Toeplitz quantization on real symmetric domains

MIROSLAV ENGLIŠ (Prague)

[joint work with HARALD UPMEIER (Marburg)]

ABSTRACT. An analogue of the Berezin-Toeplitz star product, familiar from deformation quantization, is studied in the setting of real bounded symmetric domains. The analogue turns out to be a certain invariant operator, which one might call star restriction, from functions on the complexification of the domain into functions on the domain itself. In particular, we establish the usual (i.e. semiclassical) asymptotic expansion of this star restriction, and describe real-variable analogues of several other results.

Symbols and quantizations

Quantization (operator calculus):

$$f \text{ function on } \Omega \longmapsto Q_f \in \mathrm{Op}(H)$$

Ω = manifold, H = Hilbert space.

Usually also

$$Q_{\mathbf{1}} = I.$$

Dequantization (symbol calculus): opposite direction,

$$T \in \mathrm{Op}(H) \longmapsto \widetilde{T} \text{ function on } \Omega.$$

QUANTIZATION IN PHYSICS

Traditional: Ω = symplectic manifold,

$$[Q_f, Q_g] = \frac{ih}{2\pi} Q_{\{f,g\}} \qquad (+O(h^2)),$$

irreducibility.

Here $\{\cdot,\cdot\}$ = Poisson bracket; $h \to 0$ Planck's constant.

GEOMETRIC QUANTIZATION — no-go theorems.

DEFORMATION QUANTIZATION — star product $*$:

$$\boxed{Q_f Q_g =: Q_{f*g},}$$

at least in some asymptotic sense as $h \searrow 0$.

(View f,g as formal power series in h.)

Toeplitz quantization

For a Kähler manifold Ω, with Kähler form ω,

take a real-valued potential Φ for ω
(i.e. $\frac{\partial^2 \Phi}{\partial z_j \partial \overline{z}_k} = g_{j\overline{k}}$, the metric associated to ω);

and consider the subspace L^2_{hol} of all holomorphic functions in L^2 on Ω with respect to the measure $e^{-\Phi/h} \wedge^n \omega$.

Let $P : L^2 \to L^2_{\text{hol}}$ be the orthogonal projection.

A <u>Toeplitz operator</u> with symbol $f \in L^\infty(\Omega)$ is the operator on L^2_{hol} defined by

$$T_f : u \longmapsto P(fu).$$

All these objects $(L^2, L^2_{\text{hol}}, P, T_f)$ depend on h, $h > 0$.

(Obvious variant if the potential Φ exists only locally — spaces of holomorphic L^2 sections of suitable line-bundles, which exist provided ω satisfies the appropriate integrality conditions, and h can then only assume the discrete set of values $h = 1/m$, $m = 1, 2, 3, \ldots$.)

Theorem. *$f \mapsto T_f$ is a deformation quantization in the above sense.*

$\Longrightarrow$ (Berezin-) Toeplitz quantization.

[Klimek-Lesniewski 1992] — disc, Riemann surfaces
[Coburn 1992] — Euclidean $\mathbf{C}^n$
[Bordemann-Meinrenken-Schlichenmaier 1994] — compact manifolds
([Karabegov 1996] — separation of variables)
[M.E. 1997–2002] — pseudoconvex domains with Kähler metric
$\vdots$

Also generalizations beyond Kähler case — spin structures etc.
[Ma, Marinescu, ...].

Operator fields

Operator field:

$$\Omega \ni z \longmapsto Q_z \in \mathrm{Op}(H).$$

Gives rise to operator calculus by

$$f \longmapsto Q_f := \int_\Omega f(z)\, Q_z\, dz$$

for a measure dz on Ω (e.g. the symplectic volume ω^n).

[Gracia-Bondia] (quantizers), [Ali-Doebner] (prime quantization)

representation theory [Harish-Chandra], time-frequency analysis

Toeplitz operators and operator fields

Let Ω be a domain in $\mathbf{C}^n$ and $d\mu(z)$ a measure on Ω continuous w.r.t. the Lebesgue measure dz. Consider the space

$$H = L^2_{\text{hol}}(\Omega, d\mu)$$

of all holomorphic functions on Ω square-integrable w.r.t. μ.
Then the evaluation functionals

$$H \ni f \longmapsto f(z) \in \mathbf{C}$$

are continuous, hence are given by scalar product with some $K_z \in H$:

$$f(z) = \langle f, K_z \rangle = \int_\Omega f(x) K(z,x)\, d\mu(x), \qquad K(z,x) := K_x(z) = \overline{K_z(x)}.$$

One calls H a Bergman space and $K(z,x)$ the (weighted) Bergman kernel.

Toeplitz operator on the Bergman space with symbol f:

$$T_f : H \to H, \qquad T_f \phi := P(f\phi),$$

where $P : L^2(\Omega, d\mu) \to H = L^2_{\text{hol}}(\Omega, d\mu)$ is the orthogonal projection.

NOTATION: normalized reproducing kernels

$$k_z := \frac{K_z}{\|K_z\|} = \frac{K(\cdot, z)}{K(z,z)^{1/2}}.$$

("coherent states")

Fact: $f \mapsto T_f$ is given by the rank-one operator field

$$T_z := \langle \cdot, k_z \rangle k_z$$

with respect to the measure $K(z,z)\, d\mu(z)$.

Proof: For any $u, v \in H$,

$$\begin{aligned} \langle T_f u, v \rangle &= \langle P(fu), v \rangle = \langle fu, v \rangle \\ &= \int f(z) u(z) \overline{v(z)}\, d\mu(z) \\ &= \int f(z) \langle u, K_z \rangle \langle K_z, v \rangle\, d\mu(z) \\ &= \int f(z) \langle u, k_z \rangle \langle k_z, v \rangle\, K(z,z)\, d\mu(z). \quad \square \end{aligned}$$

Equivariance

Assume that G is a group of transformations of Ω (symmetries),

$$G \ni g : \Omega \to \Omega,$$

and U is a (projective) unitary representation of G on H:

$$U : \; g \mapsto U_g \in \mathrm{Op}(H).$$

The quantization is called equivariant if

$$\boxed{Q_{f \circ g} = U_g^* Q_f U_g.}$$

On the level of operators fields: if

$$Q_f = \int_\Omega f(z)\, Q_z \, d\mu(z)$$

where $d\mu$ is *G-invariant*, equivariance corresponds to

$$\boxed{Q_{g(z)} = U_g Q_z U_g^*.}$$

Equivariance of Toeplitz quantization: for holomorphic transformations g preserving the Kähler metric, there are the representations of the form

$$U_g : \ f \longmapsto f \circ g^{-1} \cdot m_g$$

with some "multipliers" m_g (typically, powers of the Jacobian of g).

With respect to these, Toeplitz quantization is equivariant.

$$T_{f \circ g} = U_g^* T_f U_g$$

More specifically,

$$T_z = \langle \cdot, k_z \rangle k_z$$

is an equivariant operator field, and

$$K_h(z, z) \, e^{-\Phi/h} \wedge^n \omega$$

is invariant under such transformations g.

When the equivariance is most prominent: if the symmetries are rich — in particular, if G acts on Ω transitively (*homogeneous spaces*), or if even the isotropy groups are large (*symmetric spaces/domains*).

In particular, if G acts on Ω *transitively* then from

$$Q_{g(z)} = U_g Q_z U_g^*.$$

it follows that any equivariant operator field (and, hence, also the equivariant quantization induced by it) is uniquely determined by the single operator Q_0 for any fixed base-point $0 \in \Omega$. Namely, if $z \in \Omega$, by transitivity there is $g \in G$ with $g(0) = z$, and then

$$Q_z = Q_{g(0)} = U_g Q_0 U_g^*.$$

Furthermore, Q_0 must be invariant under the elements fixing 0, i.e.

$$(*) \qquad Q_0 = U_g Q_0 U_g^* \quad \text{for all } g \text{ with } g(0) = 0.$$

Applies, in particular, to symmetric domains. An equivariant quantization on such domain is thus completely determined by the single operator Q_0, satisfying $(*)$. We say that it is generated by Q_0.

Reformulation of equivariance in the language of representation theory: the quantization

$$f \mapsto Q_f, \qquad \mathrm{Func}(\Omega) \to \mathrm{Op}(H),$$

intertwines two representations of G, namely, the representation on functions given by composition:

$$f \longmapsto f \circ g,$$

and the representation on operators by conjugation:

$$X \longmapsto U_g^* X U_g.$$

Aim of this talk: extension to real, rather than complex, manifolds.

Flavour: more representation theory than physics.

Objects:

complex symmetric domains	real symmetric domains
(equivariant) operator fields	(equivariant) function fields
(Toeplitz) quantization	(Toeplitz) extension
star product	star restriction.

Bounded symmetric domains

(Hermitian symmetric spaces of non-compact type)

Domains $\Omega \subset \mathbf{C}^d$ such that $\forall x, y \in \Omega\ \exists \phi \in G = \{$the group of all biholomorphic self-maps$\}$ interchanging x and y, satisfying $\phi = \phi^{-1}$ and having only isolated fixed-points.

Has *Harish-Chandra realization* as circular, convex domain containing 0.

The group G then operates transitively, and if we denote by K the stabilizer of 0

$$K = \{g \in G : g(0) = 0\},$$

then G is a semisimple Lie group, K a maximal compact subgroup, and $\Omega \cong G/K$.

In fact $K \subset U(d)$.

Jordan-theoretic description:

$\mathbf{C}^d =: Z$ has a structure of Hermitian Jordan triple,

$\Omega =$ the unit ball of Z,

$K =$ the group of all automorphisms of Z.

Polar decomposition: $\exists e_1, \dots, e_r \in Z$ ($r =$ rank) such that any $z \in Z$ can be written in the form

$$z = k(t_1 e_1 + t_2 e_2 + \cdots + t_r e_r),$$

with $k \in K$ and unique $t_1 \geq t_2 \geq \cdots \geq t_r \geq 0$; $z \in \Omega$ if and only if $t_1 < 1$.

Jordan determinant: polynomial on $Z \times Z$, holomorphic in $x, \overline{y}$, uniquely determined by

$$h(z, z) = \prod_{j=1}^{r} (1 - t_j^2).$$

Bergman kernels on symmetric domains: the Bergman kernel

$$K(x, y) = c\, h(x, y)^{-p},$$

$c = \mathrm{vol}(\Omega)$, $p \in \{2, 3, 4, \dots\}$ — genus.

Furthermore, if we consider the measures

$$d\mu_\nu(z) := c_\nu\, h(z, z)^{\nu - p}\, dz,$$

where c_ν is normalizing constant to make $\mu_\nu(\Omega) = 1$, then the Bergman spaces

$$\mathcal{H}^\nu := L^2_{\mathrm{hol}}(\Omega, d\mu_\nu)$$

are nontrivial for all $\nu > p - 1$ and have reproducing kernels

$$K_\nu(x, y) = h(x, y)^{-\nu}.$$

Invariant measure:

$$d\mu(z) = h(z,z)^{-p}\, dz$$

is the (unique) G-invariant measure on Ω.

Representation of G: The operators

$$U_g f(z) := f(g^{-1}(z)) \cdot J_{g^{-1}}(z)^{\nu/2p},$$

where J_g denotes the complex Jacobian of g, are a projective unitary representation of G on $\mathcal{H}^\nu$.

Toeplitz calculi on bounded symmetric domains: equivariant, and generated by the operator

$$T_0 = \langle \cdot, k_0 \rangle k_0 = \langle \cdot, \mathbf{1} \rangle \mathbf{1},$$

i.e. by the rank-one projection onto the constants.

Toeplitz star product:

$$T_{f*g} = T_f T_g,$$

asymptotically as $\nu \to +\infty$. [Bortwick-Lesniewski-Upmeier]
(So $h = 1/\nu \searrow 0$ plays the role of Planck constant.)

Example of BSD: unit ball of $m \times n$ complex matrices,

$$I_{mn} := \{Z \in \mathbf{C}^{m\times n} : \|Z\|_{\mathbf{C}^n \to \mathbf{C}^m} < 1\}.$$

Group of motions/stabilizer: $\qquad G = SU(m,n),\ K = U(m) \times U(n).$

[For $n = 1$, the unit ball of $\mathbf{C}^m$; for $m = n = 1$, the unit disc.]

Genus $p = m + n$; Jordan determinant: $\qquad h(x,y) = \det(I - xy^*).$

Weighted Bergman spaces/kernels: holomorphic function square-integrable with respect to $c_\nu \det(I - zz^*)^{\nu-m-n}$; reproducing kernels:

$$K_\nu(x,y) = \det(I - xy^*)^{-\nu}. \qquad \square$$

Classification: by Cartan, any BSD is (biholomorphic to) a Cartesian product of the "building blocks" (Cartan domains):

- $I_{mn} = U_{m,n}(\mathbf{C})/U_m(\mathbf{C}) \times U_n(\mathbf{C}) = \{Z \in \mathbf{C}^{m\times n} : I - Z^*Z > 0\}$ (matrix balls);
- $II_m = Sp_{2m}(\mathbf{R})/U_m(\mathbf{C}) = \{Z \in I_{mm} : Z^t = Z\}$ (symmetric matrices);
- $III_m = O_m(\mathbf{H})/U_m(\mathbf{C}) = \{Z \in I_{mm} : Z^t = -Z\}$ (skew-symmetric matrices);
- $IV_m = SO_{m,2}/SO_{n,0} \times SO_{0,2}$ (Lie balls);
- $V = E_{6(-14)}/\operatorname{Spin}_{10} \times SO_2$, $VI = E_{7(-25)}/E_6 \times SO_2$.

Thus — we have nice equivariant quantizations on these domains in $\mathbf{C}^n$.

(Extends also to compact counterparts — Grassmann manifolds etc.)

Operator fields on reproducing kernel spaces

Assume we have a Hilbert space $\mathcal{H}$ with reproducing kernel,

$$f(z) = \langle f, K_z \rangle, \qquad f \in \mathcal{H}, z \in \Omega.$$

For any operator on $\mathcal{H}$:

$$Tf(x) = \langle Tf, K_x \rangle = \langle f, T^* K_x \rangle = \int_\Omega f(y) \, \overline{T^* K_x(y)} \, dy$$

$\Rightarrow$ any T is an *integral operator*, with integral kernel

$$\widetilde{T}(x, y) = \overline{T^* K_x(y)} = \langle T K_y, K_x \rangle = T K_y(x).$$

The correspondence $T \longleftrightarrow \widetilde{T}$ is one-to-one. Furthermore, if $\mathcal{H}$ consists of holomorphic functions (e.g. for Bergman spaces), $\widetilde{T}$ is holomorphic in $x, \overline{y}$.

In the holomorphic case, we can thus identify operators on $\mathcal{H}$ with (some) holomorphic functions on $\Omega \times \overline{\Omega}$.

(Here $\overline{\Omega}$ is Ω with the conjugate complex structure.)

Under this identification, operator calculi are just maps

$$f \longmapsto \widetilde{Q_f}, \qquad \mathrm{Func}(\Omega) \to \mathrm{Hol}(\Omega \times \overline{\Omega}).$$

(Similarly, operator fields are maps $\Omega \to \mathrm{Hol}(\Omega \times \overline{\Omega})$.)

How does equivariance show in this picture?

Equivariance:

$$Q_{f\circ g} = U_g^* Q_f U_g.$$

One has (direct computation)

$$\widetilde{(U_g^* T U_g)} = (U_g \otimes \overline{U}_g)\,\widetilde{T}.$$

Thus equivariant calculi correspond to maps $\mathrm{Func}(\Omega) \to \mathrm{Hol}(\Omega \times \overline{\Omega})$ satisfying

$$\widetilde{Q}_{f\circ g} = (U_g \otimes \overline{U}_g)\,\widetilde{Q}_f.$$

What about star products?

Recall: star product was defined (heuristically) by

$$Q_f Q_g = Q_{f*g}.$$

In order to pass to the operator-kernel formalism: need $\widetilde{Q_f Q_g}$.

Instead of a bilinear map $(f, g) \mapsto f * g$, we can view $*$ as a map

$$* : \mathrm{Func}(\Omega \times \overline{\Omega}) \to \mathrm{Func}(\Omega), \qquad (f \otimes g) \mapsto (f * g)$$

(where $(f \otimes g)(x, y) := f(x)g(y)$).

Observation: for Toeplitz operators,

$$\widetilde{T_f T_g} = T^{\Omega\times\overline{\Omega}}_{f\otimes g} K.$$

(Direct computation.) Thus the star-product $f * g$ can be defined by

$$T^{\Omega\times\overline{\Omega}}_{f\otimes g} K =: \widetilde{T^{\Omega}_{f*g}}.$$

Directly generalizes to any pair of calculi $Q^{\Omega\times\overline{\Omega}}$ and Q on $\Omega \times \overline{\Omega}$ resp. Ω.

Summary: upon identifying operators T with their operator kernels $\widetilde{T}$, the following picture emerged.

Quantization:

$$f \mapsto Q_f, \qquad \mathrm{Func}(\Omega) \to \mathrm{Hol}(\Omega \times \overline{\Omega}).$$

Operator field:

$$z \mapsto Q_z, \qquad \Omega \to \mathrm{Hol}(\Omega \times \overline{\Omega}).$$

Star-product:

$$f \otimes g \mapsto f * g, \qquad \mathrm{Func}(\Omega \times \Omega) \to \mathrm{Func}(\Omega),$$

which is a deformation of the pointwise product $f \otimes g \mapsto fg$.

Equivariance: intertwines the corresponding actions of G.

Passage from complex to real bounded symmetric domains

For real symmetric domains: use this operator-kernel formalism, and replace Ω and $\Omega\times\overline{\Omega}$ by a real BSD $\Omega_{\mathbf{R}}$ and its complexification $\Omega_{\mathbf{C}}$.

"Quantization" (better: "extension"):

$$f \mapsto Q_f, \qquad \mathrm{Func}(\Omega_{\mathbf{R}}) \to \mathrm{Hol}(\Omega_{\mathbf{C}}).$$

"Operator field" (better: "function field"):

$$z \mapsto Q_z, \qquad \Omega_{\mathbf{R}} \to \mathrm{Hol}(\Omega_{\mathbf{C}}).$$

Star-product (better: "star restriction"):

$$F \mapsto \#F, \qquad \mathrm{Func}(\Omega_{\mathbf{C}}) \to \mathrm{Func}(\Omega_{\mathbf{R}});$$

should be deformation of restriction $F \mapsto F|_{\Omega_{\mathbf{R}}}$.

Equivariance: intertwine the corresponding actions of the group.

Example: the Bargmann transform $L^2(\mathbf{R}^n) \to L^2_{\mathrm{hol}}(\mathbf{C}^n, e^{-|z|^2}dz)$.

Real bounded symmetric domains

Setup: $\Omega_{\mathbf{C}} = G/K$ a complex bounded symmetric domain

$z \mapsto z^{\#}$ involution (conjugate-linear, $z^{\#\#} = z$)

such that ${\Omega_{\mathbf{C}}}^{\#} = \Omega_{\mathbf{C}}$

$\Omega_{\mathbf{R}} := \{z \in \Omega_{\mathbf{C}} : z^{\#} = z\}$

$G_{\mathbf{R}} := \{g \in G : g(z^{\#}) = (gz)^{\#}\}$, $K_{\mathbf{R}} := G_{\mathbf{R}} \cap K$

$G_{\mathbf{R}}$ reductive, $\Omega_{\mathbf{R}} \simeq G_{\mathbf{R}}/K_{\mathbf{R}}$ — real BSD

Examples:

- $\Omega_{\mathbf{C}} = \mathbf{D}$, $z^{\#} = \overline{z}$ $\dots$ $\Omega_{\mathbf{R}} = (-1, +1)$
- $\Omega_{\mathbf{C}} = I_{mn}$, $z^{\#} = \overline{z}$ $\dots$ $\Omega_{\mathbf{R}} = I_{mn} \cap \mathbf{R}^{m \times n}$
- $\Omega_{\mathbf{C}} = I_{nn}$, $z^{\#} = z^{*}$ $\dots$ $\Omega_{\mathbf{R}} = I_{nn}^{\text{self-adjoint}}$
- $\Omega_{\mathbf{C}} = \mathbf{C}^{n}$, $z^{\#} = \overline{z}$ $\dots$ $\Omega_{\mathbf{R}} = \mathbf{R}^{n}$
- $\Omega_{\mathbf{C}} = \Omega \times \overline{\Omega}$, $(z, \overline{w})^{\#} = (w, \overline{z})$ $\dots$ $\Omega_{\mathbf{R}} = \Omega$. $\square$

$\Omega_{\mathbf{R}}$	$G_{\mathbf{R}}/K_{\mathbf{R}}$	Σ	$r_{\mathbf{R}}$	$a_{\mathbf{R}}$	$b_{\mathbf{R}}$	$c_{\mathbf{R}}$	d	$r_{\mathbf{C}}$	$\Omega_{\mathbf{C}}$
$I^{\mathbf{R}}_{r,r+b}$	$U_{r,r+b}(\mathbf{R})/U_r(\mathbf{R})\times U_{r+b}(\mathbf{R})$	D_r/B_r	r	1	b	0	$r(r+b)$	r	$I_{r,r+b}$
$I^{\mathbf{H}}_{2r,2r+2b}$	$U_{r,r+b}(\mathbf{H})/U_r(\mathbf{H})\times U_{r+b}(\mathbf{H})$	C_r/BC_r	r	4	$4b$	3	$4r(r+b)$	$2r$	$I_{2r,2r+2b}$
$V^{\mathbf{O}_0}$	$U_{2,2}(\mathbf{H})/U_2(\mathbf{H})\times U_2(\mathbf{H})$	B_2	2	3	4	0	16	2	V
$III^{\mathbf{R}}_r$	$G_r(\mathbf{R})/U_r(\mathbf{R})$	A_r	r	1	–	–	$\frac{1}{2}r(r+1)$	r	III_r
$I^{\mathbf{C}}_{r,r}$	$G_r(\mathbf{C})/U_r(\mathbf{C})$	A_r	r	2	–	–	r^2	r	$I_{r,r}$
$II^{\mathbf{H}}_{2r}$	$G_r(\mathbf{H})/U_r(\mathbf{H})$	A_r	r	4	–	–	$r(2r-1)$	r	II_{2r}
$VI^{\mathbf{O}_0}$	$G_4(\mathbf{H})/U_4(\mathbf{H})$	D_3	3	4	0	0	27	3	VI
$III^{\mathbf{H}}_{2r}$	$Sp_{2r}(\mathbf{C})/U_r(\mathbf{H})$	C_r	r	2	0	2	$r(2r+1)$	$2r$	III_{2r}
$II^{\mathbf{R}}_{2r+\varepsilon}$	$O_{2r+\varepsilon}(\mathbf{C})/U_{2r+\varepsilon}(\mathbf{R})$	D_r/B_r	r	2	2ε	0	$r(2(r+\varepsilon)-1)$	r	$II_{2r+\varepsilon}$
$IV^{\mathbf{R},q}_{p+q}$	$SO_{p,1}\times SO_{1,q}/SO_{p,0}\times SO_{0,q}$	D_2/A_2	2	n/a	0	0	$p+q$	2	IV_{p+q}
$IV^{\mathbf{R},0}_n$	$SO_{n,1}/SO_{n,0}$	C_1	1	–	0	$n-1$	n	2	IV_n
$V^{\mathbf{O}}$	$F_{4(-20)}/SO(9)$	BC_1	1	–	8	7	16	2	V
$VI^{\mathbf{O}}$	$E_{6(-26)}\times O(2)/F_4\times O(1)$	A_3	3	8	–	–	27	3	VI

(Will exclude the D_2 case in the sequel.)

Operator calculi on real BSDs

Recall: $d\mu(z) = h(z,z)^{-p}\,dz$ was a G-invariant measure on $\Omega_{\mathbf{C}}$.

Fact: $d\mu_{\mathbf{R}} := h(x,x)^{-p/2}\,dx$ is an $G_{\mathbf{R}}$-invariant measure on $\Omega_{\mathbf{R}}$.

<u>"Operator calculus"</u> on $\Omega_{\mathbf{R}}$ (rather: *operator extension*):

$$A : f \mapsto A_f, \quad C^\infty(\Omega_{\mathbf{R}}) \to L^2_{\mathrm{hol}}(\Omega_{\mathbf{C}}), \quad f \circ g \mapsto U_g^* A_f.$$

<u>"Operator field"</u> on $\Omega_{\mathbf{R}}$ (rather: *function field*):

$$x \mapsto A_x, \quad \Omega_{\mathbf{R}} \to L^2_{\mathrm{hol}}(\Omega_{\mathbf{C}}), \quad g(x) \mapsto U_g A_x.$$

Operator field gives rise to operator calculus:

$$f \mapsto A_f := \int_{\Omega_{\mathbf{R}}} f(x)\, A_x\, d\mu_{\mathbf{R}}(x).$$

Uniquely determined by A_0; A_0 fixed by all U_k, $k \in K$.

Example: real Toeplitz calculus — corresponds to the choice

$$A_0 = \mathbf{1}.$$

This yields the operator field

$$A_x = K_x / \|K_x\| =: k_x$$

(the coherent states on $\Omega_{\mathbf{R}} \subset \Omega_{\mathbf{C}}$!). Corresponding Toeplitz calculus ("Toeplitz extension") — combinations of the coherent states:

$$A_f = \int_{\Omega_{\mathbf{R}}} f(x)\, k_x \, d\mu_{\mathbf{R}}(x).$$

For $\Omega_{\mathbf{R}} = \Omega$, $\Omega_{\mathbf{C}} = \Omega \times \overline{\Omega}$, with Ω a complex BSD in $\mathbf{C}^n$ — yields the previous (usual) Toeplitz operators.

Example: For $\Omega_{\mathbf{R}} = \mathbf{R}^n$, $\Omega_{\mathbf{C}} = \mathbf{C}^n$ — yields the Bargmann transform.

Weyl calculus on $\Omega_{\mathbf{R}}$ — more complicated.

Star products on real BSDs

Recall: in the complex situation, defined by

$$T_f T_g =: T_{f*g},$$

or

$$T^{\Omega\times\overline{\Omega}}_{f\otimes g} K = \widetilde{T_f T_g} = \widetilde{T^{\Omega}_{f*g}}.$$

Analogue for real-BSDs:

$$\boxed{T^{\Omega_{\mathbf{C}}}_F \mathcal{I} = T^{\Omega_{\mathbf{R}}}_{\# F}},$$

where

$$\mathcal{I}(z) := K(z, z^{\#})^{1/2}$$

is the holomorphic function which arises from $\widetilde{I} = K$. (Or from $\#\mathbf{1} = \mathbf{1}$.)

Defines "<u>star-restriction</u>"

$$\# : F \mapsto \#F, \quad \mathrm{Func}(\Omega_{\mathbf{C}}) \to \mathrm{Func}(\Omega_{\mathbf{R}}).$$

Depends on the Planck parameter $\nu = 1/h$ through the spaces

$$\mathcal{H}^\nu \equiv L^2_{\mathrm{hol}}(\Omega_{\mathbf{C}}, d\mu_\nu), \qquad d\mu_\nu(z) = c_\nu \, h(z,z)^\nu \, d\mu(z)$$

on which the Toeplitz operators $T_F^{\Omega_{\mathbf{C}}}, T_{\#F}^{\Omega_{\mathbf{R}}}$ act.

<u>Main result</u>: existence of semiclassical limit $\nu \to +\infty$.

Theorem 1. *For any* $F \in C^\infty(\Omega_{\mathbf{C}}) \cap L^\infty(\Omega_{\mathbf{C}})$,

$$\#F = \sum_{j=0}^{\infty} \nu^{-j} \ \rho L_j F,$$

where ρ *is the operator of restriction from* $\Omega_{\mathbf{C}}$ *to* $\Omega_{\mathbf{R}}$, *and* L_j *are some differential operators which are* $G_{\mathbf{R}}$*-invariant:*

$$L_j(F \circ g) = (L_j F) \circ g \quad \forall g \in G_{\mathbf{R}}.$$

Furthermore,

$$L_0 = I,$$

and L_j *involve only holomorphic derivatives, i.e.*

$$L_j(\overline{H} F) = \overline{H} L_j F \qquad \forall H \in \mathrm{Hol}\,.$$

There is also a (kind of) explicit description for L_j.

Viewing an operator calculus on a real BSD

$$A : \text{functions on } \Omega_{\mathbf{R}} \to \text{Hol}(\Omega_{\mathbf{C}})$$

as a map from $L^2(\Omega_{\mathbf{R}}, d\mu)$ into $\mathcal{H}^\nu$, one can consider the adjoint

$$A^* : \text{Hol}(\Omega_{\mathbf{C}}) \to \text{ functions on } \Omega_{\mathbf{R}}.$$

(In the original "complex" situation, this is the dequantization map from operators to functions.)

`Example.` For operator calculi arising from an operator field,

$$A^* F(x) = \langle F, A_x \rangle.$$

In the complex case,

$$A^* T(z) = \text{tr}(A_z^* T). \quad \square$$

The composed map

$$B = A^* A : f \mapsto A_f \mapsto A^* A_f, \quad \text{functions on } \Omega_{\mathbf{R}} \to \text{ functions on } \Omega_{\mathbf{R}},$$

is called the Berezin transform.

[Berezin] [Unterberger-Upmeier] [Arazy-Orsted]

Explicitly:

$$Bf(x) = c_\nu \int_{\Omega_{\mathbf{R}}} f(\zeta)\, \frac{h(x,x)^{\nu/2} h(\zeta,\zeta)^{\nu/2}}{h(x,\zeta)^\nu}\, d\mu_{\mathbf{R}}(\zeta).$$

Example: for the Segal-Bargmann space of $\Omega_{\mathbf{R}} = \mathbf{R}^n$, $\Omega_{\mathbf{C}} = \mathbf{C}^n$ — the heat operator, $B = e^{\Delta/2\nu}$.

Stationary phase method $\rightsquigarrow$ asymptotic expansion:

$$Bf = \sum_{j=0}^\infty \nu^{-j} R_j f \qquad \text{as } \nu \to +\infty,$$

with $G_{\mathbf{R}}$-invariant differential operators R_j on $\Omega_{\mathbf{R}}$, $R_0 = \text{id}$.

Let B^{-1} denote the inverse of the right-hand side in the ring of all formal power series in ν^{-1}.

Theorem 2. *For F holomorphic,*

$$B\#F = F.$$

That is,

$$\#F = B^{-1}F.$$

In a certain precise sense, # is thus the "complexification" of the formal inverse of the Berezin transform B.

(Seems to be a new result even for the complex case.)

Finer structure of B

Consider the space $\mathcal{P}$ of all *holomorphic* polynomials on $\mathbf{C}^d \supset \Omega_{\mathbf{C}}$.

Fock inner product:

$$\langle p, q \rangle_F := \pi^{-d} \int_{\mathbf{C}^d} p(z)\overline{q(z)}\, e^{-|z|^2}\, dz = \overline{q}(\partial)p(0).$$

Under the action $p \mapsto p \circ k$ of $K_{\mathbf{C}}$, Peter-Weyl decomposition:

$$\mathcal{P} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}},$$

where $\mathbf{m} = (m_1, \dots, m_r)$, $m_1 \geq m_2 \geq \dots \geq m_r \geq 0$
(*partitions*/*signatures* of length r). [Schmid]
($r =$ the *rank* of $\Omega_{\mathbf{C}}$)

$\mathcal{P}_{\mathbf{m}} \subset$ homogeneous polynomials of degree $|\mathbf{m}| := m_1 + \dots + m_r$.

Recall:

$$\#F = \sum_j \nu^{-j}\, \rho L_j F, \qquad Bf = \sum_j \nu^{-j} R_j f,$$

with L_j, R_j $G_{\mathbf{R}}$-invariant differential operators on $\Omega_{\mathbf{C}}$, $\Omega_{\mathbf{R}}$ respectively.

Since $G_{\mathbf{R}}$ is transitive on $\Omega_{\mathbf{R}}$ $\implies$ $\rho L_j, R_j$ determined uniquely by their action at the origin, which is a $K_{\mathbf{R}}$-invariant constant coefficient differential operator on $\mathbf{C}^d$ resp. $\mathbf{R}^d$.

$\implies$ $R_j f(0) = p_j(\nabla) f(0)$ for some $K_{\mathbf{R}}$-invariant polynomial p_j on $\mathbf{R}^d$;

L_j contains only holomorphic derivatives $\implies$ $L_j F(0) = l_j(\partial)$ for some $K_{\mathbf{R}}$-invariant polynomial l_j on $\mathbf{C}^d$.

Peter-Weyl decomposition: $\quad p_j = \sum_{\mathbf{m}} p_{j,\mathbf{m}} \quad$ (& similarly for l_j),

where only those $\mathbf{m}$ occur for which $\mathcal{P}_{\mathbf{n}}^{K_{\mathbf{R}}} \neq \{0\}$. (<u>spherical</u> signatures)

Known: $\mathcal{P}_{\mathbf{m}}^{K_{\mathbf{R}}} = \mathbf{C}\phi_{\mathbf{m}}$. (Jack polynomials)

Altogether: at the origin,

$$B = \sum_j \nu^{-j} R_j = \sum_j \nu^{-j} p_j(\nabla) = \sum_{j,\mathbf{m}} \nu^{-j} p_{j,\mathbf{m}}(\nabla) =: \sum_{\mathbf{m}} \underbrace{\frac{\phi_{\mathbf{m}}(\nabla)}{[\nu]_{\mathbf{m}}}}_{B_{\mathbf{m}}} .$$

Similarly,

$$\#F(0) = \sum_{\mathbf{m}} \underbrace{\frac{\phi_{\mathbf{m}}(\partial) F(0)}{\{\nu\}_{\mathbf{m}}}}_{\#_{\mathbf{m}}} .$$

A kind of *Peter-Weyl components* of B, $\#$.

For $\Omega_{\mathbf{R}} = \Omega$: $B_{\mathbf{m}}$ described by [Arazy-Orsted].

Theorem 3. *For real symmetric domains* $\Omega_{\mathbf{R}}$ *not of type* A*,*

$$B_{\mathbf{m}} = \frac{\phi_{\mathbf{m}}(\nabla)}{[\nu]_{\mathbf{m}}}$$

is given by

$$[\nu]_{\mathbf{m}} = \frac{(2r_{\mathbf{R}}/r_{\mathbf{C}})^{2|\mathbf{m}|}}{d_{\mathbf{m}}} \left(\frac{d_X}{r_{\mathbf{R}}}\right)_{\mathbf{m}} \left(\frac{\nu r_{\mathbf{C}} + d_X - d_Y}{2r_{\mathbf{R}}}\right)_{\mathbf{m}},$$

where $d_{\mathbf{m}} = \dim \mathcal{P}_{\mathbf{n}}$, $\mathbf{n}$ *being the spherical signature associated to* $\mathbf{m}$, *and*

$$d_X = r_{\mathbf{R}} + \frac{r_{\mathbf{R}}(r_{\mathbf{R}} - 1)}{2} a_{\mathbf{R}}, \quad d_Y = r_{\mathbf{R}} c_{\mathbf{R}},$$

are certain constants depending only on $\Omega_{\mathbf{R}}$. (See the table.)

Problems.
- root system of type A ?
- $\#_{\mathbf{m}}$? (Unknown except for $\Omega_{\mathbf{R}}=\mathbf{R}^n$.)

Concluding remarks

(1) Other complexifications for symmetric spaces:

$\Omega = G/K \;\rightsquigarrow\; G^{\mathbf{C}}/K^{\mathbf{C}}$
"complex crown" (Gindinkin, Kroetz, Olafsson, Faraut, . . .)

$G^{\mathbf{C}}$ has no relation to our complexification $\Omega_{\mathbf{C}}$ of Ω.

(2) Physics applications:

symmetric domains = Riemannian symmetric spaces
$\rightsquigarrow$ symplectic symmetric spaces
WKB-type quantizations (Bieliavsky, Gutt, Pevzner, . . .)

Is there an approach like ours but based on the symplectic structure?

References

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