaldson's recent work. Note if (X, ω) is Kähler and $\mathbf{J} = J$, then $B_p(x) \in C^\infty(X, \mathrm{End}(E))$ for p large enough, thus $b_r(x) \in \mathrm{End}(E)_x$.

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Theorem If (X,ω) is Kähler and $\mathbf{J} = J$, then we have,

$$b_0 = \mathrm{Id}_E, \quad b_1 = \frac{1}{4\pi}\Big[\sqrt{-1}\sum_i R^E(e_i, Je_i) + \frac{1}{2} r^X \mathrm{Id}_E\Big].$$

here r^X is the scalar curvature of (X, g^{TX}), and $\{e_i\}$ is an orthonormal basis of (X, g^{TX}).

Lu, Wang, peak sections

Catlin, Zelditch, Charles, Szegö kernels.

Ma-Marinescu: b_1 for general symplectic manifolds (symplectic moment).

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One of the advantages of our method is that it can be easily generalized to the orbifold situation, and indeed, we deduce the explicit asymptotic expansion near the singular set of the orbifold.

Theorem

If (X,ω) is a symplectic orbifold with the singular set X', and L, E are corresponding proper orbifold vector bundles on X. Then there exist smooth coefficients $b_r(x) \in \mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E)_x$ with $b_0 = (\det \mathbf{J})^{1/2} I_{\mathbb{C}\otimes E}$, and $b_r(x)$ are polynomials in R^{TX}, $R^{\det}$, R^E (and R^L) and their derivatives with order $\leq 2r-1$ (resp. $2r$) and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x, such that for any $k, l \in \mathbb{N}$, there exist $C_{k,l} > 0$, $N \in \mathbb{N}$ such that for any $x \in X$, $p \in \mathbb{N}$,

$$\left|\frac{1}{p^n} B_p(x) - \sum_{r=0}^{k} b_r(x) p^{-r}\right|_{C^l} \leq$$

$$C_{k,l}\Big(p^{-k-1} + p^{l/2}(1+\sqrt{p}d(x,X'))^N e^{-C\sqrt{p}d(x,X')}\Big) \quad .$$

Moreover if the orbifold (X,ω) is Kähler, $\mathbf{J} = J$ and the proper orbifold vector bundles E, L are holomorphic on X, then $b_r(x) \in \mathrm{End}(E)_x$ with b_0, b_1 still given by the formula before and $b_r(x)$ are polynomials in R^{TX}, R^E and their derivatives with order $\leq 2r - 1$ at x.

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Theorem (Ma-Marinescu) There exists $C_L > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \Omega^{>0}(X, L^p \otimes E) = \bigoplus_{q \geqslant 1} \Omega^{0,q}(X, L^p \otimes E)$,

$$\|D_p s\|_{L^2}^2 \geqslant (2p\mu_0 - C_L)\|s\|_{L^2}^2 .$$

Moreover $\operatorname{spec} D_p^2 \subset \{0\} \cup [2p\mu_0 - C_L, +\infty[$.

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Finite propagation speed technique

- Transplanting the problem to $\mathbb{R}^{2n}$—the usual yoga

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- Rescaling and Taylor expansion in $\mathbb{R}^{2n}$

Now we can pretend that $X = \mathbb{R}^{2n}$ (with a curved metric) with the point of interest x_0 being identified with $0 \in \mathbb{R}^{2n}$. Let

$$t = \frac{1}{\sqrt{p}}$$

. For $s \in C^\infty(\mathbb{R}^{2n}, \mathbf{E}_{x_0})$ and $Z \in \mathbb{R}^{2n}$, set

$$(S_t s)(Z) = s(Z/t), \quad \mathbf{D}_t = S_t^{-1} t D_p S_t.$$

Then we have

$$\mathbf{D}_t = \mathbf{D}_0 + \sum_{r=1}^{m} t^r \mathbf{A}_r + O(t^{m+1}),$$

where

$$\mathbf{D}_0 = \sum_j c(e_j)\Big(\nabla_{e_j} + \frac{1}{2}R^L_{x_0}(Z, e_j)\Big),$$

and $\mathbf{A}_r$ denotes operator of the type

$$\sum_{i=1}^{2n} c(e_i)\Big(\mathcal{A}_{i,r}\nabla_{e_i} + \mathcal{B}_{i,r-1} + \mathcal{C}_{i,r+1}\Big)$$

with $\mathcal{B}_{i,r}$ (resp. $\mathcal{A}_{i,r}$, resp. $\mathcal{C}_{i,r}$) $(r \in \mathbb{N}, i \in \{1, \cdots, 2n\})$ homogeneous polynomials in Z of degree r with coefficients polynomials in R^{TX}, $R^{\det}$, R^E (resp. R^{TX}, resp. R^L, R^{TX}) and their derivatives at x_0 to order $r-1$ (resp. $r-2$, resp. $r-1$, $r-2$)

Now if we denote by $J_{r,u}$ the Taylor coefficient of order r for the heat kernel $e^{-u\mathbf{D}_t^2}$ at $t=0$, then $J_{r,u}$ is given explicitly by

$$\sum_{\sum_{i=1}^{j} r_i = r} (-1)^j \int_{u\Delta_j} e^{-(u-u_j)\mathbf{D}_0^2} \mathbf{B}_{r_j} e^{-(u_j-u_{j-1})\mathbf{D}_0^2} \cdots \mathbf{B}_{r_1} e^{-u_1\mathbf{D}_0^2} du_1 \cdots du_j,$$

where

$$\mathbf{D}_t^2 = \mathbf{D}_0^2 + \sum_{r=1}^{\infty} \mathbf{B}_r t^r.$$

Since $\mathbf{D}_0^2$ is a generalized harmonic oscillator, its heat kernel $e^{-u\mathbf{D}_0^2}$ can be computed explicitly.

Now of course there are some technical difficulties, such as the lack of usual elliptic estimate for the rescaled operators. It turns out that one can introduce a family of Sobolev norms defined by the rescaled connection on L^p. Then we can extend the functional analysis technique developed by **Bismut-Lebeau**.

Thank you!