Torus fibraitons and localization of index

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Contents

This talk is based on a joint work in progress with Mikio Furuta and Takahiko Yohida.

- **Torus fibrations and localization of index I (arXive:0804.3258, math.SG).**
- **Torus fibrations and localization of index II (in preparation).**

Riemann-Roch number

Prequantized symplectic manifold

- (M, ω) : closed symplectic manifold.
- (L, ∇) : prequantizing line bundle over (M, ω).

$$(F(\nabla) = -2\pi\sqrt{-1}\omega)$$

Fix an ω-compatible almost complex structure.

$$\rightsquigarrow D_L = D : \Omega^{0,*}(M, L) \longrightarrow \Omega^{0,*}(M, L)$$

(associated Dirac-type operator with $\mathbb{Z}/2$-grading)

Definition. (Riemann-Roch number)

$$RR(M, \omega)(= Q(L)) := \mathrm{Ind} D \in \mathbb{Z}.$$

Localization of RR-number

1. **Lagrangian fibration (w/o singular fibers)**
   $\Rightarrow RR(M,\omega) =^{\#}$ BS fibers. **(Andersen)**
2. **Non-singular toric manifold with the moment map μ**
   $\Rightarrow RR(M,\omega) =^{\#} \mu(M)_{\mathbb{Z}}$. **(Danilov)**
3. **Hamiltonian G-manifold**
   (cf. Fixed point formula (Atiyah-Bott, $\cdots$)),
   $\Rightarrow$ **"Quantization commutes with Reduction";**
   $RR(M,\omega)^G = RR(M_G,\omega_G)$
   (Ginzburg-Guillmin-Karshon, Meinrenken, $\cdots$ (cobordism, symplectic cut), Tian-Zhang (Witten deformation), etc.).

Definition. (Bohr-Sommerfeld (BS) fiber)

A smooth fiber F of a Lagrangian fibration $(M, \omega) \longrightarrow B$ with a prequntizing line bundle (L, ∇) is a *Bohr-Sommerfeld fiber*.
$\overset{\text{Def}}{\Longleftrightarrow}$ The restriction $(L, \nabla)|_F$ is a trivial flat line bundle.
$(\Longleftrightarrow \Gamma^{\text{flat}}((L, \nabla)|_F) \cong \mathbb{C}.\ )$

Theorem. (Andersen)

For a Lagrangian fibration without singular fibers over a prequantized closed symplectic manifold (M, ω), the equality

$$RR(M, \omega) =^{\#} \text{BS fibers}$$

holds.

Aim of our project (j/w Mikio Furuta and Takahiko Yoshida)

Want to

- re-prove Andersen's theorem via localization of $\mathrm{Ind}D$.
- generalize the equality for Lagrangian fibration with singular fibers or more gerenal case.

Motivated examples.

- Moduli space of flat $SU(2)$-bundles over a Riemann surface (Goldman, Jefrrey-Weitsmann).
- Gelfand-Cetlin system (Guillmin-Sternberg).

Points of our localization

• Use a structure of flat torus bundle.
(← Arnold-Liouville's theorem).
• Do not need any global group action nor symplectic structure.
• Infinite dimensional analogue of Witten deformation.
(Perturb D by a Dirac-type operator along fibers.)
• "Adiabatic limit".

Simple version

Setting

- (M, g) :Riemannian manifold (not necesalliry compact).
- V : open subset of M such that $M \setminus V$ is compact.
- $W \to M$: $\mathbb{Z}/2$-graded $\mathrm{Cl}(TM)$-module bundle.

Suppose

1. There is a Riemmanian submersion $\pi : V \to U$.
2. π is a fiber bundle s.t. $(\mathrm{fiber}, g|_{\mathrm{fiber}})$=flat torus.
3. $D_{\mathrm{fiber}} : \Gamma(W|_V) \to \Gamma(W|_V)$: family of Dirac-type operator along fibers of π such that $D_{\mathrm{fiber}} \cdot c(\tilde{X}) + c(\tilde{X}) \cdot D_{\mathrm{fiber}} = 0 \quad (\forall X \in TU)$.

Deformation

Let $(M, W, \pi : V \to U, D_{\text{fiber}})$ be the above data.
Take any Dirac-type operator $D : \Gamma(W) \to \Gamma(W)$
and a non-negative number $t \geq 0$.

Definition.

Put $D_t := D + tD_{\text{fiber}} : \Gamma(W) \to \Gamma(W)$.
(Extend D_{fiber} to whole M as 0 via a cut-off function.)

KEY

$DD_{\text{fiber}} + D_{\text{fiber}}D$ **is a differential operator along fibers of π.**

($\because$ Assumption 3.)

Definition.

$\{\pi : V \to U, D_{\text{fiber}}\}$ is an *acyclic fibration* over V.
$\overset{\text{Def}}{\Longleftrightarrow}$ $\text{Ker}\, D_{\text{fiber}} = 0$.

Proposition/Definition

Suppose $V \cong N \times \mathbb{R}_+$ and all the data are translationally invariant. If $\{\pi : V \to U, D_{\text{fiber}}\}$ is acyclic, then

$$\text{Ind}(M, V, \pi, D_{\text{fiber}}) = \text{Ind}(M, V) := \text{Ind} D_t \in \mathbb{Z}$$

is well-defined and it is deformation invariant for any $t \gg 0$, and it does not depend on $t \gg 0$. We call it the *local index* of (M, V).

General case

For any (M, V) and an acyclic fibration over V we can deform them into (M', V') and an another acyclic fibraion so that $V' \cong N \times \mathbb{R}_+$ and all the data are translationally invariant by cutting by a codimension 1 submanifold $N \subset V$. Then we put

$$\mathrm{Ind}(M, V) := \mathrm{Ind}(M', V') \in \mathbb{Z}.$$

We can check that $\mathrm{Ind}(M, V)$ does not depend on the choice of N. We call it the *local index* of (M, V).

Proposition.

$\mathrm{Ind}(M, V)$ satisfies the following properties;

1. M : closed $\Rightarrow \mathrm{Ind}(M, V) = \mathrm{Ind}(D)$.
2. Excision : $\mathrm{Ind}(M, V) = \mathrm{Ind}(M \setminus \overline{V'}, V \setminus \overline{V'})$.

Theorem. (Localization of index.)

Let M be a closed Riemmanian manifold and $W \to M$ a $\mathbb{Z}/2$-graded $\mathrm{Cl}(TM)$-module bundle. Suppose that there is an open covering $M = V \cup (\cup_{i=1}^{m} O_i)$ such that $\{O_i\}_{i=1,\cdots,m}$ are mutually disjoint and V is equipped with an acyclic fibration. Then we have

$$\mathrm{Ind}D = \sum_{i=1}^{m} \mathrm{Ind}(O_i, O_i \cap V).$$

Remark.

- If $\pi : M \to B$ is a fibration with finitely many singular fibers and each O_i is a neighborhood of a singular fiber, then the above theorem implies that $\mathrm{Ind}D$ can be computed by counting singular fibers with multiplicity.
- Computation of the multiplicity is an another problem.

Lagrangian fibration case

$$\begin{cases} M : \text{prequantized symplectic manifold}, \\ V : \text{open subset s.t } M \setminus V \text{ is compact}, \\ \pi : V \to U : \text{Lagrangian torus bundle}. \end{cases}$$

$$\Longrightarrow \begin{cases} W := \Lambda^* T^* M^{0,1} \otimes L \text{ (for an appropriate } J), \\ D_{\text{fiber}} := d_{L,\text{fiber}} + d^*_{L,\text{fiber}} : \Gamma(\Lambda^* T^* \pi \otimes L) \\ (T\pi : \text{tangent bundle along fibers of } \pi). \end{cases}$$

Remark. (KEY POINT)

- π : Lagrangian fibration $\Rightarrow$ $T^*\pi \otimes \mathbb{C} \cong T^*M^{0,1}|_V$.
- Arnold-Liouville's theorem

$\Longrightarrow$ π : flat torus bundle w.r.t $g_J = \omega(\cdot, J\cdot)$.

$\Longrightarrow$ $D_{\text{fiber}} \cdot c(\tilde{X}) + c(\tilde{X}) \cdot D_{\text{fiber}} = 0 \quad (\forall X \in TU)$.

Remark.

1. In the Lagrangian fibraiton case one has :

$$
\begin{aligned}
& \{\pi : V \to U, D_{\text{fiber}}\} \text{ is acyclic.} \\
\stackrel{\text{Def}}{\Longleftrightarrow} \quad & \operatorname{Ker} D_{\text{fiber}} = 0 \\
\Longleftrightarrow \quad & H^*(\text{Fiber}; L|_{\text{Fiber}}) = 0 \\
\Longleftrightarrow \quad & \text{Non BS condition.}
\end{aligned}
$$

2. "Non-acyclic point/fiber"
   = Singular point/fiber or regular BS point/fiber.

Theorem. (Localization of RR number I)

If a closed prequantized symplectic manifold has a structure of a Lagrangian fibration, then the Riemann-Roch number is localized at any neighborhoods of singular fibers and regular Bohr-Sommerfeld fibers.

Theorem.

If V is a neighborhood of a regular Bohr-Sommerfeld fiber F, then we have $\mathbf{Ind}(V, V \setminus F) = 1$.

Corollary. (Andersen)

For a non-singular Lagrangian fibration over a prequantized closed symplectic manifold $\pi : (M, \omega) \longrightarrow B$ we have

$$RR(M, \omega) =^{\#} \text{BS fibers.}$$

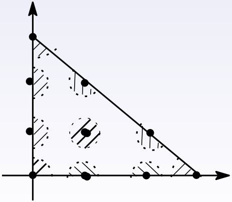
Problem of the simple version

Consider a moment map $\pi : (\mathbb{C}P^2, k\omega_{\mathrm{FS}}) \to B(\subset \mathbb{R}^2)$ for a toric action, $[z_0 : z_1 : z_2] \mapsto [z_0 : t_1 z_1 : t_2 z_2]$.

1. $RR(\mathbb{C}P^2) =^{\#} (B \cap \mathbb{Z}^2)$ (Danilov)
2. Singular fibers $\longleftrightarrow \partial B$.
3. "Non-acyclic points"

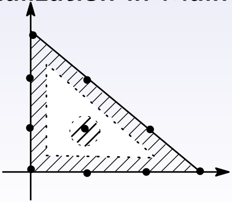
   $\longleftrightarrow$ singular points + (interior) lattice points.

○ Expected localization



(Lattice points)

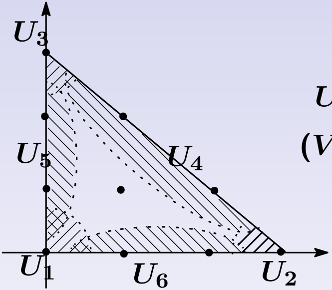
○ Localization in Main Thm. I



(Sing. points + lattice points)

Observatoin - $\mathbb{C}P^2$ as a union of torus bundles

$B =$



$U_7 :=$ int$B$.

$(V_i := \pi^{-1}(U_i))$

$$\mathbb{C}P^2 = V_1 \cup V_2 \cup \cdots \cup V_7.$$

$$V_i = \begin{cases} T^2\text{-bundle } (i = 7) \\ T^1\text{-bundles } (i = 4, 5, 6) \\ \text{trivial bundles } (i = 1, 2, 3). \end{cases}$$

$$(T^1 = (\mathrm{diag} S^1)^\perp,\ 1 \times S^1,\ S^1 \times 1 \subset T^2)$$

Compatibility over the intersections

There are commutative diagrams of torus bundles for $i > j$:

$$
\begin{array}{ccc}
 & V_i \cap V_j & \\
 \swarrow^{T_i} & & \searrow^{T_j} \\
(V_i \cap V_j)/T_i & \xleftarrow{\;T_i/T_j\;} & (V_i \cap V_j)/T_j
\end{array}
$$

$$(T_i, T_j = T^2 \text{ or } T^1 \text{ or point. })$$

Remark. (IMPORTANT)

One can see that each V_i has a structure of an acyclic fibration outside of the inverse images of integral lattice points.

Definition. (Compatible fibration)

Let V be a Riemannian manifold.

$\{\pi_\alpha : V_\alpha \to U_\alpha \mid \alpha \in A\}$ is a *compatible fibration on* V. $\overset{\text{Def}}{\Longleftrightarrow}$

1. $V = \cup_{\alpha \in A} V_\alpha$ is an open covering.
2. Each $\pi_\alpha : V_\alpha \to U_\alpha$ is a flat torus bundle.
3. $V_\alpha \cap V_\beta = \pi_\alpha^{-1}\pi_\alpha(V_\alpha \cap V_\beta) = \pi_\beta^{-1}\pi_\beta(V_\alpha \cap V_\beta)$.
4. If $V_\alpha \cap V_\beta \neq \emptyset$ and $\alpha \neq \beta$, then there exist a commutative diagram of flat torus bundles,

$$
\begin{array}{ccc}
 & V_\alpha \cap V_\beta & \\
\swarrow^{\pi_\alpha|_{V_\alpha \cap V_\beta}} & & \searrow^{\pi_\beta|_{V_\alpha \cap V_\beta}} \\
\pi_\alpha(V_\alpha \cap V_\beta) & \xleftarrow{\exists} & \pi_\beta(V_\alpha \cap V_\beta)
\end{array}
$$

or with the converse direction in the bottom line.

Remark.

For a manifold equipped with a torus action we can construct a structure of a compatible fibration.

- M : Riemannian manifold.
- V : open subset of M such that $M \setminus V$ is compact.
- $W \to M$: $\mathbb{Z}/2$-graded $\mathrm{Cl}(TM)$-module bundle .
- $\{\pi_\alpha : V_\alpha \to U_\alpha \mid \alpha \in A\}$: a compatible fibration over V.

Definition. (Compatible system of Dirac-type operators)

$\{D_\alpha : \Gamma(W|_{V_\alpha}) \to \Gamma(W|_{V_\alpha}) \mid \alpha \in A\}$ is a *compatible system of Dirac-type operators.* $\overset{\text{Def}}{\Longleftrightarrow}$
Each D_α is a Dirac-type operator along fibers of π_α and it anti-commutes with the Clifford action of base direction of π_α.

Definition. (Acyclic compatible fibration)

$\{\pi_\alpha : V_\alpha \to U_\alpha, D_\alpha \mid \alpha \in A\}$ is an acyclic compatible fibration $\overset{\text{Def}}{\Longleftrightarrow}$ $\mathrm{Ker}\, D_\alpha = 0$ for all $\alpha \in A$.

Theorem. (Localization of index II)

Let M be a closed Riemannian manifold and $W \to M$ a $\mathbb{Z}/2$-graded $\mathrm{Cl}(TM)$-module bundle. Suppose that there is an open covering $M = V \cup (\cup_{i=1}^{m} O_i)$ such that $\{O_i\}_{i=1,\cdots,m}$ are mutually disjoint and V is equipped with an acyclic compatible fibration. Then we can define the *local index* $\mathrm{Ind}(O_i, O_i \cap V)$ and we have

$$\mathrm{Ind}(D) = \sum_{i=1}^{m} \mathrm{Ind}(O_i, O_i \cap V).$$

Theorem. (Localization of RR number II)

Suppose that (M, ω) is a prequantized closed symplectic manifold equipped with a Hamiltonian torus action. Then the Riemann-Roch number $RR(M, \omega)$ is localized at any neighborhoods of the inverse images of the integral lattice points in the moment polytope.

Remarks/Problems

1. We do not know how we can construct a structure of a compatible fibration in more general cases, e.g., moduli space of flat bundles or Gelfand-Cetlin system.
2. There are examples of compatible fibrations which do not admit global torus action, and we can apply our localization theorem (Yoshida's next talk).
3. Compute the multiplicity of sigular fibers which do not appear in the toric case.