Holomorphic functions and subelliptic heat kernels over lie groups.

Later parts of the talk describes work with Leonard Gross, Laurent Saloff-Coste.

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GEOQUANT

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Some papers

- D., Gross, Saloff-Coste. Holomorphic functions and subelliptic heat kernels over Lie groups. *J. European Mathematical Society*, **11**, 941—-978 (2009).

- D., Gross, Saloff-Coste. Surjectivity of the Taylor map for complex nilpotent Lie groups. *Math. Proc. Camb. Phil. Soc.*, **146**, 177–195. (2009)

- D., Gross, Saloff-Coste. Growth of Taylor coefficients over complex homogeneous spaces. *Preprint.*

Basic Notation

- $(G, o) =$ a pointed complex manifold
- $\mathcal{H}(G) := \{f : G \to \mathbb{C} | \ f$ is holomorphic$\}$
- λ be a measure on G,

$$(f, g)_{L^2(\lambda)} := \int_G f \cdot \bar{g} \, d\lambda$$

$$\|f\|^2_{L^2(\lambda)} := (f, f)_{L^2(\lambda)} = \int_G |f|^2 \, d\lambda$$

$$\mathcal{H}L^2(G) := \left\{ f \in \mathcal{H}(G) : \|f\|_{L^2(\lambda)} < \infty \right\}$$

Fock, Kakutani, Itô, Segal, Bargmann, Gross, Hall

- Suppose $G =$Compact type Lie group and $o = e$
- $\mathcal{D}_t$ is a space of derivatives of holomorphic functions
- $Tf := \{$"derivatives" of f at $e\}$

$$\begin{array}{ccccc} & & T \circ e^{t\Delta/2} & & \\ \mathrm{L}^2(G, p_t) & & - - - - - - - - - - \longrightarrow & & \mathcal{D}_t \\ & \searrow & & \nearrow & \\ \mathcal{A} \circ e^{t\Delta/2} & & \mathcal{H}\mathrm{L}^2(G^{\mathbb{C}}, \mu_t) & & T \end{array} \tag{1}$$

Two Basic Questions

Let

$$\mathcal{D} := \{\text{“derivatives” of } f \text{ at } o : f \in \mathcal{H}(G)\}$$

be the **derivative space** associated to $\mathcal{H}(G)$ and let $T : \mathcal{H}(G) \to \mathcal{D}$ be the “**Taylor map;**”

$$Tf := \{\text{“derivatives” of } f \text{ at } o\}.$$

1. Characterize the derivative space, $\mathcal{D}$.

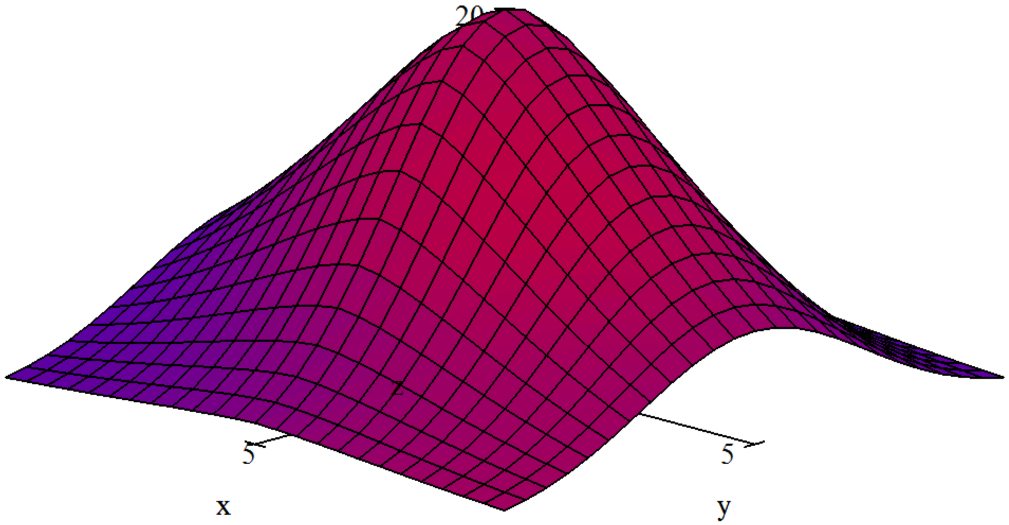
2. Find the norm, $\|\cdot\|_{\mathcal{D}}$, on $\mathcal{D}$ such that

$$\boxed{\int_G |f|^2 \, d\lambda = \|Tf\|_{\mathcal{D}}^2 \text{ for all } f \in \mathcal{H}(G).}$$

- We will begin with the case, $G = \mathbb{C}$ and $o = 0$.

The Case $G = \mathbb{C}$

- Let $G = \mathbb{C}$ and $o = 0$
- $z = x + iy$
- $dm(z) = dxdy$
- $d\lambda = \rho(z)\, dm(z)$ with $\rho \in C(\mathbb{C}, (0, \infty))$.



$$\rho \in C(\mathbb{C}, (0, \infty))$$

Notation (Taylor map)**.** Given a function, f, which is holomorphic near 0, let

$$Tf = \left\{f^{(n)}(0)\right\}_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}_0}.$$

Notation (Derivative Space)**.** Let

$$\mathcal{D} := \left\{\alpha := \{\alpha_n\}_{n=0}^{\infty} \subset \mathbb{C} : \limsup_{n\to\infty} \left|\frac{\alpha_n}{n!}\right|^{1/n} = 0\right\}.$$

Theorem 1 (Taylor's Theorem & Root Test)**.** *If $f \in \mathcal{H}(\mathbb{C})$ then*

- ***The Root Test:*** $Tf \in \mathcal{D}$,
- ***Taylor's Theorem:*** $T : \mathcal{H}(\mathbb{C}) \to \mathcal{D}$ *is a linear isomorphism with inverse,*

$$T^{-1}(\alpha)(z) = \sum_{n=0}^{\infty} \frac{\alpha_n}{n!} z^n \text{ for all } z \in \mathbb{C}.$$

Goals

1. Develop some basic properties of $\mathcal{H}L^2(\lambda)$.

2. Identify the norm on $\mathcal{D}$ which makes

$$T|_{\mathcal{H}L^2(\lambda)} : \mathcal{H}L^2(\lambda) \to \mathcal{D} \quad \textbf{isometric.}$$

3. Characterize the image, $T\left(\mathcal{H}L^2(\lambda)\right) \subset \mathcal{D}$, of the Taylor map.

Complex Analysis Basics

- $\partial_x := \frac{\partial}{\partial x}$ and $\partial_y := \frac{\partial}{\partial y}$.
- $\partial := \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\bar{\partial} := \partial_{\bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ so
- $\partial_x = \partial + \bar{\partial}$ and $\partial_y = i\left(\partial - \bar{\partial}\right)$.
- If $\Delta := \partial_x^2 + \partial_y^2$, then

$$\Delta = \left(\partial + \bar{\partial}\right)^2 - \left(\partial - \bar{\partial}\right)^2 = 4\partial\bar{\partial}.$$

Corollary 2. *If $f \in \mathcal{H}(\mathbb{C})$, then $|f|^2$ is sub-harmonic,*

$$\Delta |f|^2 = 4|\partial f|^2 = 4|\partial_x f|^2 \geq 0.$$

Proof: The Cauchy Riemann equations imply,

$$\bar{\partial} f = 0 \text{ and } \partial \bar{f} = 0$$

and therefore,

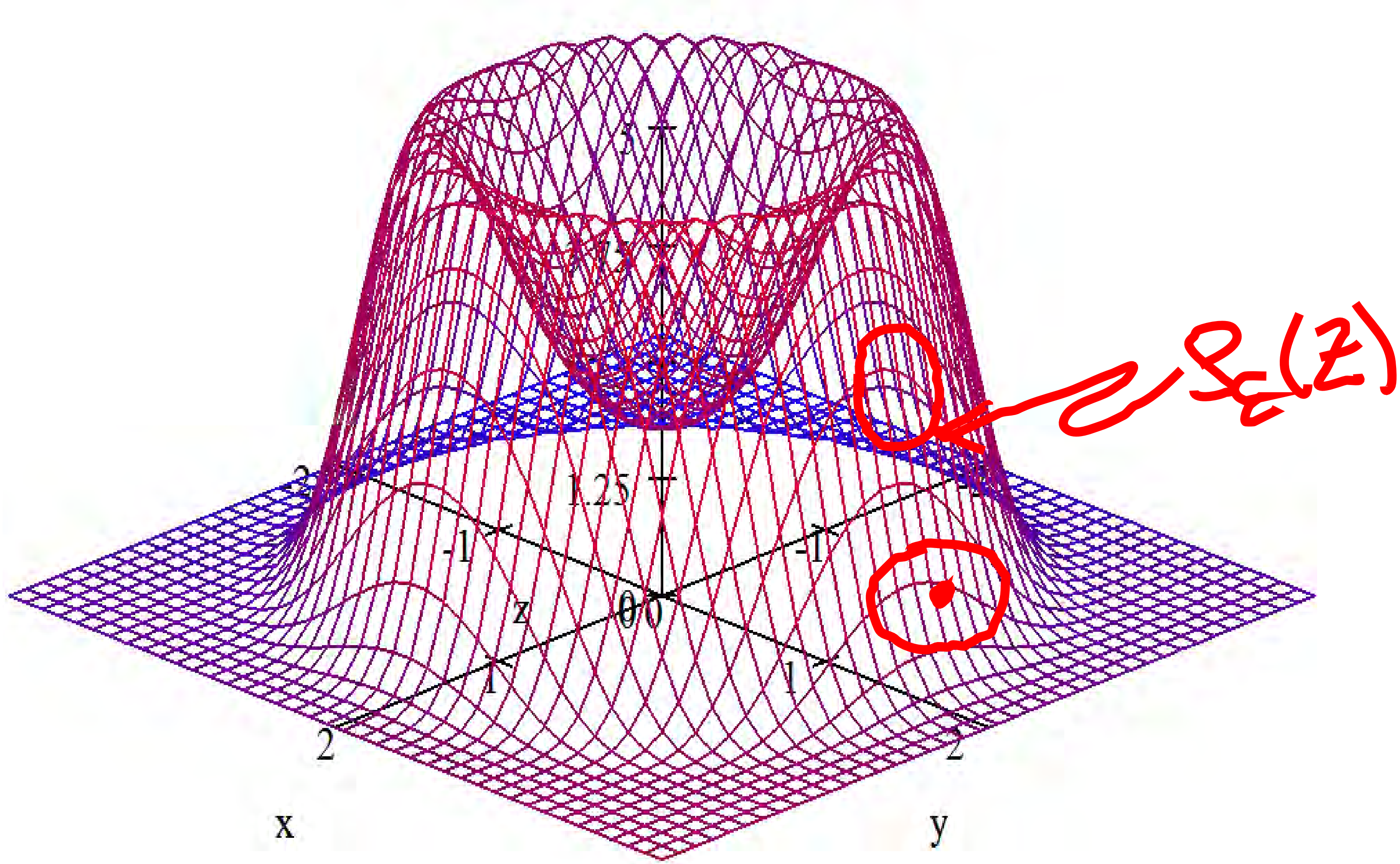
$$\Delta |f|^2 = 4\partial\bar{\partial}\left(f\ \bar{f}\right) = 4\partial f\ \bar{\partial}\bar{f}.$$

Q.E.D.

Pointwise Bounds

Notation. For every $\varepsilon > 0$, let

$$\rho_\varepsilon(z) := \min\{\rho(w) : w \in D(z,\varepsilon)\}.$$



Theorem 3 (Crude Pointwise Bounds)**.** *Suppose that $g \geq 0$ is a sub-harmonic (i.e. $\Delta g \geq 0$), then*

$$g(z) \leq \|g\|_{L^1(\lambda)} \frac{1}{\pi\varepsilon^2} \frac{1}{\rho_\varepsilon(z)} \; \forall\, z \in \mathbb{C}. \tag{2}$$

In particular if $f \in \mathcal{H}(\mathbb{C})$, then

$$|f(z)|^2 \leq \frac{1}{\pi\varepsilon^2} \|f\|^2_{L^2(\lambda)} \frac{1}{\rho_\varepsilon(z)} \; \forall\, z \in \mathbb{C}. \tag{3}$$

Proof: By the mean value inequality (42),

$$\begin{aligned} g(z) &\leq \fint_{D(z,\varepsilon)} g(w)\, dm(w) \\ &= \fint_{D(z,\varepsilon)} g(w) \frac{1}{\rho(w)} \rho(\omega)\, dm(w) \\ &\leq \frac{1}{\rho_\varepsilon(z)} \fint_{D(z,\varepsilon)} g(w)\, \rho(\omega)\, dm(w) \\ &= \frac{1}{\pi\varepsilon^2} \frac{1}{\rho_\varepsilon(z)} \|g\|_{L^1(\lambda)}. \end{aligned}$$

For Eq. (3), apply (2) with $g(z) := |f(z)|^2$. Q.E.D.

Liouville's Theorem

Definition 4. Let $\mathcal{HP}(\mathbb{C})$ denote the space of holomorphic polynomials. Further let,

$$\mathcal{HP}_k = \{p \in \mathcal{HP}(\mathbb{C}) : \deg(p) \leq k\} = \left\{ p(z) = \sum_{n=0}^{k} a_n z^k : a_n \in \mathbb{C} \right\}.$$

Corollary 5 (Louiville's Theorem)**.** *Suppose there exists $c < \infty$ and $n \in \mathbb{N}_0$ such that*

$$\rho(z) \geq \frac{c}{|z|^{2n} + 1} \text{ for all } z \in \mathbb{C}.$$

Then $\mathcal{H}L^2(\rho) = \mathcal{HP}_k$ for some $k < n$ where $\mathcal{HP}_k := \{0\}$ if $k \leq 0$.

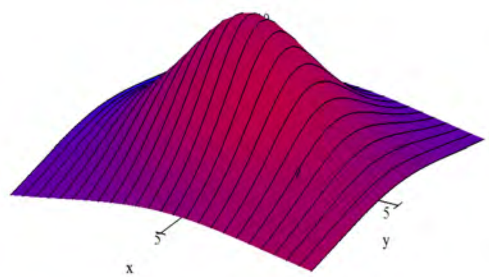


Figure 1: Here, $\rho(z) = 10/\left(1 + 0.1 \cdot |z|^2\right)$ and $\mathcal{H}L^2(\lambda) \cong \mathbb{C}$.

Proof: Use the pointwise bounds along with Cauchy estimates (see 44). Q.E.D.

A Non-Uniform Decay Example

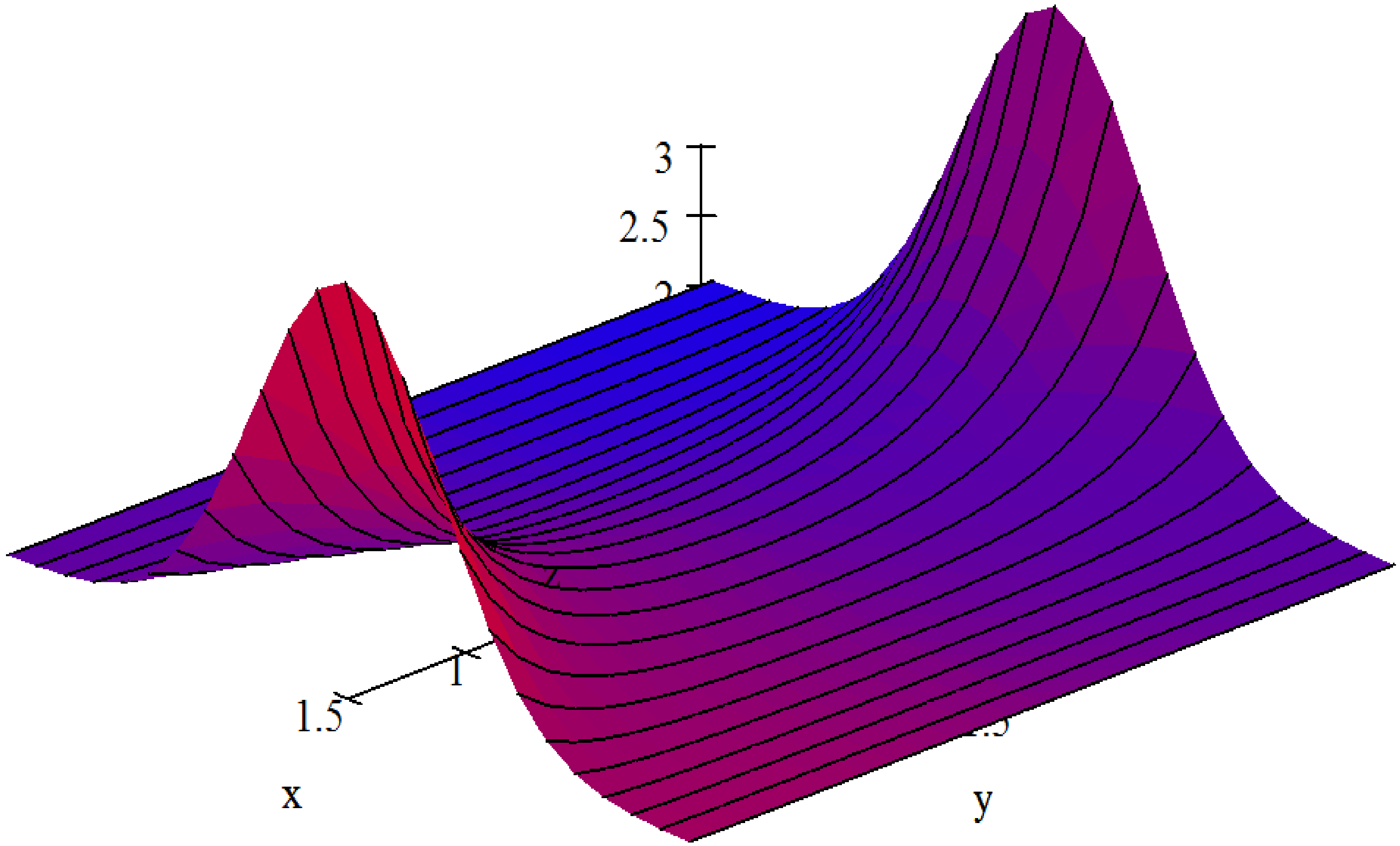


Figure 2: Plot of $d\lambda_c/dm$ for $c = 2$.

- For $c > 0$, let

$$d\lambda_c(z) = \frac{1}{\pi} \exp\left(-\left((1-c)\,x^2 + (1+c)\,y^2\right)\right) dx\, dy.$$

- If $c > 1, \mathcal{H}L^2(\lambda_c)$ does **not** contain any polynomials other than 0.

- Nevertheless, $\dim \mathcal{H}L^2(\lambda_c) = \infty$ since

$$\mathcal{H}L^2(\lambda_0) \ni f(z) \to e^{\frac{c}{2}z^2} f(z) \in \mathcal{H}L^2(\lambda_c)$$

  is unitary.

$\mathcal{H}L^2(\lambda)$ is a Hilbert Space

Theorem 6 ($\mathcal{H}L^2(\lambda)$ is Hilberitan). *$\mathcal{H}L^2(\lambda)$ is a closed subspace of $L^2(\lambda)$ and hence is a **Hilbert** space.*

Proof: For $\varepsilon, r > 0$ let

$$C(r,\varepsilon) := \frac{1}{\pi\varepsilon^2} \cdot \sup_{|w| \leq r} \frac{1}{\rho_\varepsilon(w)} = \frac{1}{\pi\varepsilon^2} \cdot \frac{1}{\min_{|w| \leq r+\varepsilon} \rho_\varepsilon(w)}.$$

Then by the pointwise bounds,

$$\sup_{|w| \leq r} |f(w)|^2 \leq C(r,\varepsilon) \|f\|^2_{L^2(\lambda)} \text{ for all } f \in \mathcal{H}L^2(\lambda).$$

So if $\{f_n\}_{n=1}^\infty \subset \mathcal{H}L^2(\lambda)$ and $f_n \to f \in L^2(\lambda)$, we have,

$$\sup_{|w| \leq r} |f_n(w) - f_m(w)|^2 \leq C(r,\varepsilon) \|f_n - f\|^2_{L^2(\lambda)} \to 0 \text{ as } m,n \to \infty.$$

So $\{f_n\}_{n=1}^\infty$ is uniformly convergent on compacts and therefore $f \in \mathcal{H}(\mathbb{C})$. Q.E.D.

The Reproducing Kernel

Theorem 7. *There exists a unique function,* $k(z,w) = k_\lambda(z,w) \in \mathbb{C}$ *such that for all* $w \in \mathbb{C}$, *there exists a unique* $k(\cdot,w) \in \mathcal{H}L^2(\lambda)$ *such that*

$$f(w) = (f, k(\cdot,w))_{L^2(\lambda)} \ \forall\, f \in \mathcal{H}L^2(\lambda). \tag{4}$$

Moreover (see 45)

1. $k(w,z) = (k(\cdot,z), k(\cdot,w))$ *and hence* $\overline{k(w,z)} = k(z,w)$.
2. $k(z,\bar{w})$ *is a holomorphic function of* (z,w).
3. *If* $\{\varphi_n\}_{n=0}^\infty \subset \mathcal{H}L^2(\lambda)$ *is any orthonormal basis, then*

$$k(z,w) = \sum_{n=0}^{\infty} \varphi_n(z)\,\overline{\varphi_n(w)}. \tag{5}$$

   The sum is absolutely convergent.

4. *For all* $w, z \in \mathbb{C}$,

$$\|k(\cdot,z)\|_{L^2(\lambda)}^2 = k(z,z) \le \frac{1}{\pi\varepsilon^2}\frac{1}{\rho_\varepsilon(z)} \text{ and}$$

$$|k(z,w)| \le \sqrt{k(z,z)\cdot k(w,w)} \le \frac{1}{\pi\varepsilon^2}\frac{1}{\sqrt{\rho_\varepsilon(z)\cdot\rho_\varepsilon(w)}}.$$

Optimal Pointwise Bounds

Corollary 8 (Optimal Pointwise Bounds)**.** *For all* $f \in \mathcal{H}L^2(\lambda)$,

$$|f(w)|^2 \le k(w,w)\,\|f\|^2_{L^2(\lambda)} \text{ for all } w \in \mathbb{C}.$$

These pointwise bounds are optimal.

Proof: By the Cauchy-Schwarz inequality,

$$|f(w)|^2 = \left|(f, k(\cdot,w))_{L^2(\lambda)}\right|^2$$

$$\le \|k(\cdot,w)\|^2_{L^2(\lambda)}\,\|f\|^2_{L^2(\lambda)} = k(w,w)\,\|f\|^2_{L^2(\lambda)}.$$

The function $f(z) := k(z,w)$ saturates this inequality. Q.E.D.

The Radial Symmetric Case

Theorem 9. *Suppose that* $\rho(z) = \rho(|z|)$ *and* $\mathcal{HP} \subset \mathcal{H}L^2(\lambda)$, *i.e.*

$$a_n^2 := \int_{\mathbb{C}} |z|^{2n} \rho(z)\, dm(z) < \infty \text{ for all } n \in \mathbb{N}_0.$$

Then

1. $\left\{\frac{z^n}{a_n}\right\}_{n=0}^{\infty}$ *forms an orthonormal basis for* $\mathcal{H}L^2(\lambda)$.

2. *For any* $f \in \mathcal{H}L^2(\lambda)$,

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

converges pointwise and $L^2(\lambda)$.

Proof

If $f \in \mathcal{H}L^2(\lambda)$, then (using Taylor's theorem to evaluate the angular integral)

$$(f, z^n) = \int_0^\infty \left(\int_{-\pi}^{\pi} f\left(re^{i\theta}\right) r^n e^{-in\theta} d\theta \right) \rho(r)\, r dr$$

$$= \int_0^\infty \left(2\pi r^{2n} \frac{f^{(n)}(0)}{n!} \right) \rho(r)\, r dr = a_n^2 \frac{f^{(n)}(0)}{n!}.$$

From this it follows that

$$\left\{ \frac{z^n}{a_n} \right\}_{n=0}^{\infty} \text{ is orthonormal subset of } \mathcal{H}L^2(\lambda).$$

Let $P : \mathcal{H}L^2(\lambda) \to \mathcal{H}L^2(\lambda)$ be orthogonal projection onto $\overline{\mathcal{HP}}$. Then

$$Pf = \sum \frac{1}{a_n^2} (f, z^n) z^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \; \forall\, f \in \mathcal{H}L^2(\lambda)$$

converges in $L^2(\lambda)$ and pointwise to f (by Taylor's theorem) and so $f = Pf \in \overline{\mathcal{HP}}$.

Density of Polynomials

Corollary 10 (Density of Polynomials). *When* $\rho(z) = \rho(|z|)$, $\mathcal{HP}$ *is dense in* $\mathcal{H}L^2(\lambda)$.

Proof: See the above proof or see 47 for an alternate proof. Q.E.D.

Question. Under what conditions on ρ is $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$?

Remark. We know (see 53 or 10) $\mathcal{HP}$ is dense in $\mathcal{H}L^2(\lambda)$ if $\rho(z) = \tilde{\rho}(|az+b|)$ for some $a \neq 0$. It is also true if

$$\rho(z) = C \exp\left(-\left(ax^2 + 2bxy + cy^2\right)\right)$$

for some $a, b > 0$ and $c \in \mathbb{R}$ such that $b^2 - ac < 0$.

Radial Symmetric Case Summary

Notation. The **Taylor map** is: $Tf := \alpha \in \mathcal{D}$, where $\alpha_n := f^{(n)}(0)$. Let,

$$a_n^2 := \int_{\mathbb{C}} |z|^{2n} d\lambda(z), \qquad \|\alpha\|_\rho^2 := \sum_{n=0} |\alpha_n|^2 \left(\frac{a_n}{n!}\right)^2, \text{ and}$$

$$J(\lambda) := \left\{\alpha = (\alpha_0, \alpha_1, \dots) \in \mathbb{C}^{\mathbb{N}_0} : \|\alpha\|_\rho^2 < \infty\right\}.$$

Theorem 11 (Radial Case). *If $\rho(z) = \rho(|z|)$, then $T : \mathcal{H}L^2(\lambda) \to J(\lambda)$ is **unitary**.*

Moreover, for all $f \in \mathcal{H}(\mathbb{C})$,

$$\int_{\mathbb{C}} |f(z)|^2 \rho(z)\, dm(z) = \sum_{n=0} \left|f^{(n)}(0)\right|^2 \left(\frac{a_n}{n!}\right)^2 \qquad \text{(\textbf{Isometry Property.})}$$

and

$$|f(z)|^2 \le \|f\|_{L^2(\lambda)}^2 \left(\sum_{n=0}^{\infty} \frac{1}{a_n^2} |z|^{2n}\right). \qquad \text{(\textbf{Optimal Pointwise Bounds.})}$$

$$k(z,w) = k_\lambda(z,w) = \sum_{n=0}^{\infty} \frac{1}{a_n^2} (z\bar{w})^n \qquad \text{(\textbf{Reproducing Kernel.})}$$

Proof (Skip)

The fact that $T : \mathcal{H}L^2(\lambda) \to J(\lambda)$ is unitary is a translation of the fact that $\left\{\frac{z^n}{a_n}\right\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}L^2(\lambda)$ and the identity,

$$(f, z^n) = a_n^2 \frac{f^{(n)}(0)}{n!}.$$

To see the isometry property is valid for all $f \in \mathcal{H}(\mathbb{C})$, use $T : \mathcal{H}L^2(\lambda) \to J(\lambda)$ is unitary, Taylor's theorem, and **Fatou's lemma**, to show;

$$\begin{aligned}
\int_{\mathbb{C}} |f(z)|^2 \rho(z)\, dm(z) &= \int_{\mathbb{C}} \liminf_{N\to\infty} \left| \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n \right|^2 \rho(z)\, dm(z) \\
&\leq \liminf_{N\to\infty} \int_{\mathbb{C}} \left| \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} z^n \right|^2 \rho(z)\, dm(z) \\
&= \liminf_{N\to\infty} \sum_{n=0}^{N} a_n^2 \left| \frac{f^{(n)}(0)}{n!} \right|^2 = \sum_{n=0}^{\infty} a_n^2 \left| \frac{f^{(n)}(0)}{n!} \right|^2 .
\end{aligned}$$

Exponential Examples

Notation. For $\kappa > 0$, let

$$\rho_\kappa(z) := \frac{\kappa}{2\pi}\exp\left(-|z|^\kappa\right) \text{ and } \Gamma(z) := \int_0^\infty t^z e^{-t}\frac{dt}{t}$$

Theorem 12. *If $\rho = \rho_\kappa$, then*

$$a_n^2 = \Gamma\left(\frac{2n+2}{\kappa}\right), \qquad k(z,w) = \sum_{n=0}^\infty \frac{1}{\Gamma\left(\frac{2n+2}{\kappa}\right)}(z\bar{w})^n,$$

and for all $f \in \mathcal{H}(\mathbb{C})$,

$$\int_{\mathbb{C}} |f(z)|^2 \frac{\kappa}{2\pi}\exp\left(-|z|^\kappa\right) dm(z) = \sum_{n=0} \left|f^{(n)}(0)\right|^2 \frac{\Gamma\left(\frac{2n+2}{\kappa}\right)}{(n!)^2}$$

and

$$|f(z)|^2 \le \|f\|^2_{L^2(\rho_\kappa dm)}\left(\sum_{n=0}^\infty \frac{|z|^{2n}}{\Gamma\left(\frac{2n+2}{\kappa}\right)}\right).$$

Example $(\kappa = 1)$

$$k(z,w) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} (z\bar{w})^n = \frac{\sinh\left(\sqrt{z\bar{w}}\right)}{\sqrt{z\bar{w}}}$$

For all $f \in \mathcal{H}(\mathbb{C})$,

$$\frac{1}{2\pi} \int_{\mathbb{C}} |f(z)|^2 \exp(-|z|)\, dm(z) = \sum_{n=0} \left| f^{(n)}(0) \right|^2 \frac{(2n+1)!}{(n!)^2},$$

and

$$|f(z)|^2 \leq \|f\|^2_{L^2(\lambda)} \frac{\sinh(|z|)}{|z|} \leq \|f\|^2_{L^2(\lambda)} \frac{1}{2|z|} e^{|z|}.$$

Example $(\kappa = 2)$

$$d\lambda(z) = \frac{1}{\pi} \exp\left(-|z|^2\right) dm(z)$$

$$k(z,w) = \sum_{n=0}^{\infty} \frac{1}{n!} (z\bar{w})^n = e^{z\bar{w}}.$$

For all $f \in \mathcal{H}(\mathbb{C})$,

$$\frac{1}{\pi} \int_{\mathbb{C}} |f(z)|^2 \exp\left(-|z|^2\right) dm(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left|f^{(n)}(0)\right|^2,$$

and

$$|f(z)|^2 \leq \|f\|_{L^2(\lambda)}^2 e^{|z|^2}.$$ (**Bargmann's** Pointwise Bounds)

References: V.A. Fock (1932) [Fock, 1928], Segal [Segal, 1956, Segal, 1962] and Bargmann [Bargmann, 1961]. (See also Gross and Malliavin [Gross & Malliavin, 1996] for more history.)

Heat Kernel Interpretation for $\kappa = 2$

Fact.

$$\left(e^{t\Delta/4}g\right)(z) = \int_{\mathbb{C}} \frac{1}{\pi t} \exp\left(-\left|z-w\right|^2/t\right) g(w)\, dm(\omega).$$

In particular taking $t = 1$ and $z = 0$ implies,

$$\int_{\mathbb{C}} \left|f(z)\right|^2 \frac{1}{\pi} \exp\left(-\left|z\right|^2\right) dm(z) = \left(e^{\Delta/4}\left|f\right|^2\right)(0).$$

Recalling that $\Delta = 4\partial\bar{\partial}$ and that $\partial\bar{\partial}\left|f\right|^2 = \left|\partial f\right|^2$, we have formally,

$$e^{\Delta/4}\left|f\right|^2 = e^{\partial\bar{\partial}}\left|f\right|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial\bar{\partial}\right)^n \left|f\right|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \left|\partial^n f\right|^2.$$

Combining these last two equations explains why (in this case) that

$$\int_{\mathbb{C}} \left|f(z)\right|^2 \frac{1}{\pi} \exp\left(-\left|z\right|^2\right) dm(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \left|f^{(n)}(0)\right|^2.$$

The Segal – Bargmann Transform

Theorem 13 (The Segal – Bargmann isometry)**.** *For all* $f \in L^2(\mathbb{R}, d\lambda)$,

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)|^2 e^{-x^2/2} dx = \frac{1}{\pi} \int_{\mathbb{C}} \left| \left(e^{\frac{1}{2}\partial_x^2} f \right)_a (z) \right| \exp\left(-\frac{1}{4} |z|^2 \right) dm(z) .$$

Also see 52.

Proof: Let us recall,

$$\partial_x^2 = \left(\partial + \bar{\partial}\right)^2 = \partial^2 + \bar{\partial}^2 + 2\partial\bar{\partial}.$$

By density of $\mathcal{H}L^2(\mathbb{C}, \lambda)$ in $L^2(\mathbb{R}, \lambda)$, it suffice to assmue $f \in \mathcal{H}L^2(\mathbb{C}, \lambda)$. In this case,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)|^2 e^{-x^2/2} dx &= \left(e^{\frac{1}{2}\partial_x^2} |f|^2 \right)(0) = \left(e^{\frac{1}{2}\left[\partial^2 + \bar{\partial}^2 + 2\partial\bar{\partial}\right]} |f|^2 \right)(0) \\
&= \left(e^{\partial\bar{\partial}} e^{\frac{1}{2}\partial^2} e^{\frac{1}{2}\bar{\partial}^2} \left[f \cdot \bar{f} \right] \right)(0) = e^{\partial\bar{\partial}} \left(e^{\frac{1}{2}\partial^2} f \cdot e^{\frac{1}{2}\bar{\partial}^2} \bar{f} \right)(0) \\
&= e^{\Delta_{\mathbb{C}}/4} \left(\left| e^{\frac{1}{2}\partial^2} f \right|^2 \right)(0) \\
&= \frac{1}{\pi} \int_{\mathbb{C}} \left| \left(e^{\frac{1}{2}\partial_x^2} f|_{\mathbb{R}} \right)_a (z) \right| \exp\left(-\frac{1}{4} |z|^2 \right) dm(z) .
\end{aligned}$$

Q.E.D.

Generalizations to Lie Groups

- $G =$ complex simply connected Lie group (e.g. $SL(n, \mathbb{C})$)
- $\mathfrak{g} = T_e G$ its Lie algebra (e.g. $sl(n, \mathbb{C})$)
- $\mathfrak{g}^* =$ the dual space of $\mathfrak{g}$
- $q =$ a non-negative Hermitian form on $\mathfrak{g}^*$ (e.g. $q(A, B) = \mathrm{tr}(B^* A)$)

Fact. There exists $m \leq \dim_{\mathbb{C}}(\mathfrak{g})$ and a linearly independent set, $\{X_l\}_{l=1}^m$, such that

$$q(\alpha, \beta) = \sum_{l=1}^{m} \alpha(X_l) \overline{\beta(X_l)}$$

for all $\alpha, \beta \in \mathfrak{g}^*$.

Definition 14 (Horizontal subspace)**.** The horizontal subspace associated to q is $H = H(q) := \mathrm{span}(X_l : 1 \leq l \leq m)$ with the inner product: $(X_l, X_k)_H := \delta_{lk}$.

Derivative Spaces

- $q^{\otimes k}$ = the extension of q to $(\mathfrak{g}^{\otimes k})^*$, i.e.

$$q^{\otimes k}(\alpha) = \sum_{l_1,\dots,l_k=1}^{m} \left|\alpha\left(X_{l_1} \otimes \cdots \otimes X_{l_k}\right)\right|^2$$

- $T(\mathfrak{g})$ is the tensor algebra over $\mathfrak{g}$ and $T(\mathfrak{g})' = \Pi_{k=0}^{\infty}(\mathfrak{g}^{\otimes k})^*$.

- For each $t > 0$ define

$$q_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} q^{\otimes k} \tag{6}$$

- $J = \langle \xi \otimes \eta - \eta \otimes \xi - [\xi, \eta] : \xi, \eta \in \mathfrak{g} \rangle \subset T(\mathfrak{g})$

- $J^0 = \{\alpha \in T(\mathfrak{g})' : \alpha|_J \equiv 0\}$ – the "Derivative Space."

- $$J_t^0 := \left\{\alpha \in J^0 : q_t(\alpha) < \infty\right\}.$$

Two Algebraic Theorems

Definition 15 (*Hörmander's condition*)**.** We say q satisfies *Hörmander's condition* if $\mathrm{Lie}\,(H\,(q)) = \mathfrak{g}$.

Theorem 16 (D., Gross, Saloff-Coste)**.** *The following are equivalent:*

1. *Hörmander's condition holds, i.e.* $\mathrm{Lie}\,(H) = \mathfrak{g}$.
2. $T\,(\mathfrak{g}) = T\,(H) + J$.
3. *for any* $t > 0$, $q_t|_{J_t^0}$ *is an inner product on* J_t^0.

Theorem 17 (D., Gross, Saloff-Coste)**.** *If* $\mathfrak{g}$ *is "stratified," then the finite rank tensors in* J^0 *are dense in* J_t^0.

Remark. For general $\mathfrak{g}$ there are typically **no** finite rank tensors in J^0, see [Gross, 1998].

The Heat Kernel

- $\tilde{A}(g) = L_{g*}A$ for all $A \in g$ and $g \in G$

- (**Laplacian**) $\Delta = \Delta_q := \sum_{l=1}^{m} \left[\tilde{X}_l^2 + \left(\widetilde{iX_l}\right)^2\right]$

- (**Heat Kernel**) Let $\rho_t : G \to (0, \infty)$ satisfy,

$$\left(e^{t\Delta/4}f\right)(e) = \int_G f(g)\,\rho_t(g)\,dg.$$

where dg denotes a *right Haar* measure on G.

Fact. The heat kernel, ρ_t, satisfies:

$$\begin{cases} \partial\rho_t(x,\cdot)/\partial t = (1/4)\Delta\rho_t(x,\cdot) \\ \rho_t(x,y)dy \to \delta_x(dy) \text{ (weakly) as } t \to 0. \end{cases} \tag{7}$$

$\rho_t \in C^\infty(G, (0,\infty))$ by Hörmander's theorem [Hörmander, 1967].

The Taylor Isomorphism

- Let $\mathcal{H}$ = the holomorphic functions on G
- For $\beta = A_1 \otimes \cdots \otimes A_n \in T(\mathfrak{g})$, let $\tilde{\beta} = \tilde{A}_1 \dots \tilde{A}_n$
- For $f \in \mathcal{H}$ and $x \in G$, let

$$\langle \hat{f}(x), \beta \rangle = (\tilde{\beta} f)(x) \; \forall \, \beta \in T(\mathfrak{g}) \,. \tag{8}$$

- $\hat{f}(x) \in J^0$ is the **Taylor coefficient** at x.
- *Taylor map* $\left(Tf := \hat{f}(e)\right)$,

$$\mathcal{H} \cap L^2(G, \rho_t) \ni f \xrightarrow{T} \hat{f}(e) \in J_t^0. \tag{9}$$

Theorem 18 (D., Gross, Saloff-Coste)**.** *If G is simply connected and q satisfies Hörmander's condition, then the* ***Taylor map,*** $T : \mathcal{H}L^2(\rho_t) \to J_t^0$ ***is unitary****. Moreover,*

$$\int_G |f(g)|^2 \rho_t(g)\, dg = \left\| \hat{f}(e) \right\|_t^2 \text{ for all } f \in \mathcal{H}(G) \,.$$

The “Classical” Example

- $G = \mathbb{C}^d$ with additive group structure
- $H = \mathfrak{g} = \mathbb{C}^d$, $X_l = e_l$ for $l = 1, 2, \ldots, d = m$
- $q(\alpha) = \sum_{l=1}^d |\alpha(e_l)|^2$
- $d(w, z) = |z - w|$
- $\Delta = \sum_{l=1}^d \left(\frac{\partial^2}{\partial x_l^2} + \frac{\partial^2}{\partial y_l^2} \right)$ where $z = x + iy$.

$$\rho_t(z) = \left(\frac{1}{\pi t}\right)^d \exp\left(-|z|^2/t\right)$$

- J^0 = Symmetric Tensors = Bosonic Fock Space
- For $f \in \mathcal{H}$, $\hat{f}(z) \in J^0$ since mixed partial derivatives commute.
- References: V.A. Fock (1932) [Fock, 1928], Segal [Segal, 1956, Segal, 1962] and Bargmann [Bargmann, 1961]. (See also Gross and Malliavin [Gross & Malliavin, 1996] for more history.)
- For proofs, go to 54 and 55.

Some History

The Taylor Isomorphism Theorem 18 was known to hold for *non-degenerate* q in the following cases:

1. $G = K_{\mathbb{C}}$: Driver [Driver, 1995] (inspired by B. Hall [Hall, 1994])
2. G arbitrary: Driver and Gross [Driver & Gross, 1997]
3. $G =$ infinite dimensional complex Hilbert-Schmidt orthogonal group: M. Gordina, [Gordina, 2000b] and [Gordina, 2000a]
4. $G =$ invertible operators in a factor of type II_1: M. Gordina in [Gordina, 2002]
5. $G =$ path and loop groups of a "stratified" Lie group: M. Cecil, in [Cecil, 2006].
6. $G =$ infinite dimensional Heisenberg like groups, see [Driver & Gordina, 2008b, Driver & Gordina, 2008a, Driver & Gordina, 2008c].
7. For the case presented here see, [Driver *et al.*, 2009b], [Driver *et al.*, 2009c], and [Driver *et al.*, 2009a].

Isometry Proof

- Working analogously to the "$\kappa = 2$ example" above one sees, **formally,** that

$$\|f\|^2_{L^2(\rho_t dm)} = e^{t\Delta/4} |f|^2 (e) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l_1,\dots,l_k=1}^{m} \left|\tilde{X}_{l_1} \dots \tilde{X}_{l_k} f(e)\right|^2 = \left\|\hat{f}\right\|_t^2 .$$

- To make this rigorous takes a fair amount of work and requires:

1. Gaussian heat kernel bounds which involve the "Carnot-Caratheodory" distance on G associated to q (see 56).
2. Good a-priori pointwise bounds for f and there derivatives.
3. Careful attention to the fact that finite rank tensors are not dense in $J_t^0(\mathfrak{g})$ in general.
4. Similarly we must deal with the complication of not knowing a simple to use dense subset of $\mathcal{H}L^2(\rho_t dm)$.

Surjectivity Proof

- The surjectivity proof require the reconstruction of a holomorphic function from its derivatives, $\alpha \in J_t^0(\mathfrak{g})$.

Notation (Rolling Map)**.** Associated to a finite energy path, $g : [0,1] \to G$, from e to $z \in G$, let

$$b(s) = b(g, s) := \int_0^s L_{g(t)^{-1}*}\dot{g}(t)\,dt \in \mathfrak{g}.$$

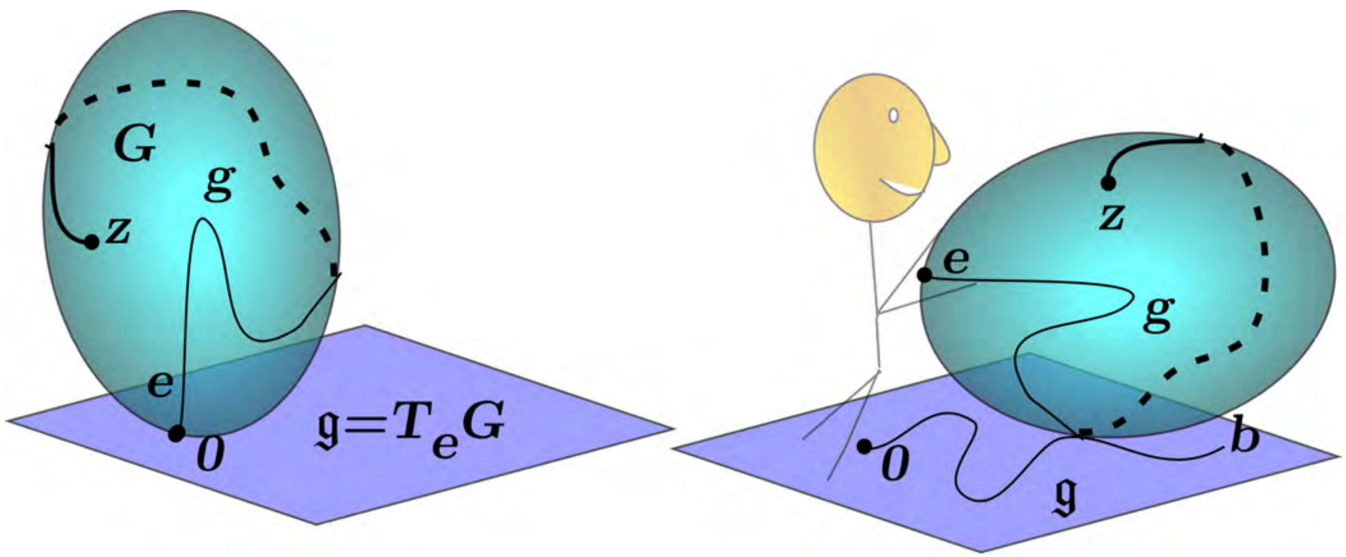


Figure 3: Cartan's rolling map.

Group Taylor Series

Theorem 19 (A Reconstruction Theorem). *Suppose;*

1. $g:[0,1]\to G$ *such that* $g(0)=e$ *and* $g(1)=z$,
2. $f\in\mathcal{H}$ *or* f *is holomorphic near* $e\in G$,
3. $\Psi(g):=\sum_{n=0}^{\infty}\Psi_n(g)$ *where*

$$\Psi_n(g):=\int_{0\le s_1\le\cdots\le s_n\le 1} db(s_1)\otimes db(s_2)\otimes\cdots\otimes db(s_n).$$

Then

$$f(z)=\left\langle\hat{f}(e),\Psi(g)\right\rangle=\sum_{n=0}^{\infty}\left\langle\hat{f}(e),\Psi_n(g)\right\rangle \tag{10}$$

and if g *is horizontal, i.e.* $b(s)\in H$, *we have the* ***pointwise bounds,***

$$|f(z)|^2\le\|\hat{f}(e)\|_t^2 e^{d_H^2(e,z)/t}\le\|\hat{f}(e)\|_t^2 e^{\ell_H^2(g)/t} \tag{11}$$

An Exponential Path Example

Suppose that $b(s) = sA$ for some $A \in \mathfrak{g}$. Then

$$g(s) = e^{sA}$$

and Eq. (10) reduces to the familiar formula,

$$f\left(e^{A}\right) = \left\langle \hat{f}(e), \sum_{n=0}^{\infty} \frac{1}{n!} A^{\otimes n} \right\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\tilde{A}^{n} f\right)(e).$$

Proof

For $b \in H^1(\mathfrak{g})$ (the **finite energy paths** in $\mathfrak{g}$) let $g_t(b)$ solve (see Figure 3)

$$\dot{g}_t(b) = L_{g_t(b)*}\dot{b}(t) \text{ with } g_0(b) = e.$$

1. The map $H^1(\mathfrak{g}) \ni b \to g_1(b) \in G$ is holomorphic.

2. The map $H^1(\mathfrak{g}) \ni b \to f(g_1(b)) \in \mathbb{C}$ is holomorphic.

3. By **Taylor's Theorem**,

$$f(g_1(b)) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial_b^n (f \circ g_1)(0).$$

4. A direct but involved computation shows,

$$\frac{1}{n!} \partial_b^n (f \circ g_1)(0) = \langle D^n f(e), \Psi_n(g) \rangle = \left\langle \hat{f}(e), \Psi_n(g) \right\rangle.$$

5. The pointwise bounds in (11) follow from (10) and the Cauchy-Schwarz inequality.

Horizontal Reconstruction Theorem

Theorem 20 (Horizontal Reconstruction)**.** *Given* $\alpha \in J_t^0$*, there exists* $f \in \mathcal{H}$ *such that* $\hat{f}(e) = \alpha$.

Proof Ideas

- We must define f by
$$f(g(1)) := \langle \alpha, \Psi(g) \rangle \tag{12}$$
for all paths, g, such that $g(0) = e$.

- However, in the degenerate case, we only know *a priori* that $\langle \alpha, \Psi(g) \rangle$ is well defined when g is *horizontal.*

- How do we show $g \to \langle \alpha, \Psi(g) \rangle$ only depends on $g(1)$?

- **Answer:** we first construct local version of f and then use an analytic continuation argument to patch them together.

Local Reconstruction Theorem

Theorem 21 (Local Reconstruction). *There exists open neighborhoods, $0 \in \Omega \subset \mathbb{C}^d$ and $e \in U \subset G$ such that:*

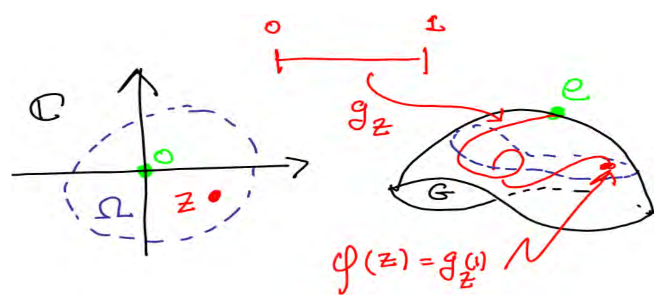
1. *for $z \in \Omega$ there exists a horizontal paths, $g_z(t) \in G$, depending holomorphically on z, such that*

2. *if $\varphi(z) := g_z(1)$, then $\varphi : \Omega \to U$ is biholomoprhic.*

3. *The function $f : U \to \mathbb{C}$ defined by*

$$f(\varphi(z)) := \langle \alpha, \Psi(g_z) \rangle$$

is holomorphic and $\hat{f}(e) = \alpha$.



Example: Complex Heisenberg Group

$G = \mathbb{C}^3$ with group law;

$$\begin{aligned}&(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') \\ &= \left(z_1 + z_1', z_2 + z_2', z_3 + z_3' + \frac{1}{2}(z_1 z_2' - z_2 z_1')\right).\end{aligned}$$

- $\mathfrak{g} = \mathbb{C}^3$, $H = \mathbb{C}^2 \times \{0\}$, $X_l = e_l$ for $l = 1, 2$.
- $q(\alpha) = \sum_{l=1}^{2} |\alpha(e_l)|^2$
- $\Delta = \tilde{X}_1^2 + \tilde{X}_2^2 + \tilde{Y}_1^2 + \tilde{Y}_2^2$ where $Y_l = iX_l$.
- $$\Delta_H = \Delta_{z_1} + \Delta_{z_2} + \frac{|z_1|^2 + |z_2|^2}{4}\Delta_{z_3} + L\frac{\partial}{\partial x_3} + S\frac{\partial}{\partial y_3}$$
- L and S are angular momentum ops. on $\mathbb{C}^2 \times \{0\}$.

Heat Kernel and Horizontal Paths

Theorem 22 (Hypoelliptic Heat Kenrel)**.** *The heat kernel for the complex Heisenberg group setup above is given by,*

$$\rho_t(z) = \left(\frac{1}{2\pi}\right)^4 \int_{\mathbb{C}} \frac{|w|^2}{\sinh^2(|w|\, t/4)} \times \exp\left(-\frac{1}{4}|w| \coth(|w|\, t/4)\left(|z_1|^2 + |z_2|^2\right)\right) \times e^{i\,\mathrm{Re}(w\bar{z}_3)} dm(w).$$

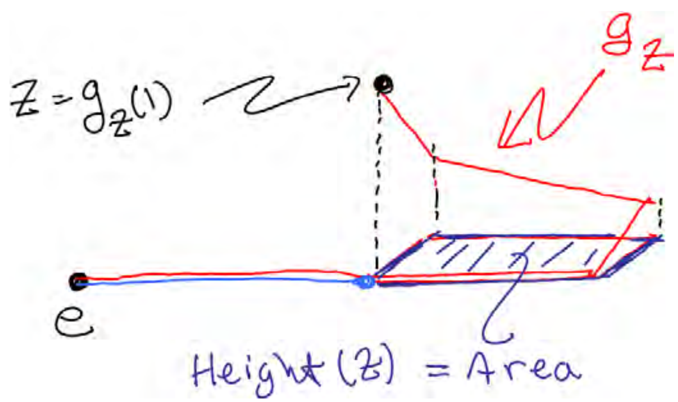


Figure 4: The path g_z for the Heisenberg group. The rectangular region is long and skinny.

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The Taylor map on homogenous spaces

- Let $K \subset G$ be a connected, closed, complex subgroup of G, $\mathfrak{k} = \mathrm{Lie}(K)$.
- $M = K \setminus G$ M be the space of right K cosets,
- $\pi : G \to M$ be the associated quotient map,

Notation. The formula,

$$\dot{A}(m) := \frac{d}{dt}|_0 \left(me^{tA}\right) \text{ for all } m \in M \text{ and } A \in \mathfrak{g} \tag{13}$$

defines a linear map, $\mathfrak{g} \ni A \to \dot{A} \in \mathrm{Vect}(M)$, where $\mathrm{Vect}(M)$ denotes the linear space of smooth vector fields on M.We may also define a sub-Laplacian Δ_M on M given by

$$\Delta_M = \sum_{j=1}^{m} \left(\dot{X}_j^{\,2} + \dot{Y}_j^2\right), \text{ where } Y_j := iX_j. \tag{14}$$

Definition 23. Let $\lambda_t(dm)$ be **heat kernel** measure on M given by

$$\lambda_t(dm) = (\pi_* \rho_t)(dm). \tag{15}$$

Theorem 24. *The family of probability measures* $\{\lambda_t : t \in (0,\infty)\}$, *is the unique family of probability measures on* M *such that, for all* $\phi \in C_c^\infty(M)$, *the function* $t \to \lambda_t(\phi) := \int_M \phi d\lambda_t$ *is continuously differentiable and satisfies*

$$\frac{d}{dt}\lambda_t(\phi) = \frac{1}{4}\lambda_t(\Delta_M \phi) \quad \textit{and} \quad \lim_{t\downarrow 0}\lambda_t(\phi) = \phi(o). \tag{16}$$

Definition 25 (G – space Taylor map)**.** For $u \in \mathcal{H}(M)$, define $\hat{u} \in T'$ by; $\langle \hat{u}, 1\rangle = u(Ke)$ and for all $n \in \mathbb{N}$,

$$\langle \hat{u}, \xi_1 \otimes \cdots \otimes \xi_n \rangle = (\dot{\xi}_1 \cdots \dot{\xi}_n u)(Ke) = \tilde{\xi}_1 \dots \tilde{\xi}_n [u \circ \pi](e) \text{ for all } \xi_j \in \mathfrak{g}. \tag{17}$$

The map $\mathcal{H}(M) \ni u \to \hat{u} \in T'$ is called the Taylor map on M.

Theorem 26 (The quotient Taylor map)**.** *For all* $t > 0$, *the Taylor map*

$$\mathcal{H}(M) \supset \mathcal{H}L^2(M, \lambda_t) \ni u \to \hat{u} \in (J + \mathfrak{k}T)_t^0 \subset T'$$

is a unitary map, where

$$(J + \mathfrak{k}T)_t^0 = \{\alpha \in T' : \langle \alpha, J + \mathfrak{k}T\rangle = \{0\}\}.$$

Two examples

The Grushin complex 2-space

Notation (Complex Heisenberg group). Let $H_3^{\mathbb{C}} = \mathbb{C}^3$ with the group law

$$(z_1, z_2, z_3) \cdot (z_1', z_2', z_3') = (z_1 + z_1', z_2 + z_2', z_3 + z_3' + (1/2)(z_1 z_2' - z_2 z_1'))\,.$$

We take

- Let $q(\alpha) := |\alpha(e_1)|^2 + |\alpha(e_2)|^2$ for all $\alpha \in \left(\mathbb{C}^3\right)^*$.
- $K = \mathbb{C} = \{(z_1, z_2, z_3) : z_2 = z_3 = 0\} \subset H_3^{\mathbb{C}}$
- $M = K \backslash H_3^{\mathbb{C}} \cong \mathbb{C}^2 \cong \mathbb{R}^4;\ \xi = (w, z) = (u + iv, x + iy) \in M,$
- It turns out that
$$\Delta_M = (\partial/\partial u)^2 + (\partial/\partial v)^2 + (u^2 + v^2)((\partial/\partial x)^2 + (\partial/\partial y)^2).$$
- The heat kerne density, $\lambda_t(\xi)$ satisfies
$$\frac{c_1}{V(\sqrt{t})} \exp\left(-C_1 \frac{\delta(\xi)^2}{t}\right) \le \lambda_t(\xi) \le \frac{C_2}{V(\sqrt{t})} \exp\left(-c_2 \frac{\delta(\xi)^2}{t}\right) \tag{18}$$

- $\delta(\xi)$ is the subelliptic distance between the origin and $\xi = (w, z)$,

$$c(|u| + |v| + \sqrt{|x|} + \sqrt{y|}) \leq \delta(\xi) \leq C(|u| + |v| + \sqrt{|x|} + \sqrt{|y|}).$$

- $V(r) \simeq r^6, \quad r > 0$

- $$\Omega\left(m,n\right) = m! \cdot \binom{|n|_1}{m_1} \cdot \binom{|n|_2 - |m|_1}{m_2} \cdot \binom{|n|_3 - |m|_2}{m_3} \cdots \binom{|n|_k - |m|_{k-1}}{m_k}$$

Corollary 27. *Suppose that f is a holomorphic function on $\mathbb{C}^2$, Then*

$$\|f\|_t^2 = \|\hat{f}\|_t^2 = \sum_{N=0}^{\infty} \frac{t^N}{N!} \sum_{(m,n)\in I(N)} \Omega^2\left(m,n\right) \left| \left(\partial_w^{|n|-|m|} \partial_z^{|m|} f\right)(0,0) \right|^2. \tag{19}$$

Example. When $f(w,z) = g(z) = z^3$,the only non-zero derivative at $(0,0)$ is $(\partial^3 f/\partial^3 z)(0,0) = 6$. So according to Eq. (**??**),

$$\int_M (x^2+y^2)^3 \lambda_t(d\xi) = \|\hat{f}\|_t^2 = \frac{61}{20} t^6. \tag{20}$$

Example. For $f(w,z) = wz^3$, we have

$$\| \int_M |w|^2 |z|^6 \lambda_t(d\xi) = \|\hat{f}\|_t^2 = \frac{277}{28} t^7. \tag{21}$$

A one-dimensional complex G - space

Notation. We let $G := \mathbb{C}^{\times} \ltimes \mathbb{C}$ with group law,

$$(a, b)(a', b') = (aa', ab' + b)$$

Take:

- $q(\alpha) = |\langle \alpha, e_1 \rangle|^2 + |\langle \alpha, e_2 \rangle|^2$ (this is positive!),
- $K = \mathbb{C}^{\times} \times \{0\}$, $M := K \backslash G \cong \mathbb{C}$.
- $\Delta_M = \left[1 + x^2 + y^2\right] \left(\partial_x^2 + \partial_y^2\right)$
- The associated heat kernel satisfies,

$$\frac{c_\epsilon}{\ln(\cosh^2 \sqrt{t})} e^{-(1+\epsilon)(\sinh^{-1}|\xi|)^2/t} \leq \lambda_t(\xi) \leq \frac{C_\epsilon}{\ln(\cosh^2 \sqrt{t})} e^{-(1-\epsilon)(\sinh^{-1}|\xi|)^2/t}. \tag{22}$$

Corollary 28. *Suppose that f is a holomorphic function on $\mathbb{C}$, then*

$$\int_{\mathbb{C}} |f(\xi)|^2 \lambda_t(\xi)\, d\xi = \|f\|_t^2 = \sum_{m=0}^{\infty} c_m(t) \left| f^{(m)}(0) \right|^2 \tag{23}$$

where

$$\begin{aligned} c_0(t) &\equiv 1 \\ c_1(t) &= e^t - 1, \\ c_2(t) &= \frac{1}{12} e^{4t} - \frac{1}{3} e^t + \frac{1}{4} \\ c_m(t) &\gtrapprox \frac{1}{\sqrt{2\pi t}} \frac{e^{m^2 t}}{m^{2m+1}} \end{aligned}$$