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Functorial A_∞-coproduct of combinatorial simplicial chains that induces itself under barycentric subdivision

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Plan

1. Recallments on A_∞-things
   - Gradings, signs, tensor algebra derivations, A_∞-coproducts...
   - Transferring A_∞-coproducts along deformation retractions
2. Recallments on combinatorial simplicial topology
   - Combinatorial simplicial complexes and Kolmogorov's problem
   - Flags and barycentric subdivisions
3. Functorial barycentric retraction of combinatorial simplex
   - Some precise combinatorial formulas
4. Functorial A_∞-coproduct transferred to itself via barycentric retraction
   - Recursive formulas
   - Computational example: $\dim = 1$ case
5. Final comments (if the time allows)
   - Open questions
   - What does 'sum over trees' formula come from

Grading and sign conventions

We fix a ground field $\Bbbk$ of characteristic zero and work with graded vector spaces $V = \bigoplus_{n\in\mathbb{Z}} V_n$ over $\Bbbk$. A linear map $V \xrightarrow{\varphi} W$ has degree k, if $\forall\ i$ $\varphi(V_i) \subset W_{i+k}$. Tensor products of operators are composed and applied to tensor products of elements by means of the Koszul sign rule:

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_m) \circ (g_1 \otimes g_2 \otimes \cdots \otimes g_m) = (-1)^{\varepsilon(f;g)} (f_1 \circ g_1) \otimes (f_2 \circ g_2) \otimes \cdots \otimes (f_m \circ g_m) ,$$

$$f_1 \otimes f_2 \otimes \cdots \otimes f_m (v_1 \otimes v_2 \otimes \cdots \otimes v_m) = (-1)^{\varepsilon(f;v)} f_1(v_1) \otimes f_2(v_2) \otimes \cdots \otimes f_m(v_m) .$$

The sign depends on two ordered collections of degrees:

$$\varepsilon(\alpha_1, \alpha_2, \dots, \alpha_m; \beta_1, \beta_2, \dots, \beta_m) = \deg\alpha_m \cdot (\deg\beta_1 + \cdots + \deg\beta_{m-1}) + \deg\alpha_{m-1} \cdot (\deg\beta_1 + \cdots + \deg\beta_{m-2}) + \cdots + \deg\alpha_2 \cdot \deg\beta_1 .$$

We always write $[f,g] \stackrel{\text{def}}{=} fg - (-1)^{\deg f \deg g} gf$ for graded commutator.
A differential on V is degree -1 map $V \xrightarrow{\partial} V$ satisfying

$$\partial^2 = \frac{1}{2}[\partial, \partial] = 0\,.$$

Graded vector space V equipped with a differential ∂_V is called a complex.
Shift by k takes V to $V[k]$ that has

$$V[k]_i = V_{i+k} \quad \text{and} \quad \partial_{V[k]} = (-1)^k \partial_V\ .$$

Thus, the identity map $s : V \longrightarrow V[1]$ has degree -1 and commutes with differentials (in graded sense): $s \circ \partial_V = -\partial_{V[1]} \circ s$.
We write $\mathrm{T}V \stackrel{\text{def}}{=} \prod_{n\geqslant 1} V^{\otimes n}$ for completed reduced tensor algebra, that is algebra of formal non-commutative power series $\tau = \sum_{k\geqslant 1} \tau_k\,,\ \tau_k \in V^{\otimes k}$, without constant term. We refer to k as tensor degree of τ_k, whereas total degree $\deg \tau_k$ comes from $\deg(v_1 \otimes v_2 \otimes \cdots \otimes v_k) = \sum \deg(v_\nu)\,.$

Derivations

A derivation $\mathrm{T}V \xrightarrow{D} \mathrm{T}V$ is a $\Bbbk$-linear map satisfying the Leibnitz rule

$$D \circ \mu = \mu \circ (D \otimes 1 + 1 \otimes D),$$

where $\mathrm{T}V \otimes \mathrm{T}V \xrightarrow{\mu} \mathrm{T}V$ is the tensor multiplication. Being applied to elements, this looks like:

$$D(\omega_1 \otimes \omega_2) = (D\omega_1) \otimes \omega_2 + (-1)^{\deg D \cdot \deg \omega_1} \omega_1 \otimes (D\omega_2).$$

There is $\Bbbk$-linear isomorphism between derivations $\mathrm{T}V \xrightarrow{D} \mathrm{T}V$ and linear maps $V \xrightarrow{\delta} \mathrm{T}V$. It takes D to its restriction

$$\delta_D = D|_V : V \xrightarrow{\delta} \mathrm{T}V .$$

Backwards, it extends δ by the Leibnitz rule to the map D_δ defined on the whole of $\mathrm{T}V$. Under this bijection the (graded) commutator of derivations $[D_{\delta_1}, D_{\delta_2}]$ turns to the Gerstenhaber bracket of maps $\{\delta_1, \delta_2\}$ defined by

$$D_{\{\delta_1,\delta_2\}} \stackrel{\text{def}}{=} [D_{\delta_1}, D_{\delta_2}] = D_{\delta_1} D_{\delta_2} - (-1)^{\deg \delta_1 \cdot \deg \delta_2} D_{\delta_2} D_{\delta_1} .$$

A_∞-coproduct

A_∞-coproduct on V is a derivation $\mathrm{T}(V[1]) \xrightarrow{D} \mathrm{T}(V[1])$ of degree -1 such that $D^2 = 0$. It defines and is uniquely defined by $\Bbbk$-linear map

$$\delta = \sum_{n \geqslant 1} \delta_n = D|_{V[1]} \,:\, V[1] \longrightarrow \mathrm{T}(V[1])$$

of degree -1 such that $\{\delta, \delta\} = 0$. In terms of V itself, the homogeneous components of δ are the maps $V \xrightarrow{\widetilde{\delta}_n} V^{\otimes n}$, of degree $n-2$, fitted into commutative diagram:

$$\begin{array}{ccccc}
V[1] & \xrightarrow{\delta_n} & V[1]^{\otimes n} & & (\deg \delta_n = -1) \\
\uparrow s & & \uparrow s^{\otimes n} & & (\deg s = -1) \\
V & \xrightarrow{\widetilde{\delta}_n} & V^{\otimes n} & & (\deg \widetilde{\delta}_n = n-2)
\end{array}$$

where $V \xrightarrow{s} V[1]$ is the identity map.

The equation $\{\delta,\delta\}=0$ can be expanded as a system of quadratic relations on the maps $V \xrightarrow{\widetilde{\delta}_n} V^{\otimes n}$. For $n=1,\,2,\,3,\,\dots$ they are:

$$\{\delta_1,\delta_1\}=0 \iff \widetilde{\delta}_1^2=0$$

$$\{\delta_1,\delta_2\}+\{\delta_2,\delta_1\}=0 \iff \widetilde{\delta}_2\widetilde{\delta}_1=(\widetilde{\delta}_1\otimes 1+1\otimes\widetilde{\delta}_1)\,\widetilde{\delta}_2$$

$$\delta_2^2+\{\delta_1,\delta_3\}=0 \iff (\widetilde{\delta}_2\otimes 1)\otimes\widetilde{\delta}_2-(1\otimes\widetilde{\delta}_2)\otimes\widetilde{\delta}_2= (\widetilde{\delta}_1\otimes 1\otimes 1+1\otimes\widetilde{\delta}_1\otimes 1+1\otimes 1\otimes\widetilde{\delta}_1)\,\widetilde{\delta}_3+\widetilde{\delta}_3\widetilde{\delta}_1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The first equation says that $\widetilde{\delta}_1 : V \longrightarrow V$ is a differential on V. The second says that $\widetilde{\delta}_1$ satisfies the co-Leibnitz rule w.r.t. co-multiplication $\widetilde{\delta}_2 : V \longrightarrow V\otimes V$. The third says that the co-associator of this co-multiplication $V \xrightarrow{(\widetilde{\delta}_2\otimes 1)\otimes\widetilde{\delta}_2-(1\otimes\widetilde{\delta}_2)\otimes\widetilde{\delta}_2} V^{\otimes 3}$ is zero homotopic by means of contracting homotopy $\widetilde{\delta}_3$, and so on. Thus, A_∞-coproduct with just two non-zero components: $\delta=\delta_1+\delta_2$ is the standard co-associative DG-coalgebra structure (or just coalgebra structure, if $\delta_1=0$).

In terms of dual space, the dual map $\mathrm{T}(V^*[-1]) \xrightarrow{\delta^*} V^*[-1]$ provides V^* with a series of n-ary operations

$$\underbrace{V^* \otimes \cdots \otimes V^*}_{n} \xrightarrow{\widetilde{\delta}_n^*} V^*, \; n = 1, 2, 3, \dots, \; \deg \widetilde{\delta}_n^* = 2 - n$$

called higher multiplications and satisfying quadratic relations dual to above (they say that $\widetilde{\delta}_1^*$ and $\widetilde{\delta}_2^*$ provide V^* with the DGA-structure, possibly non-associative but with the associator homotopic to zero by means of 'triple product' $\widetilde{\delta}_3^*$, e.t.c.) Such dual data is called A_∞-algebra structure.

Remark

In physics, products $\widetilde{\delta}_n^*(v_1^*, v_2^*, \dots, v_n^*)$ are known as 'correlators'. Typically, they are expressed by some integrals. Quadratic relations between them come from integration tricks (often not well defined). Higher correlators appearing in 'effective theory' attached to some space W usually are computed in terms of appropriate 'ground theory' attached to another space V connected with W by means of some 'reduction'. Mathematically, this is formalized as transferring A_∞-structures along deformation retractions.

SDR-data

Strong deformation retraction between complexes (V, ∂_V), (W, ∂_W) (SDR-data for short) is a diagram $\gamma \circlearrowright V \underset{\sigma}{\overset{\pi}{\rightleftarrows}} W$, where π, σ commute with the differentials, homotopy $\gamma : V \longrightarrow V$ has degree 1, and

$$\pi\sigma = 1_W\,, \quad \sigma\pi = 1_V + \partial_V\gamma + \gamma\partial_V\,, \quad \gamma^2 = 0\,, \quad \pi\gamma = 0\,, \quad \gamma\sigma = 0\,.$$

Example (Retraction onto homology)

Let $V = A \oplus B \oplus C$, where $B = \operatorname{im}\partial_V$ is the space of boundaries, $C \subset \ker\partial_V$ is transversal to B in $\ker\partial_V$, and $A \subset V$ is transversal to $\ker\partial_V$ in V. Thus, ∂_V maps A isomorphically onto B and annihilates $B \oplus C$, $C \cong H(V)$ represents the homologies. Then SDR-data is given by $\gamma \circlearrowright V \underset{\sigma}{\overset{\pi}{\rightleftarrows}} C$, where C is considered as complex with zero differential, π, σ come from splitting $V = A \oplus B \oplus C$, and homotopy γ takes B isomorphically onto A via $-\partial_V|_A^{-1}$ and annihilates $C \oplus A$. Relation $\sigma\pi = 1_V + [\partial_V, \gamma]$ holds, because $-[\partial_V, \gamma]$ projects V onto $A \oplus B$ along C. Other relations are evident.

Transferring A_∞-coproducts along SDR-data

Let $\gamma \looparrowright V[1] \underset{\sigma}{\overset{\pi}{\rightleftarrows}} W[1]$ be SDR-data and $\mathsf{T}(V[1]) \xrightarrow{D_\delta} \mathsf{T}(V[1])$ be an A_∞-coproduct associated with power series $\delta = \partial_{V[1]} + \sum_{n\geqslant 2} \delta_n$, where

$$\delta_n : V[1] \longrightarrow V[1]^{\otimes n}$$

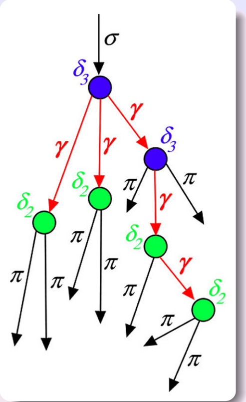
and linear term $V[1] \xrightarrow{\delta_1} V[1]$ coincides with differential $\partial_{V[1]} = -\partial_V$.
Then transferred A_∞-structure on W is given by derivation

$$\mathsf{T}(W[1]) \xrightarrow{D_{\delta_{\mathrm{ind}}}} \mathsf{T}(W[1])$$

associated with power series $\delta_{\mathrm{ind}} = \partial_{W[1]} + \sum_{n\geqslant 2} \delta_{\mathrm{ind},n}$ whose n-th degree component $W[1] \xrightarrow{\delta_{\mathrm{ind},n}} W[1]^{\otimes n}$ for $n \geqslant 2$ is the sum over trees

$$\delta_{\mathrm{ind},n} = \sum_{\Gamma} \delta_\Gamma \,.$$

Sum over trees



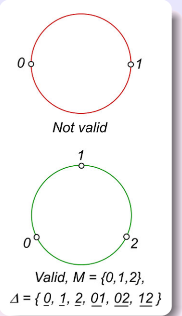
The summation goes over all planar trees Γ with one root, n leaves, and internal vertices of valency $\geqslant 3$. Each tree Γ should be oriented from the root to the leaves and decorated by operators

- $W \xrightarrow{\sigma} V$ on the incoming root-edge;
- $V \xrightarrow{\pi} W$ on all outgoing leaf-edges;
- $V \xrightarrow{\gamma} V$ on all internal edges;
- $V \xrightarrow{\delta_k} V^{\otimes k}$ on all $(k+1)$-valent vertexes.

Their composition along the tree gives operator $W[1] \xrightarrow{\delta_\Gamma} W[1]^{\otimes n}$ corresponding to Γ in the sum $\delta_{\text{ind},n} = \sum_{\Gamma} \delta_\Gamma$.

Implication $\{\delta,\delta\} = 0 \implies \{\delta_{\text{ind}},\delta_{\text{ind}}\} = 0$ is not obvious and was re-established by many authors under various assumptions. If we'll have time, we'll sketch some easy arguments in appendix.

Combinatorial simplicial complexes



Combinatorial simplicial complex is topological space properly triangulated by simpleces in such a way that each simplex is uniquely determined by its vertexes.

Formally, it is defined by vertex set M and set of simpleces Δ which is a subset in the set of all subsets in M such that

- Δ contains all elements of M;
- Δ contains all subsets of each $\sigma \in \Delta$.

We write $\underline{x_1\, x_2\, \dots\, x_k}$ for simpleces $\{x_1, x_2, \dots, x_k\} \in \Delta$ considered as non-ordered non-oriented sets. They form full subcategory in category $\mathscr{S}(M)$ of all subsets in M with inclusions as the morphisms.

We write $[X_1, X_2, \dots, X_k]$ for totally ordered collections of objects.

An oriented simplex $\overline{x_1\, x_2\, \dots\, x_k}$ is the orbit of $[x_1, x_2, \dots, x_k]$ under the action of alternating subgroup $\mathfrak{A}_k \subset \mathfrak{S}_k$ by the permutations of x_ν's. So, each simplex produces two oriented simpleces.

Simplicial chain complex

We fix a finite set M of cardinality m and write $\overline{C} = \overline{C}^{[M]}$ for chain complex

$$0 \longrightarrow \overline{C}_{m+1} \xrightarrow{\overline{\partial}} \overline{C}_m \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \overline{C}_2 \xrightarrow{\overline{\partial}} \overline{C}_1 \longrightarrow 0$$

consisting of vector spaces $\overline{C}_k$ spanned by oriented simpleces $\overline{x_1\,x_2\,\ldots\,x_k}$ modulo the relation identifying orientation change with change of sign,

$$\overline{\partial}\,:\,\overline{x_1\,x_2\,\ldots\,x_k} \longmapsto \sum_{\nu=1}^{k}(-1)^{\nu}\cdot\overline{x_1\,\ldots\,x_{\nu-1}\,x_{\nu+1}\,\ldots\,x_k}\,.$$

Warning

Simplicial chain complex commonly used in topology is $\overline{C}[1]$. Topological degree of simplex σ equals its dimension $\dim\sigma$. Our combinatorial degree is the cardinality $|\sigma| = \dim\sigma + 1$. Thus, an A_∞-coproduct of topological chains is given by degree -1 map $\overline{C}[2] \xrightarrow{\delta} \mathrm{T}(\overline{C}[2])$ whose homogeneous components $\overline{C} \xrightarrow{\widetilde{\delta}_n} \overline{C}^{\otimes n}$ in combinatorial grading have degree $2n-3$ and are odd (satisfy sign-less quadratic relations).

Kolmogorov's problem

Let $\overline{C}^{[M]}$ be simplicial chain complex associated with the standard combinatorial simplex $\Delta = \mathscr{S}(M)$ with vertex set M. We are interesting in A_∞-coproducts on topological simplicial chains that are functorial w.r.t. inclusions of finite sets $M_1 \longrightarrow M_2$.

Such a coproduct δ is $\mathrm{Aut}\,(M)$-equivariant and provides all combinatorial simplicial complexes Δ with naturally compatible A_∞-coalgebra structures such that $\delta(\sigma) \subset \mathscr{S}(\sigma)$ for each simplex σ.

Early in the development of topology, at Moscow's conference in 1935, when multiplicative structures on cohomologies were just under construction, Kolmogorov asked about co-associative co-product δ_2 of this sort and proposed a candidate. To write it down, let us fix some total ordering $[x_1, x_2, \dots, x_m]$ on M and write $\overline{Y}$ for $Y \subset M$ oriented by this ordering. For two disjoint subsets $\overline{Y} = \overline{y_1\, y_2\, \dots\, y_k}$, $\overline{Z} = \overline{z_1\, z_2\, \dots\, z_\ell}$ let $\overline{X \cdot Y} \overset{\text{def}}{=} \overline{y_1\, y_2\, \dots\, y_k z_1\, z_2\, \dots\, z_\ell}$. Finally, let $\overline{M}_i \overset{\text{def}}{=} \overline{M \smallsetminus \{x_i\}}$.

Kolmogorov's coproduct δ_2^{K} takes $\overline{M}$ to

$$\frac{\sum_{i=1}^{m}(-1)^{i+1}\sum_{Y\subseteq M_i}\operatorname{sgn}(Y)\binom{|M|-1}{|Y|}^{-1}\overline{Y\cdot\{x_i\}}\otimes\overline{\{x_i\}\cdot(M_i\smallsetminus Y)}}{|M|} \qquad \text{(K)}$$

where the second sum runs over subsets $Y\subset M_i = M\smallsetminus\{x_i\}$ (including $\varnothing$ and M_i), $\binom{*}{*}^{-1}$ stays for inverse binomial coefficient, and $\operatorname{sgn}(Y)$ means the sign of shuffle permutation $\overline{M}_i\leftrightarrow\overline{Y\cdot(M_i\smallsetminus Y)}$. For example:

$$\delta_2^{\mathrm{K}}(\overline{01}) = \frac{+\left(\overline{10}\otimes\overline{0}+\overline{0}\otimes\overline{01}\right)-\left(\overline{01}\otimes\overline{1}+\overline{1}\otimes\overline{10}\right)}{2} = -\frac{1}{2}\,\mathrm{ad}_{\overline{01}}\left(\overline{0}+\overline{1}\right)$$

where $b\xmapsto{\mathrm{ad}_a} a\otimes b - b\otimes a$ is commutation operator in tensor algebra.

Coproduct (K) is well defined, functorial w.r.t. inclusions of finite sets $M_1\hookrightarrow M_2$, and compatible with differential $\delta_1=\overline{\partial}$.
However, it turns to be non-associative.

At the same time an associative product on cohomologies was constructed by means of much simpler Alexander-Whitney coproduct

$$\delta_2^{\mathrm{AW}}([x_1, x_2, \dots, x_k]) = \sum_{i=1}^{k} [x_1, x_2, \dots, x_i] \otimes [x_i, x_{i+1}, \dots, x_k] \,,$$

defined for totally ordered chains only and neither functorial nor palpable in $\overline{C}$, but it does not matter for homological problems stated up to homotopy. When this local goal of those years was achieved, Kolmogorov's question has fallen outside the mainstream of topology for some time.

However, it comes back in focus when we have to compute, say, higher Massey products needed for recovering the homotopy type of manifold from its cohomologies, not to mention pure combinatorial significance of problem. In modern setup, Kolmogorov's question could be treated as:

How to extend the Kolmogorov's coproduct (K) to some functorial A_∞-coproduct in most natural way?

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However, it comes back in focus when we have to compute, say, higher Massey products needed for recovering the homotopy type of manifold from its cohomologies, not to mention pure combinatorial significance of problem. In modern setup, Kolmogorov's question could be treated as:

Are there functorial A_∞-coalgebra structures on $\overline{C}$ inducing higher Massey products on $H(\overline{C})$?

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If so, can we choose some preferable ones? (Say, having especially nice combinatorial or symmetric properties.)

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However, it comes back in focus when we have to compute, say, higher Massey products needed for recovering the homotopy type of manifold from its cohomologies, not to mention pure combinatorial significance of problem. In modern setup, Kolmogorov's question could be treated as:

Or in physical cant: What is most elegant ground theory on $\overline{C}$ producing on $H(\overline{C})$ an effective theory whose correlators are the Massey products?

We attack these questions using barycentric subdivisions. We construct functorial in M SDR-data between chain complex $\overline{C}^{[M]}$ of standard simplex $\mathscr{S}(M)$ and chain complex $\overline{C}^{[B(M)]}$ of its barycentric subdivision $B(M)$

$$\gamma \looparrowright \overline{C}^{[B(M)]} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \overline{C}^{[M]} \qquad \text{(BR)}$$

This barycentric retraction is essentially unique. Each functorial in M A_∞-coproduct $\delta^{[M]}$ on $\overline{C}^{[M]}$, by functoriality, provides $\overline{C}^{[B(M)]}$ with A_∞-coproduct $\delta^{[B(M)]}$, which can be transferred along (BR) back to $\overline{C}^{[M]}$. Thus, we get new functorial in M A_∞-coproduct $\delta^{[M]}_{\text{ind}}$ on $\overline{C}^{[M]}$.

Definition

Functorial A_∞-coproduct of combinatorial simplicial chains is called barycentrically stable, if it is transferred to itself under this procedure.

In the rest of talk we explain why such a product δ^{bs} should exist, be unique up to a constant factor, and have $\delta^{\text{bs}}_1 = \overline{\partial}$ and $\delta^{\text{bs}}_2 = \delta^{\mathrm{K}}$ from (K). We write explicit recursive formula expressing δ^{bs}_k through $\delta^{\text{bs}}_{<k}$ and deduce nice closed formula for $\delta^{\text{bs}}(\overline{01})$.

Flags and barycentric subdivisions

A flag of length k in M is a chain of strictly embedded non-oriented sets

$$F_1 \subset F_2 \subset \cdots \subset F_k \tag{F}$$

Flag (F) defines and is uniquely defined by ordered collection of graded components $[G_1, G_2, \ldots, G_k]$, $G_1 \stackrel{\text{def}}{=} F_1$, $G_i \stackrel{\text{def}}{=} F_i \smallsetminus F_{i-1}$ for $i \geqslant 2$. We often use $[G_1, G_2, \ldots, G_k]$ as alternative notation for flag (F).

Definition

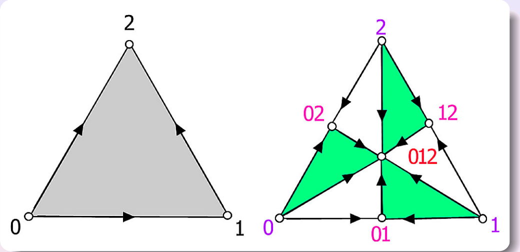
Flag is called saturated, if all its graded components have cardinality one.

Saturated flags (F) stay in bijection with total orderings on F_k.

Definition

Barycentric subdivision $B(M)$, of a simplex with vertex set M, is a simplicial complex whose vertexes are non-empty subsets of M and k-vertex simpleces are the flags of length k in M.

Thus, vertexes of $B(M)$ are the objects of the category $\mathscr{S}(M)$ (subsets of M with inclusions as the morphisms), oriented edges of $B(M)$ are the morphisms $F_1 \subset F_2$ in $\mathscr{S}(M)$, 2-dimensional faces are the pairs of consequent morphisms $F_1 \subset F_2 \subset F_3$ in $\mathscr{S}(M)$, e.t.c.



In mechanical terms, vertex $F \subset M$ depicts the centre of mass for the points of F. Geometrically, we put new vertex in the barycenter of each face and join it with all other vertexes of that face.

Notation

Chain complex of simplicial complex $B(M)$ is called the barycentric chain complex and is denoted by $\underline{C}^{[M]} = \overline{C}^{[B(M)]}$.

Barycentric chain complex looks like

$$0 \longrightarrow \underline{C}_{m+1} \xrightarrow{\partial} \underline{C}_m \xrightarrow{\partial} \cdots \xrightarrow{\partial} \underline{C}_2 \xrightarrow{\partial} \underline{C}_1 \longrightarrow 0\,.$$

Basis of vector space $\underline{C}_k$ consists of length k flags $F_1 \subset F_2 \subset \cdots \subset F_k$ and differential $\underline{C}_k \xrightarrow{\partial} \underline{C}_{k-1}$ takes this flag to

$$\sum_{\nu=1}^{k}(-1)^{\nu} F_1 \subset \cdots \subset F_{\nu-1} \subset F_{\nu+1} \subset \cdots \subset F_k$$

(ν th term of the filtration is omitted for $\nu = 1, 2, \ldots, k$). In terms of graded components this is written as

$$\sum_{\nu=1}^{k-1}(-1)^{\nu}\,[G_1, \ldots, G_{\nu-1}, G_{\nu} \sqcup G_{\nu+1}, G_{\nu+2}, \cdots, G_k]$$

(ν th comma is replaced by union).

Functorial retraction

Theorem

Over a field $\Bbbk$ of zero characteristics there exists strong deformation retraction

$$\gamma \circlearrowright \overline{C}^{[B(M)]} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \overline{C}^{[M]}$$

functorial w.r.t. inclusions of finite sets $M_1 \hookrightarrow M_2$. It is unique up to rescaling $\sigma \mapsto t\sigma$, $\pi \mapsto t^{-1}\pi$ by some $t \in \Bbbk$.

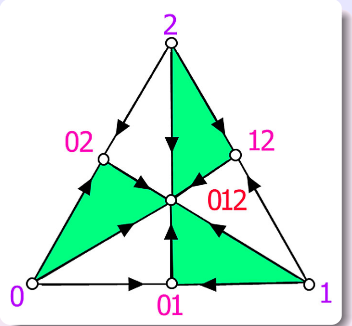
Note that functoriality means existence of the same retraction for any combinatorial simplicial complex $\Delta \subset \mathscr{S}(M)$. It also forces maps σ, π, γ to be equivariant w.r.t. the action of the permutation group $\mathrm{Aut}(M)$. Actually, in a presence of this equivariance, the SDR-data relations

$$\pi\sigma = 1_W \,, \quad \sigma\pi = 1_V + \partial_V\gamma + \gamma\partial_V \,, \quad \gamma^2 = 0 \,, \quad \pi\gamma = 0 \,, \quad \gamma\sigma = 0$$

produce quite over-determined system of linear equations and this is a kind of luck that it turns to be solvable at all.

Functorial inclusion $\overline{C} \xhookrightarrow{\sigma} \underline{C}$

There is unique up to rescalling intertwiner from the sign representation of symmetric group Aut (M) on the oriented faces of M to the tabloid representation on the flags. Geometrically, it takes each simplex to the oriented chain formed by all its barycentric pieces. In combinatorial terms,



$$\sigma\left(\overline{x_1 x_2 \ldots x_k}\right) = \sum_{g \in \mathfrak{S}_k} \operatorname{sgn}(g) [x_{g(1)} x_{g(2)} \ldots x_{g(k)}]$$

(alternated sum of saturated flags built from $\{x_1, x_2, \ldots, x_k\}$). For example:

$$\sigma(\overline{012}) = ([0,1,2] + [1,2,0] + [2,1,0]) - ([0,2,1] + [2,1,0] + [1,0,2]) ,$$

Note that $\overline{C} \xhookrightarrow{\sigma} \underline{C}$ obviously commutes with differentials.

Functorial projection $\underline{C} \xrightarrow{\pi} \overline{C}$

This formula is less obvious and requires denominator:

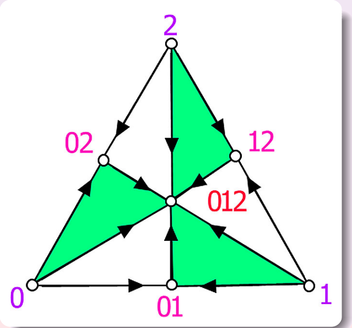
$$\pi\,(\,F_1 \subset F_2 \subset \;\cdots\; \subset F_k\,) = \frac{1}{\prod\limits_{\nu=1}^{k} |F_\nu|} \cdot \sum_{\substack{(x_1, x_2, \dots, x_k) \\ \text{running through} \\ G_1 \times G_2 \times \cdots \times G_k}} \overline{x_1\, x_2\, \dots\, x_k}\,.$$

Thus, we sum oriented simpleces $\overline{x_1\, x_2\, \dots\, x_k}$ for all possible choices of $x_i \in F_i \smallsetminus F_{i-1}$ and divide the result by the product of cardinalities of the flag sets $|F_i|$. For example:

$$\pi([\underline{01}]) \;=\; \tfrac{1}{2}\left(\overline{0} + \overline{1}\right), \quad \pi([0,1]) \;=\; \tfrac{1}{2}\,\overline{01}\,,$$

$$\pi([\underline{01},2]) = \tfrac{1}{6}\,\left(\overline{02} + \overline{12}\right),$$

$$\pi([0,\underline{12}]) = \tfrac{1}{3}\,\left(\overline{01} + \overline{02}\right).$$



Functorial degree 1 homotopy $\gamma : \underline{C} \longrightarrow \underline{C}$

annihilates saturated flags; non-saturated $F_1 \subset F_2 \subset \cdots \subset F_k$ goes to

$$\sum_{\substack{i: \\ |F_i|>i}} (-1)^i \cdot \prod_{\nu=1}^{i} |F_\nu|^{-1} \cdot \sum_{\substack{(x_1,x_2,\dots,x_i) \\ \text{running through} \\ G_1\times G_2\times\cdots\times G_i}} \sum_{g\in\mathfrak{S}_i} \operatorname{sgn}(g)\cdot F^{(i)}(x,g)\,,$$

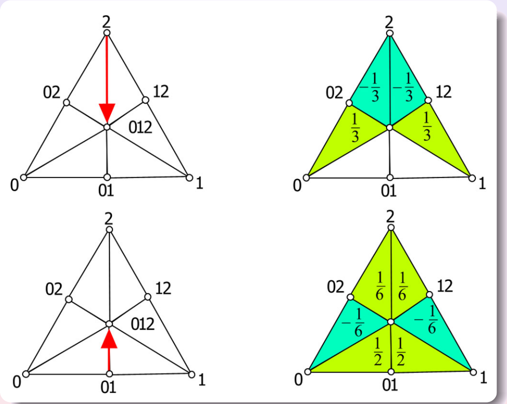
where $F^{(i)}(x,g)$ denotes following flag of length $(k+1)$:

$$\underline{x_{g(1)}} \subset \underline{x_{g(1)}x_{g(2)}} \subset \cdots \subset \underline{x_{g(1)}\,x_{g(2)}\,\dots\,x_{g(i)}} \subset F_i \subset F_{i+1} \subset \cdots \subset F_k\,.$$

Thus, for each i such that sub-flag $F_1 \subset F_2 \subset \cdots \subset F_i$ is not saturated we consider all ordered collections $(x_1, x_2, \dots, x_i) \in G_1 \times G_2 \times \cdots \times G_i$ and form alternated sum of all saturated flags built of them; then we extend each saturated flag to the right side by $\subset F_i \subset F_{i+1} \subset \cdots \subset F_k$ and divide the sum by $(-1)^i \prod_{\nu=1}^{i} |F_\nu|$; finally, we add together these weighted sums coming from all i's.

For example, γ acts on two combinatorially different internal edges of the barycentric subdivision of the triangle as follows:

$$\gamma([\underline{2},\underline{01}] = -\frac{1}{3}\Big([\underline{2},\underline{0},\underline{1}] - [\underline{0},\underline{2},\underline{1}] + [\underline{2},\underline{1},\underline{0}] - [\underline{1},\underline{2},\underline{0}]\Big)$$



$$\gamma([\underline{01},\underline{2}] = \frac{1}{2}\Big([\underline{0},\underline{1},\underline{2}]+[\underline{1},\underline{0},\underline{2}]\Big)-\frac{1}{6}\Big([\underline{0},\underline{2},\underline{1}]-[\underline{2},\underline{0},\underline{1}]+[\underline{1},\underline{2},\underline{0}]-[\underline{2},\underline{1},\underline{0}]\Big)$$

Next formulas show how does γ act on points, edges and triangles (all flags containing non-proper inclusions are declared to be zeros)

$$\gamma(X) = \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X)$$

$$\gamma(X \subset Y) = \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X \subset Y) - \frac{1}{|X||Y|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X}} \Big[(\underline{x} \subset \underline{xy} \subset Y) - (\underline{y} \subset \underline{xy} \subset Y) \Big]$$

Next formulas show how does γ act on points, edges and triangles (all flags containing non-proper inclusions are declared to be zeros)

$$\gamma(X \subset Y \subset Z) = \frac{1}{|X|} \sum_{x \in X} (\underline{x} \subset X \subset Y \subset Z)$$

$$- \frac{1}{|X||Y|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X}} \Big[(\underline{x} \subset \underline{xy} \subset Y \subset Z) - (\underline{y} \subset \underline{xy} \subset Y \subset Z) \Big]$$

$$+ \frac{1}{|X||Y||Z|} \sum_{\substack{x \in X \\ y \in Y \smallsetminus X \\ z \in Z \smallsetminus Y}} \Big[(\underline{x} \subset \underline{xy} \subset \underline{xyz} \subset Z) - (\underline{y} \subset \underline{xy} \subset \underline{xyz} \subset Z) + (\underline{y} \subset \underline{yz} \subset \underline{xyz} \subset Z) - (\underline{z} \subset \underline{yz} \subset \underline{xyz} \subset Z) + (\underline{z} \subset \underline{xz} \subset \underline{xyz} \subset Z) - (\underline{x} \subset \underline{xz} \subset \underline{xyz} \subset Z) \Big]$$

Analyzing 'sum over trees' formula

Let $\overline{C}[2] \xrightarrow{\delta^{\mathrm{bs}}} \mathrm{T}(\overline{C}[2])$ provide topological chain complex $\overline{C}[1]$ with functorial A_∞-coproduct transferred to itself along functorial SDR-data

$$\gamma \circlearrowright \overline{C}^{[B(M)]} \underset{\sigma}{\overset{\pi}{\rightleftarrows}} \overline{C}^{[M]}.$$

Then each homogeneous component $\widetilde{\delta}^{\mathrm{bs}}_n : \overline{C} \longrightarrow \overline{C}^{\otimes n}$ satisfies

$$\widetilde{\delta}^{\mathrm{bs}}_n = \sum_{\Gamma} \widetilde{\delta}^{\mathrm{bs}}_{\Gamma} = A(\widetilde{\delta}^{\mathrm{bs}}_n) + B(\widetilde{\delta}^{\mathrm{bs}}_{<n})$$

The middle sum runs over trees with one input and n output slots decorated by σ and π's, internal edges decorated by γ's, and vertexes decorated $\widetilde{\delta}^{\mathrm{bs}}_i$.

$$A(\widetilde{\delta}^{\mathrm{bs}}_n) \stackrel{\mathrm{def}}{=} \pi^{\otimes n} \circ \widetilde{\delta}^{\mathrm{bs}} \circ \sigma$$

stays for the summand contributed by one vertex tree (corolla with 1 root and n leaves). It is the only summand depending on $\widetilde{\delta}^{\mathrm{bs}}_n$ and this dependence is linear. The sum of all other terms is denoted by $B(\widetilde{\delta}^{\mathrm{bs}}_{<n})$.

Thus, barycentric stability of functorial A_∞-coproduct δ^{bs} implies

$$(1 - A) \cdot \widetilde{\delta}_n^{bs} = B(\widetilde{\delta}_{<n}^{bs}) \quad \text{for all } n \geqslant 3. \tag{BS}$$

For $n = 1, 2$ the barycentric stability is expressed by equations

$$\widetilde{\delta}_1^{bs} = \pi \circ \widetilde{\delta}_1^{bs} \circ \sigma \ , \quad \widetilde{\delta}_2^{bs} = (\pi \otimes \pi) \circ \widetilde{\delta}_2^{bs} \circ \sigma$$

Claim

Pairs of compatible tensors $\overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_1^{[M]}} \overline{C}^{[M]}$, $\overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_2^{[M]}} \overline{C}^{[M]^{\otimes 2}}$ *of combinatorial degrees* -1 *and* $+1$ *that are functorial in* M *and go to itself via linear transformations* $\delta_1^{[M]} \mapsto \pi \circ \delta_1^{[B(M)]} \circ \sigma$ *and* $\delta_2^{[M]} \mapsto \pi^{\otimes 2} \circ \delta_2^{B([M])} \circ \sigma$ *form 1-dimensional subspace spanned by the simplicial chain differential* $\overline{\partial}$ *and the Kolmogorov co-product* (K)

One could try to recover the whole of barycentrically stable A_∞-coproduct δ^{bs} from its starting terms δ_1^{bs}, δ_2^{bs} by solving (BS) w.r.t. δ_n^{bs}.

Recursive formula for δ^{bs}

Conjecture

Eigenvalues of linear operator $A : \delta_n^{[M]} \mapsto \pi^{\otimes n} \circ \delta_n^{B([M])} \circ \sigma$, *which acts on functorial in* M *tensors* $\overline{C}^{[M]} \xrightarrow{\delta_n^{[M]}} \overline{C}^{[M]^{\otimes n}}$ *of combinatorial degree* $2n-2$, *never are equal to 1 for* $n \geqslant 3$ *and decrease exponentially as* $n \to +\infty$. *Thus, barycentrically stable functorial* A_∞*-coproduct of combinatorial simplicial chains is unique up to rescalling and can be computed by recursion:*

$$\delta_n^{bs} = (1-A)^{-1} \cdot B(\delta_{<n}^{bs})$$

where $B(\delta_{<n}^{bs})$ *is the sum over planar trees with one incoming slot,* n *outgoing slots, and internal vertexes of valency* $3 \leqslant v \leqslant (n-1)$ *oriented from the input to the outputs and decorated by* σ *on input,* π*'s on outputs,* γ*'s on internal edges and* δ_ν*'s on internal vertexes.*

Computational example: dim $= 1$ case

The combinatorial degree of $\widetilde{\delta}_n : \overline{C} \longrightarrow \overline{C}^{\otimes n}$ is $2n-3$. Hence, the coproduct of zero dimensional (i.e., of cardinality 1) simplex $\overline{0}$ has just one non-zero component $\widetilde{\delta}_2(\overline{0}) = \overline{0} \otimes \overline{0}$ (up to scalar factor taken to be 1).

For 1-dimensional (i.e., of cardinality 2) simplex $\overline{01}$, coproduct's n-th component $\overline{C} \xrightarrow{\widetilde{\delta}_n} \overline{C}^{\otimes n}$ should be in the linear span of maps

$$\begin{array}{l} \overline{01} \longmapsto \overline{01}^{\otimes\alpha} \otimes \overline{0} \otimes \overline{01}^{\otimes\beta} \\ \overline{01} \longmapsto \overline{01}^{\otimes\alpha} \otimes \overline{1} \otimes \overline{01}^{\otimes\beta} \end{array} \quad \text{(where } \alpha, \beta \geqslant 0,\ \alpha + \beta = n-1\text{)}$$

Among them, the functorial ones (commuting with transposition $0 \leftrightarrow 1$) are spanned by

$$\delta_n^\alpha \ : \ \overline{01} \longmapsto \overline{01}^{\otimes\alpha} \otimes \left(\overline{0} + (-1)^n \cdot \overline{1}\right) \otimes \overline{01}^{\otimes\beta},$$

which are, at the same time, the eigenvectors of linear operator

$$A_n : \ \delta_n^{[\overline{01}]} \mapsto \pi^{\otimes n} \circ \delta_n^{[B(\overline{01})]} \circ \sigma$$

with eigenvalues $(1/2)^{n-1}$ for odd n and $(1/2)^{n-2}$ for even n.

dim $= 1$ case: action of A_2

Indeed, $\delta_n^\alpha \circ \sigma\left(\overline{01}\right) = \delta_n^\alpha([0,1] - [1,0]) =$

$$[0,1]^{\otimes\alpha} \otimes (\,\underline{0} + (-1)^n \cdot \underline{01}\,) \otimes [0,1]^{\otimes\beta} - [1,0]^{\otimes\alpha} \otimes (\,\underline{1} + (-1)^n \cdot \underline{01}\,) \otimes [1,0]^{\otimes\beta}$$

and, applying $\pi^{\otimes n}$, we get $\pi^{\otimes n} \circ \delta_n^\alpha \circ \sigma\left(\overline{01}\right) =$

$$\frac{1}{2^{n-1}} \cdot \overline{01}^{\otimes\alpha} \otimes \left(\overline{0} + (-1)^n \cdot \frac{\overline{0} + \overline{1}}{2}\right) \otimes \overline{01}^{\otimes\beta} -$$

$$- \frac{1}{2^{n-1}} \cdot \overline{10}^{\otimes\alpha} \otimes \left(\overline{1} + (-1)^n \cdot \frac{\overline{0} + \overline{1}}{2}\right) \otimes \overline{10}^{\otimes\beta} =$$

$$= \frac{3 + (-1)^n}{2^n} \cdot \overline{01}^{\otimes\alpha} \otimes \left(\,\overline{0} + (-1)^n \cdot \overline{1}\,\right) \otimes \overline{01}^{\otimes\beta} = \frac{3 + (-1)^n}{2^n} \cdot \delta_n^\alpha\left(\overline{01}\right).$$

This agrees with the above claim that $\widetilde{\delta}_1^{\text{bs}} = \delta_1^1 = \overline{\partial} \,:\, \overline{0}1 \longmapsto \overline{0} - \overline{1}$ is the only functorial A-invariant differential and forces $\widetilde{\delta}_2^{\text{bs}}$ to take

$$\overline{01} \longmapsto x \cdot \left(\,\overline{0} + \overline{1}\,\right) \otimes \overline{01} + y \cdot \overline{01} \otimes \left(\,\overline{0} + \overline{1}\,\right)$$

dim $= 1$ case: δ_2^{bs} and recursion

Evaluating $\widetilde{\delta}_2^{\mathrm{bs}} \circ \widetilde{\delta}_1^{\mathrm{bs}} + \left(1 \otimes \widetilde{\delta}_1^{\mathrm{bs}} + \widetilde{\delta}_1^{\mathrm{bs}} \otimes 1\right) \circ \widetilde{\delta}_2^{\mathrm{bs}} = 0$ at $\overline{01}$, we get

$$\begin{aligned} 0 &= \overline{0} \otimes \overline{0} - \overline{1} \otimes \overline{1} - x \cdot (\overline{0} + \overline{1}) \otimes (\overline{0} - \overline{1}) + y \cdot (\overline{0} - \overline{1}) \otimes (\overline{0} + \overline{1}) \\ &= (1 - x + y) \cdot (\overline{0} \otimes \overline{0} - \overline{1} \otimes \overline{1}) + (x + y) \cdot (\overline{1} \otimes \overline{0} - \overline{0} \otimes \overline{1}) \end{aligned}$$

(we used cardinality 1 component $\delta_2^{\mathrm{bs}}(*) = * \otimes *$ fixed before).

Thus, $\quad \widetilde{\delta}_2^{\mathrm{bs}}(\overline{01}) = \dfrac{(\overline{0} + \overline{1}) \otimes \overline{01} - \overline{01} \otimes (\overline{0} + \overline{1})}{2} = -\dfrac{1}{2} \cdot \mathrm{ad}_{\overline{01}}(\overline{0} + \overline{1})$

Since for $n \geqslant 3$ the eigenvalues of A_n never equal 1, all higher components of δ^{bs} are uniquely recovered by means of recursive procedure described above:

$$\widetilde{\delta}_n^{\mathrm{bs}} = (1 - A)^{-1} \cdot B(\widetilde{\delta}_{<n}^{\mathrm{bs}})$$

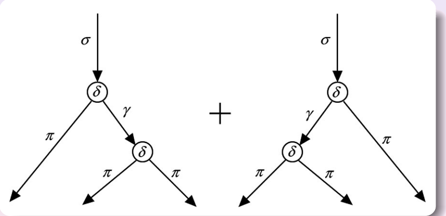
where $B(\widetilde{\delta}_{<n}^{\mathrm{bs}})$ is the sum over oriented planar trees with one root, n leaves, and vertexes of valency $3 \leqslant v \leqslant (n-1)$ decorated by σ on root, π's on leaves, γ's on edges, and $\widetilde{\delta}_\nu^{\mathrm{bs}}$'s on vertexes.

dim $= 1$ case: computing $\widetilde{\delta}_3^{\mathrm{bs}}$

For $n = 3$ there are 2 trees growing from corolla $\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma$, which takes

$$\overline{01} \xmapsto{\sigma} [0,1] - [1,0] \xmapsto{\widetilde{\delta}_2^{\mathrm{bs}}} \frac{\mathrm{ad}_{[1,0]}(\underline{1} + \underline{01}) - \mathrm{ad}_{[0,1]}(\underline{0} + \underline{01})}{2}$$

Homotopy γ annihilates everything except for the only non-saturated flag $\underline{01}$. Thus, we are forced to apply π to all factors $[0,1]$, $[1,0]$, may forget about $\underline{0}$, $\underline{1}$, and should replace $\underline{01}$ by $\widetilde{\delta}_2^{\mathrm{bs}} \circ \gamma(\underline{01})$:



$$\underline{01} \xmapsto{\gamma} \frac{[0,1] + [1,0]}{2} \xmapsto{\widetilde{\delta}_2^{\mathrm{bs}}} -\frac{\mathrm{ad}_{[0,1]}(\underline{0} + \underline{01}) + \mathrm{ad}_{[1,0]}(\underline{1} + \underline{01})}{2}$$

It is productive to think of $\widetilde{\delta}_3^{\mathrm{bs}}$ as composition of two 'propagators'

$$\overline{C} \xrightarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma]} \overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C} \xrightarrow{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \gamma]} \overline{C}^{\otimes 3}$$

The first $\overline{C} \xrightarrow{[\widetilde{\delta}_2^{\text{bs}} \circ \sigma]} \overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C}$ is obtained from $\widetilde{\delta}_2^{\text{bs}} \circ \sigma$ by removing from the result all occurrences of $\underline{0}, \underline{1} \in \ker \gamma$ and replacing all $[0,1]$, $[1,0]$ by $\pi([0,1]) = \overline{01}/2$, $\pi([1,0]) = -\overline{01}/2$. It takes

$$\overline{01} \xmapsto{[\widetilde{\delta}_2^{\text{bs}} \circ \sigma]} \frac{1}{4} \left(\text{ad}_{\overline{10}} (\underline{01}) - \text{ad}_{\overline{01}} (\underline{01}) \right) = -\frac{1}{2} \text{ad}_{\overline{01}} (\underline{01})$$

Then the second $\overline{C} \otimes \underline{C} \oplus \underline{C} \otimes \overline{C} \xrightarrow{[\widetilde{\delta}_2^{\text{bs}} \circ \gamma]} \overline{C}^{\otimes 3}$ replaces each $\underline{01}$ by

$$(\pi \otimes \pi) \circ \widetilde{\delta}_2^{\text{bs}} \circ \gamma (\underline{01}) = -\frac{1}{2} \pi \otimes \pi \left(\text{ad}_{[0,1]} (\underline{0} + \underline{01}) + \text{ad}_{[1,0]} (\underline{1} + \underline{01}) \right)$$
$$= -\frac{1}{8} \text{ad}_{\overline{01}} (\overline{0} - \overline{1})$$

Thus, the sum over trees sends $\overline{01} \longmapsto \text{ad}^2_{\overline{01}} (\overline{0} - \overline{1}) / 16$ and $(1 - A)^{-1}$ multiplies this result by its eigenvalue $\left(1 - \frac{1}{4} \right)^{-1} = \frac{4}{3}$:

$$\widetilde{\delta}_3^{\text{bs}}(\overline{01}) = \frac{1}{12} \cdot \text{ad}^2_{\overline{01}} (\overline{0} - \overline{1})$$

$\dim = 1$ case: closed formula for δ^{bs}

Theorem

For all $n \geqslant 3$ $\quad \widetilde{\delta}_n^{bs}\left(\overline{01}\right) = \dfrac{B_{n-1}}{(n-1)!} \cdot ad_{\overline{01}}^{n-1}\left(\overline{0} - \overline{1}\right) =$

$$= \frac{B_{n-1}}{(n-1)!} \cdot \sum_{\beta=0}^{n-1} (-1)^{\beta} \binom{n-1}{\beta} \cdot \overline{01}^{\otimes(n-1-\beta)} \otimes (\overline{0} - \overline{1}) \otimes \overline{01}^{\otimes\beta}$$

where B_{n-1} is the Bernoulli number and $\quad ad_a : b \mapsto a \otimes b - b \otimes a \quad$ is the commutation operator in the tensor algebra.

Since $B_k = 0$ for all odd $k \geqslant 3$, all the components of even tensor degree do vanish except for

$$\widetilde{\delta}_2^{\text{bs}}(\overline{01}) = B_1 \cdot \text{ad}_{\overline{01}}\left(\overline{0} + \overline{1}\right)$$

All the other components can be combined into one operator

$$\left(1 + \sum_{k \geqslant 2} \frac{B_k}{k!}\, \text{ad}_{\overline{01}}^{k}\right) \circ \overline{\partial}$$

Plan of the proof

Our proof goes by induction over n and consists of two steps:

1. In the recursive formula for $\widetilde{\delta}_n^{\mathrm{bs}}$ non-zero contribution to the sum over trees comes only from one trunk trees with γ's staying along the trunk. We write the contribution of such a tree as composition of propagators $\underline{C} \longrightarrow \underline{C} \cdot \overline{C}^{\otimes k} = \bigoplus_{\mu+\nu=k-1} \overline{C}^{\otimes \mu} \otimes \underline{C} \otimes \overline{C}^{\otimes \nu}$ as we done in the computation of $\widetilde{\delta}_3^{\mathrm{bs}}$ and use the inductive assumptions on $\widetilde{\delta}_{<n}^{\mathrm{bs}}$ to compute this contribution precisely in terms of Bernoulli numbers.
2. Formula obtained on the first step will reproduce the required value for $\widetilde{\delta}_n^{\mathrm{bs}}$, because of following relation on Bernoulli numbers:

$$\frac{B_m}{m!} = \frac{1}{16} \cdot \left(1 - \frac{1}{2^m}\right)^{-1} \cdot \sum_{\substack{m-2= \\ k_1+\cdots+k_i}} \left(\frac{-B_{k_1}}{2^{k_1} k_1!}\right) \cdot \cdots \cdot \left(\frac{-B_{k_i}}{2^{k_i} k_i!}\right) \quad \text{(BN)}$$

where summation runs over all compositions of $(m-2)$ into a sum of numbered positive even integers. We verify (BN) at the last step of proof.

dim = 1 sum over trees: root propagator

Contribution of each tree is a composition of propagators. The first applied to $\overline{01}$ is the root propagator $\overline{C} \xrightarrow{[\widetilde{\delta}_r^{\mathrm{bs}} \circ \sigma]} \underline{C} \cdot \overline{C}^{\otimes(r-1)}$ taking

$$\overline{01} \xmapsto{\widetilde{\delta}_r^{\mathrm{bs}} \circ \sigma} \frac{B_{r-1}}{(r-1)!} \left(\mathrm{ad}_{[0,1]}^{r-1}(\underline{0} - \underline{01}) - \mathrm{ad}_{[1,0]}^{r-1}(\underline{1} - \underline{01}) \right)$$

then applying π to all tensor factors $[0,1]$, $[1,0]$ and reducing the remaining factor modulo $\ker\gamma$. For odd r this gives

$$\mathrm{ad}_{\overline{01}}^{r-1}(\underline{0} - \underline{1}) \equiv 0 \pmod{\ker\gamma}$$

Hence, the root propagator necessary has even tensor degree, that is 2, because of $B_{2k+1} = 0$ for $k \geqslant 1$. Thus, the root propagator always takes

$$\overline{01} \xmapsto{[\widetilde{\delta}_2^{\mathrm{bs}} \circ \sigma]} -\frac{\mathrm{ad}_{\overline{01}}(\underline{01})}{2} \pmod{\ker\gamma}.$$

dim $= 1$ sum over trees: trunk propagators

The root propagator is followed by trunk propagators $\underline{C} \xrightarrow{[\widetilde{\delta}_k^{\mathrm{bs}} \circ \gamma]} \underline{C} \cdot \overline{C}^{\otimes(k-1)}$

taking $\underline{01} \xmapsto{\widetilde{\delta}_k^{\mathrm{bs}} \circ \gamma} \dfrac{B_{k-1}}{(k-1)!} \cdot \dfrac{\mathrm{ad}_{[0,1]}^{k-1}(\underline{0} - \underline{01}) + \mathrm{ad}_{[1,0]}^{k-1}(\underline{1} - \underline{01})}{2}$ and then applying π to all $[0,1]$'s and $[1,0]$'s. This gives

$$\frac{B_{k-1}}{2^k (k-1)!} \cdot \left| \begin{array}{ll} \mathrm{ad}_{\overline{01}}^{k-1}\left(\underline{0} + \underline{1} - 2 \cdot \underline{01}\right) & \text{(for odd } k\text{)} \\ \mathrm{ad}_{\overline{01}}^{k-1}\left(\underline{0} - \underline{1}\right) & \text{(for even } k\text{)} \end{array} \right.$$

Since $\underline{0} - \underline{1} \in \ker \gamma$, each trunk propagator except for the last one has odd tensor degree k and takes $\underline{01} \xmapsto{[\widetilde{\delta}_k^{\mathrm{bs}} \circ \gamma]} \dfrac{-B_{k-1}}{2^{k-1}(k-1)!} \cdot \mathrm{ad}_{\overline{01}}^{k-1}\left(\underline{01}\right)$.

Since $\underline{0} + \underline{1} - 2 \cdot \underline{01} \in \ker \pi$, the last trunk propagator has even tensor degree, that is 2, and takes $\underline{01} \xmapsto{(\pi \otimes \pi) \circ \widetilde{\delta}_2^{\mathrm{bs}} \circ \gamma} -\dfrac{1}{8} \cdot \mathrm{ad}_{\overline{01}}\left(\overline{0} - \overline{1}\right)$.

Inductive step

We conclude that for even $n \geqslant 4$ the sum over threes vanishes and $\delta_n^{\mathrm{bs}} = 0$. For odd n the sum over threes equals

$$\frac{1}{16} \sum_{\substack{n-3= \\ k_1+\cdots+k_i}} \left(\frac{-B_{k_1}}{2^{k_1} k_1!} \right) \cdot \; \cdots \; \cdot \left(\frac{-B_{k_i}}{2^{k_i} k_i!} \right) \cdot \mathrm{ad}_{\overline{01}}^{n-1}(\overline{0} - \overline{1})$$

where the sum runs over all distributions of $n-3$ valences between interior (neither the root nor the last) trunk propagators. Since the eigenvalue of A on this eigenvector is $1/2^{n-1}$, it follows from recursion (BN) that

$$\delta_n^{\mathrm{bs}}\left(\overline{01}\right) = \frac{1}{16} \left(1 - \frac{1}{2^{n-1}} \right)^{-1} \sum_{\substack{n-3= \\ k_1+\cdots+k_i}} (-1)^i \prod_{\nu=1}^{i} \frac{B_{k_\nu}}{2^{k_\nu} k_\nu!} \cdot \mathrm{ad}_{\overline{01}}^{n-1}(\overline{0} - \overline{1}) =$$

$$= \frac{B_{n-1}}{(n-1)!} \cdot \mathrm{ad}_{\overline{01}}^{n-1}\left(\overline{0} - \overline{1}\right)$$

To complete the proof it remains to verify (BN).

Prof of recursion (BN) for Bernoulli numbers

The Bernoulli numbers B_i with $i \geqslant 3$ come from 'cotangensum'

$$(t/2)\cdot \mathrm{cth}(t/2) = 1 + \sum_{k\geqslant 3} (B_k/k!)\cdot t^k\ .$$

Obvious relation $\mathrm{cth}(t) = \frac{1}{2}\left(\mathrm{cth}(t/2) + \mathrm{th}(t/2)\right)$ implies the identity

$$t\cdot \mathrm{cth}(t) - (t/2)\cdot \mathrm{cth}(t/2) = (t^2/4)\cdot \left(\,(t/2)\cdot \mathrm{cth}(t/2)\,\right)^{-1}\ .$$

Expanding $(1+\sum (B_k/k!)t^k)^{-1}$ as geometric progression and comparing coefficients at t^m, we get recursive formula

$$(2^m-1)\cdot \frac{B_m}{m!} = \frac{1}{4}\sum_{\substack{m-2=\\ k_1+\dots+k_i}} (-1)^i \prod_{\nu=1}^{i} \frac{B_{k_\nu}}{k_\nu!}\,,$$

It remains to multiply both sides by

$$(2^m-1)^{-1} = \left(1-\frac{1}{2^m}\right)^{-1}\cdot \frac{1}{2^{k_1}}\cdot\ \cdots\ \cdot \frac{1}{2^{k_i}}\cdot \frac{1}{4}$$

This completes the proof of the theorem.

Open questions

Question 1

Compute eigenvectors and eigenvalues of linear operator

$$A_n \;:\; \widetilde{\delta}_n^{[M]} \mapsto \pi^{\otimes n} \circ \widetilde{\delta}_n^{B([M])} \circ \sigma$$

acting on functorial in M tensors $\overline{C}^{[M]} \xrightarrow{\widetilde{\delta}_n^{[M]}} \overline{C}^{[M]^{\otimes n}}$ of combinatorial degree $2n-3$.

We expect elegant generating series (over n) for such eigentensors. Conjecturally, they should be closely connected with quasi-symmetric functions and Malvenuto–Reutenauer Hopf algebra of permutations as well as with its partner — non-commutative symmetric functions investigated extensively by Gelfand, Lascoux, Retakh, and others.

Open questions

Question 2

Find closed formula for the barycentrically stable functorial A_∞-coproduct

$$\delta^{\mathrm{bs}} : \overline{C}[2] \longrightarrow \mathrm{T}(\overline{C}[2])$$

in all higher dimensions.

It follows from general Koszul duality for operads that the image of δ^{bs} lies in subalgera of Lie power series. Thus, we expect close connections between δ_n^{bs}'s and projectors onto the subspaces of Lie polynomials. A closed formula for δ^{bs} costs, probably, the same prise as the Kampbell–Hausdorf formula.

Appendix: what does sum over trees formula come from

It comes in two steps. At first, each SDR-data $\gamma \looparrowright V \underset{\sigma}{\overset{\pi}{\rightleftarrows}} W$ canonically provide us with SDR-data $\gamma_{\mathrm{T}} \looparrowright \mathrm{T}(V) \underset{\sigma_{\mathrm{T}}}{\overset{\pi_{\mathrm{T}}}{\rightleftarrows}} \mathrm{T}(W)$, where

- $\mathrm{T}(V)$, $\mathrm{T}(W)$ are equipped with differentials D_{∂_V}, D_{∂_W} extending ∂_V, ∂_W by the Leibnitz rule;
- π_{T}, σ_{T} extend π, σ to the homomorphisms of tensor algebras;
- $\mathrm{T}(V) \xrightarrow{\gamma_{\mathrm{T}}} \mathrm{T}(W)$ extends $V \xrightarrow{\gamma} V$ to the whole of $\mathrm{T}(V)$ by the twisted Leibnitz rule $\quad \gamma_{\mathrm{T}} \circ \mu = \mu \circ ((f-g) \otimes \gamma_{\mathrm{T}} + \gamma_{\mathrm{T}} \otimes 1)$

(SDR-data relations are easily verified, e.g. see Eilenberg–MacLane).

Further, each operator $\mathrm{T}(V) \xrightarrow{D} \mathrm{T}(V)$ with $D^2 = 0$ can be considered as perturbation of the differential D_{∂_V}. It can be extended canonically to the perturbation of the whole SDR-data by precise formulas going back to Kadeishvili. This is the second step.

Differential perturbation (a version of)

Let SDR-data $\gamma \circlearrowright V \underset{\sigma}{\overset{\pi}{\rightleftarrows}} W$ and a map $V \xrightarrow{\varepsilon} V$ satisfy two conditions:

1. perturbed differential $\partial'_V \overset{\text{def}}{=} \partial_V + \varepsilon$ is again a differential on V (that is, satisfies ${\partial'_V}^2 = 0$)
2. well defined endomorphisms of V are provided by power series

$$(1-\gamma\varepsilon)^{-1} \overset{\text{def}}{=} 1 + \sum_{m\geqslant 1} (\gamma\varepsilon)^m\,, \quad (1-\varepsilon\gamma)^{-1} \overset{\text{def}}{=} 1 + \sum_{m\geqslant 1} (\varepsilon\gamma)^m\,.$$

Remark

Technical condition (2) holds when $\gamma\varepsilon$, $\varepsilon\gamma$ are locally nilpotent (say, by reasons of grading — this is the case we deal with). More generally, it holds when $\text{End}(E)$ is complete in some norm such that $||fg|| \leqslant ||f|| \cdot ||g||$ and $||\varepsilon|| \ll 1$.

Then the perturbation $\partial_V \mapsto \partial'_V = \partial_V + \varepsilon$ is extended canonically to the perturbation of the whole SDR-data

$$\gamma \circlearrowright (V, \partial_V) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (W, \partial)$$

$$\downarrow$$

$$\gamma' \circlearrowright (V, \partial'_V) \underset{\sigma'}{\overset{\pi'}{\rightleftarrows}} (W, \partial'_W)$$

where $\partial'_W = \partial_W + \varepsilon_{\text{ind}}$ and

$$\begin{aligned}
\sigma' &= (1 - \gamma\varepsilon)^{-1}\sigma = \sigma + \gamma\varepsilon\sigma + \gamma\varepsilon\gamma\varepsilon\sigma + \gamma\varepsilon\gamma\varepsilon\gamma\varepsilon\sigma + \ \cdots \ , \\
\pi' &= \pi(1 - \varepsilon\gamma)^{-1} = \pi + \pi\varepsilon\gamma + \pi\varepsilon\gamma\varepsilon\gamma + \pi\varepsilon\gamma\varepsilon\gamma\varepsilon\gamma + \ \cdots \ , \\
\gamma' &= (1 - \gamma\varepsilon)^{-1}\gamma = \gamma(1 - \varepsilon\gamma)^{-1} = \gamma + \gamma\varepsilon\gamma + \gamma\varepsilon\gamma\varepsilon\gamma + \ \cdots \ , \\
\varepsilon_{\text{ind}} &= \pi\varepsilon\sigma' = \pi'\varepsilon\sigma = \pi\varepsilon\sigma + \pi\varepsilon\gamma\varepsilon\sigma + \pi\varepsilon\gamma\varepsilon\gamma\varepsilon\sigma + \ \cdots \ .
\end{aligned}$$

Checks of all SDR-data relations are straightforward as well.

Proof of 'sum over trees' formula

Assume we are given with SDR-data $\gamma \looparrowright (V,\partial_V) \underset{\sigma}{\overset{\pi}{\rightleftarrows}} (W,\partial)$. Then each A_∞-coproduct $T(V[1]) \xrightarrow{D_\delta} T(V[1])$, whose linear component coincides with the differential $\delta_1 : V[1] \xrightarrow{-\partial_V} V[1]$, perturbs differential $\partial_{T(V[1])}$ in SDR-data

$$\gamma_T[1] \looparrowright (T(V[1],\partial_{T(V[1])}) \underset{\sigma_T[1]}{\overset{\pi_T[1]}{\rightleftarrows}} (T(W[1],\partial_{T(W[1])}) \qquad (T)$$

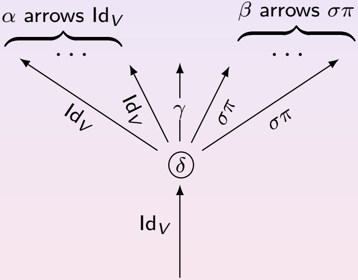
induced by the given in accordance with step 1 and shifted by 1. Write this perturbation as $D = \partial_{T(V[1])} + D_\delta = D_{\partial_{V[1]}+\delta}$ and extend by step 2 to the perturbation of the whole of SDR-data (T). The resulting perturbed data

$$\gamma'_T \looparrowright (T(V[1],D) \underset{\sigma'}{\overset{\pi'}{\rightleftarrows}} (T(W[1],D_{\delta_{ind}})$$

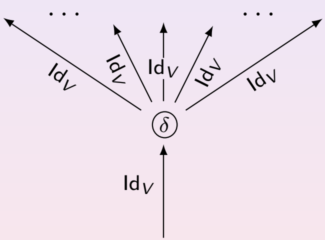
contain the required A_∞-coproduct $D_{\delta_{ind}} = D_{\partial_{T(W[1])}} + D_{\delta_{ind}} = D_{\partial_{W[1]}+\delta_{ind}}$ associated with map $\delta_{ind} = \partial_{W[1]} + \sum_{n\geqslant 2} \delta_{ind,n}$.

It has $\widetilde{\delta}_{\text{ind},n} = \pi^{\otimes n} \circ (\, \delta + \delta\gamma_{\mathsf{T}}\delta + \delta\gamma_{\mathsf{T}}\delta\gamma_{\mathsf{T}}\delta + \delta\gamma_{\mathsf{T}}\delta\gamma_{\mathsf{T}}\delta\gamma_{\mathsf{T}}\delta + \cdots \,)_{n,1} \circ \sigma$, where $(*)_{n,1} : V \longrightarrow V^{\otimes n}$ means those component of bracketed operator that sends $V \subset \mathsf{T}(V)$ to $V^{\otimes n} \subset \mathsf{T}(V)$. It equals the sum of all oriented trees with one input and n outputs composed from corollas

$(\, 1^{\otimes\alpha} \otimes \gamma \otimes (\sigma\pi)^{\otimes\beta}\,) \circ \delta_{\alpha+\beta+1}$ from $\gamma_{\mathsf{T}}\delta$:



δ_k from δ :



with the right ones allowed only as the last elements of the composition. Since total number of γ's in $\delta\gamma_{\mathsf{T}}\delta \ldots \gamma_{\mathsf{T}}\delta$ equals the number of internal edges in the corresponding trees and outgoing γ's are killed by the final $\pi^{\otimes n}$ applied to all leaves, we get what we want.

FIN:

THANKS FOR YOUR ATTENTION!