Bergman kernel and Geometric quantization (joint with Weiping Zhang)

Xiaonan Ma

Université Paris 7

University of Lexembourg, Sepember 7, 2009

Pre-quantum line bundle

Pre-quantum line bundle

- (X, ω) a compact symplectic manifold.

Pre-quantum line bundle

- (X, ω) a compact symplectic manifold.
- (L, h^L) a Hermitian line bundle over X carrying an Hermitian connection ∇^L such that

$$\frac{\sqrt{-1}}{2\pi}\left(\nabla^L\right)^2 = \omega.$$

L the pre-quantum line bundle on (X, ω).

Dirac operators

Dirac operators

- J an almost complex structure on TX such that

$$g^{TX}(v,w) = \omega(v, Jw)$$

defines a J-invariant Riemmannian metric on TX.

Dirac operators

- J an almost complex structure on TX such that

$$g^{TX}(v,w) = \omega(v, Jw)$$

defines a J-invariant Riemmannian metric on TX.

- spin^c Dirac operator

$$D^L : \Omega^{0,\frac{\text{even}}{\text{odd}}}(X,L) \longrightarrow \Omega^{0,\frac{\text{odd}}{\text{even}}}(X,L)$$

Self-adjoint first order elliptic operator.

Dirac operators

- J an almost complex structure on TX such that
$$g^{TX}(v, w) = \omega(v, Jw)$$
defines a J-invariant Riemmannian metric on TX.
- spin^c Dirac operator
$$D^L : \Omega^{0,\frac{\text{even}}{\text{odd}}}(X, L) \longrightarrow \Omega^{0,\frac{\text{odd}}{\text{even}}}(X, L)$$
Self-adjoint first order elliptic operator.
- $D^L_{\pm} := D^L|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(X,L)}$.

Dirac operators

- J an almost complex structure on TX such that
$$g^{TX}(v,w)=\omega(v,Jw)$$
defines a J-invariant Riemmannian metric on TX.
- spin^c Dirac operator
$$D^L:\Omega^{0,\frac{\text{even}}{\text{odd}}}(X,L)\longrightarrow\Omega^{0,\frac{\text{odd}}{\text{even}}}(X,L)$$
Self-adjoint first order elliptic operator.
- $D^L_\pm := D^L|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}(X,L)}$.
- When (X,ω,J) is Kähler, and L holomorphic line bundle over X,
$$D^L=\sqrt{2}\left(\overline{\partial}^L+\left(\overline{\partial}^L\right)^*\right).$$

Index of D^L

- $\operatorname{Ker}\left(D_+^L\right)$, $\operatorname{Ker}\left(D_-^L\right)$ are finite dimensional.

Index of D^L

- $\operatorname{Ker}\left(D^L_+\right)$, $\operatorname{Ker}\left(D^L_-\right)$ are finite dimensional.
- Quantization space of L is the formal difference

$$Q(L) := \operatorname{Ind}(D^L) = \operatorname{Ker}\left(D^L_+\right) - \operatorname{Ker}\left(D^L_-\right).$$

It does not depend on the choice of J and the metric and connection on L.

Index of D^L

- $\operatorname{Ker}\left(D^L_+\right)$, $\operatorname{Ker}\left(D^L_-\right)$ are finite dimensional.
- Quantization space of L is the formal difference

$$Q(L) := \operatorname{Ind}(D^L) = \operatorname{Ker}\left(D^L_+\right) - \operatorname{Ker}\left(D^L_-\right).$$

  It does not depend on the choice of J and the metric and connection on L.
- When (X, ω, J) is Kähler and L is holomorphic, then

$$Q(L) = H^{0,\text{even}}(X, L) - H^{0,\text{odd}}(X, L).$$

Index of D^L

- $\mathrm{Ker}\left(D_+^L\right)$, $\mathrm{Ker}\left(D_-^L\right)$ are finite dimensional.
- Quantization space of L is the formal difference

$$Q(L) := \mathrm{Ind}(D^L) = \mathrm{Ker}\left(D_+^L\right) - \mathrm{Ker}\left(D_-^L\right).$$

  It does not depend on the choice of J and the metric and connection on L.
- When (X, ω, J) is Kähler and L is holomorphic, then

$$Q(L) = H^{0,\mathrm{even}}(X, L) - H^{0,\mathrm{odd}}(X, L).$$

- Atiyah-Singer : $Q(L) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(L)$

$$= \int_X \det\left(\frac{e^{\sqrt{-1}R^{T^{(1,0)}X}/2\pi}}{1 - e^{-\sqrt{-1}R^{T^{(1,0)}X}/2\pi}}\right) e^{\omega}.$$

Index as a virtual representation

- G compact connected Lie group with Lie algebra $\mathfrak{g}$.

Index as a virtual representation

- G compact connected Lie group with Lie algebra $\mathfrak{g}$.
- G acts on X, and its action lifts on L, and commutes with J, h^L, ∇^L.

Index as a virtual representation

- G compact connected Lie group with Lie algebra $\mathfrak{g}$.
- G acts on X, and its action lifts on L, and commutes with J, h^L, ∇^L.
- $\mathrm{Ker}\left(D^L_{\pm}\right)$ are finite dimensional G-representations.

$$Q(L) = \mathrm{Ker}\left(D^L_{+}\right) - \mathrm{Ker}\left(D^L_{-}\right) \in R(G).$$

Index as a virtual representation

- G compact connected Lie group with Lie algebra $\mathfrak{g}$.
- G acts on X, and its action lifts on L, and commutes with J, h^L, ∇^L.
- $\mathrm{Ker}\left(D^L_\pm\right)$ are finite dimensional G-representations.

$$Q(L) = \mathrm{Ker}\left(D^L_+\right) - \mathrm{Ker}\left(D^L_-\right) \in R(G).$$

- $\Lambda^*_+ \subset \mathfrak{g}^*$ the set of dominant weights, V^G_γ the irreducible representation of G with highest weight $\gamma \in \Lambda^*_+$. Then

$$Q(L) = \bigoplus_{\gamma \in \Lambda^*_+} Q(L)^\gamma \cdot V^G_\gamma.$$

Index as a virtual representation

- G compact connected Lie group with Lie algebra $\mathfrak{g}$.
- G acts on X, and its action lifts on L, and commutes with J, h^L, ∇^L.
- Ker $\left(D^L_\pm\right)$ are finite dimensional G-representations.

$$Q(L) = \operatorname{Ker}\left(D^L_+\right) - \operatorname{Ker}\left(D^L_-\right) \in R(G).$$

- $\Lambda^*_+ \subset \mathfrak{g}^*$ the set of dominant weights, V^G_γ the irreducible representation of G with highest weight $\gamma \in \Lambda^*_+$. Then

$$Q(L) = \bigoplus_{\gamma\in\Lambda^*_+} Q(L)^\gamma \cdot V^G_\gamma.$$

- $Q(L)^\gamma$ the multiplicity of V^G_γ in $Q(L)$.
  How to compute $Q(L)^\gamma$?

Symplectic reduction

- Moment map $\mu : X \to \mathbf{g}^*$ is defined by

$$2\sqrt{-1}\pi\mu(K) = \nabla^L_{K^X} - L_K, \quad K \in \mathfrak{g}.$$

K^X the vector field on X generated by $K \in \mathfrak{g}$.

Symplectic reduction

- Moment map $\mu : X \to \mathbf{g}^*$ is defined by

$$2\sqrt{-1}\pi\mu(K) = \nabla^L_{K^X} - L_K, \quad K \in \mathfrak{g}.$$

  K^X the vector field on X generated by $K \in \mathfrak{g}$.

- One has

$$i_{K^X}\omega = d\,\mu(K),$$

Symplectic reduction

- Moment map $\mu : X \to \mathfrak{g}^*$ is defined by

$$2\sqrt{-1}\pi\mu(K) = \nabla^L_{K^X} - L_K, \quad K \in \mathfrak{g}.$$

K^X the vector field on X generated by $K \in \mathfrak{g}$.

- One has

$$i_{K^X}\omega = d\,\mu(K),$$

- For a regular value $\nu \in \mathfrak{g}^*$ of μ, symplectic reduction :

$$X_\nu = \mu^{-1}(G\cdot\nu)/G$$

X_ν is a compact symplectic orbifold.
$J,\ \omega,\ L \Longrightarrow J_\nu, \omega_\nu, L_\nu$ on X_ν.

Guillemin-Sternberg conjecture I

Guillemin-Sternberg conjecture I

- If (X, ω, J) is Kähler, L is holomorphic, $\Longrightarrow$ $(X_\nu, \omega_\nu, J_\nu)$ is Kähler, L_ν is holomorphic over X_ν.

Guillemin-Sternberg conjecture I

- If (X,ω,J) is Kähler, L is holomorphic, $\Longrightarrow$ (X_ν,ω_ν,J_ν) is Kähler, L_ν is holomorphic over X_ν.
- The quantization of L_ν is defined by

$$Q\left(L_\nu\right)=\operatorname{Ker}\left(D_+^{L_\nu}\right)-\operatorname{Ker}\left(D_-^{L_\nu}\right).$$

Guillemin-Sternberg conjecture I

- If (X, ω, J) is Kähler, L is holomorphic, $\Longrightarrow$ $(X_\nu, \omega_\nu, J_\nu)$ is Kähler, L_ν is holomorphic over X_ν.
- The quantization of L_ν is defined by

$$Q\left(L_\nu\right) = \operatorname{Ker}\left(D_+^{L_\nu}\right) - \operatorname{Ker}\left(D_-^{L_\nu}\right).$$

- Guillemin-Sternberg conjecture : For any $\gamma \in \Lambda_+^*$,

$$Q(L)^\gamma = Q\left(L_\gamma\right).$$

Equivalently,

$$Q(L) := \operatorname{Ind}(D^L) = \bigoplus_{\gamma \in \Lambda_+^*} Q\left(L_\gamma\right) \cdot V_\gamma^G.$$

Guillemin-Sternberg conjecture II

- When G is abelian, Meinrenken (JAMS 1996) and Vergne (DMJ 1996).

Guillemin-Sternberg conjecture II

- When G is abelian, Meinrenken (JAMS 1996) and Vergne (DMJ 1996).
- General G, Meinrenken, Meinrenken-Sjamaar, technique of *symplectic cut* of Lerman, 1998
  Yonglian Tian - Weiping Zhang,
  Pure analytic approach, 1998, work for a general vector bundle E verifying certain positivity condition.
  For manifolds with boundary, etc.

Guillemin-Sternberg conjecture II

- When G is abelian, Meinrenken (JAMS 1996) and Vergne (DMJ 1996).
- General G, Meinrenken, Meinrenken-Sjamaar, technique of *symplectic cut* of Lerman, 1998
  Yonglian Tian - Weiping Zhang,
  Pure analytic approach, 1998, work for a general vector bundle E verifying certain positivity condition. For manifolds with boundary, etc.
- Other proofs : Duistermaart-Guillemin-Meinrenken-Wu (for circle actions) and Jeffrey-Kirwan (for non-abelian group actions with certain extra conditions)
  Paradan, using the transversal index theory, 2001.
  Etc ...

Guillemin-Sternberg conjecture III

- Assume (X, ω, J) is Kähler, L holomorphic.

Guillemin-Sternberg conjecture III

- Assume (X, ω, J) is Kähler, L holomorphic.
- Guillemin-Sternberg, 1982 : G acts freely on $\mu^{-1}(0)$, $X_G := X_0$.

$$H^{0,0}(X,L)^G \simeq H^{0,0}(X_G, L_G).$$

Guillemin-Sternberg conjecture III

- Assume (X, ω, J) is Kähler, L holomorphic.
- Guillemin-Sternberg, 1982 : G acts freely on $\mu^{-1}(0)$, $X_G := X_0$.

$$H^{0,0}(X,L)^G \simeq H^{0,0}(X_G, L_G).$$

- Teleman, Braverman, Weiping Zhang, 2000 : for any j,

$$H^{0,j}(X,L)^G \simeq H^{0,j}(X_G, L_G).$$

Guillemin-Sternberg conjecture III

- Assume (X, ω, J) is Kähler, L holomorphic.
- Guillemin-Sternberg, 1982 : G acts freely on $\mu^{-1}(0)$, $X_G := X_0$.

$$H^{0,0}(X, L)^G \simeq H^{0,0}(X_G, L_G).$$

- Teleman, Braverman, Weiping Zhang, 2000 : for any j,

$$H^{0,j}(X, L)^G \simeq H^{0,j}(X_G, L_G).$$

- Weiping Zhang, 2000 : For any E holomorphic, if 0 is a regular values of μ,

$$H^{0,0}(X, L^k \otimes E)^G \simeq H^{0,0}(X_G, L_G^k \otimes E_G) \text{ for } k \gg 1.$$

Transversal index

- Assume X is non-compact and $\mu : X \to \mathfrak{g}^*$ is proper.

Transversal index

- Assume X is non-compact and $\mu : X \to \mathfrak{g}^*$ is proper.
- We can define a transversally elliptic symbol $\sigma^X_{L,\mu}$ in the sense of Atiyah (1974).
  $\sigma^M_{L,\mu}$ has an index :

$$\mathrm{Ind}\left(\sigma^X_{L,\mu}\right) = \bigoplus_{\gamma \in \Lambda^*_+} \mathrm{Ind}_\gamma\left(\sigma^X_{L,\mu}\right) \cdot V^G_\gamma.$$

  The set $\{\gamma \in \Lambda^*_+ : \mathrm{Ind}_\gamma\left(\sigma^X_{L,\mu}\right) \neq 0\}$ can be infinite.

Transversal index

- Assume X is non-compact and $\mu : X \to \mathfrak{g}^*$ is proper.
- We can define a transversally elliptic symbol $\sigma^X_{L,\mu}$ in the sense of Atiyah (1974).
  $\sigma^M_{L,\mu}$ has an index :

  $$\mathrm{Ind}\left(\sigma^X_{L,\mu}\right) = \bigoplus_{\gamma\in\Lambda^*_+} \mathrm{Ind}_\gamma\left(\sigma^X_{L,\mu}\right)\cdot V^G_\gamma .$$

  The set $\{\gamma \in \Lambda^*_+ : \mathrm{Ind}_\gamma\left(\sigma^X_{L,\mu}\right) \neq 0\}$ can be infinite.

Transversal index

- Assume X is non-compact and $\mu : X \to \mathfrak{g}^*$ is proper.
- We can define a transversally elliptic symbol $\sigma^X_{L,\mu}$ in the sense of Atiyah (1974).
  $\sigma^M_{L,\mu}$ has an index :

$$\operatorname{Ind}\left(\sigma^X_{L,\mu}\right) = \bigoplus_{\gamma\in\Lambda^*_+} \operatorname{Ind}_\gamma\left(\sigma^X_{L,\mu}\right) \cdot V^G_\gamma.$$

  The set $\{\gamma \in \Lambda^*_+ : \operatorname{Ind}_\gamma\left(\sigma^X_{L,\mu}\right) \neq 0\}$ can be infinite.
- In many cases,

$$\operatorname{Ind}\left(\sigma^X_{L,\mu}\right) = \operatorname{Ker}_{L^2}(D^L_+) - \operatorname{Ker}_{L^2}(D^L_-) \in R[G].$$

Vergne's conjecture

Vergne's conjecture

- Vergne's conjecture (ICM 2006, plenary lecture) : If $\mu : X \to \mathfrak{g}^*$ is proper and if $\{x \in X : \mu^X(x) = 0\}$ is compact, then

$$\mathrm{Ind}\left(\sigma^X_{L,\mu}\right) = \bigoplus_{\gamma \in \Lambda^*_+} Q\left(L_\gamma\right) \cdot V^G_\gamma .$$

Vergne's conjecture

- Vergne's conjecture (ICM 2006, plenary lecture) : If $\mu : X \to \mathfrak{g}^*$ is proper and if $\{x \in X : \mu^X(x) = 0\}$ is compact, then

$$\mathrm{Ind}\left(\sigma^X_{L,\mu}\right) = \bigoplus_{\gamma \in \Lambda^*_+} Q\left(L_\gamma\right) \cdot V^G_\gamma .$$

- Ma-Zhang 2008 : Vergne's conjecture holds even when $\{x \in X : \mu^X(x) = 0\}$ is non-compact.

Motivations and results

Motivations and results

- Results : Asymptotic expansion of G-invariant Bergman kernel on a compact symplectic manifold equipped with Hamiltonian action of a compact connected Lie group.

Motivations and results

- Results : Asymptotic expansion of G-invariant Bergman kernel on a compact symplectic manifold equipped with Hamiltonian action of a compact connected Lie group.
- Motivations :
  Version from the local index theory for Geometric quantization commutes with reduction ?

Motivations and results

- Results : Asymptotic expansion of G-invariant Bergman kernel on a compact symplectic manifold equipped with Hamiltonian action of a compact connected Lie group.
- Motivations :
  Version from the local index theory for Geometric quantization commutes with reduction ?
- Applications in Donaldson's program ?

Geometric data

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.
- ∇^L, ∇^E Hermitian connections on (L, h^L), (E, h^E).

Geometric data

- (X,ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.
- ∇^L, ∇^E Hermitian connections on (L, h^L), (E, h^E).
- Fundamental hypothesis :

$$\omega = c_1(L, \nabla^L) = \frac{\sqrt{-1}}{2\pi}(\nabla^L)^2.$$

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.
- ∇^L, ∇^E Hermitian connections on (L, h^L), (E, h^E).
- Fundamental hypothesis :

$$\omega = c_1(L, \nabla^L) = \frac{\sqrt{-1}}{2\pi}(\nabla^L)^2.$$

- g^{TX} Riemannian metric on TX, J almost complex structure on TX, preserves ω and g^{TX}.

Geometric data

- (X, ω) compact symplectic manifold, $\dim_{\mathbb{R}} X = 2n$.
- (L, h^L) Hermitian (complex) line bundle on X.
- (E, h^E) Hermitian (complex) vector bundle on X.
- ∇^L, ∇^E Hermitian connections on (L, h^L), (E, h^E).
- Fundamental hypothesis :

$$\omega = c_1(L, \nabla^L) = \frac{\sqrt{-1}}{2\pi}(\nabla^L)^2.$$

- g^{TX} Riemannian metric on TX, J almost complex structure on TX, preserves ω and g^{TX}.
- Simplification : $g^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$.
(Our results work without this assumption).

G-invariant Bergman kernel

- Spinors : $\Lambda^{\cdot} := \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^{\pm}) \longrightarrow \mathscr{C}^\infty(X, E_p^{\mp})$,

$$D_p = \sum_i c(e_i) \nabla^{E_p}_{e_i}.$$

G-invariant Bergman kernel

- Spinors : $\Lambda^{\cdot} := \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^{\pm}) \longrightarrow \mathscr{C}^\infty(X, E_p^{\mp})$,
  $$D_p = \sum_i c(e_i) \nabla^{E_p}_{e_i}.$$
- P_p^G orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $(\operatorname{Ker} D_p)^G$.

G-invariant Bergman kernel

- Spinors : $\Lambda^{\cdot} := \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^{\pm}) \longrightarrow \mathscr{C}^\infty(X, E_p^{\mp})$,
  $$D_p = \sum_i c(e_i) \nabla^{E_p}_{e_i}.$$
- P_p^G orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $(\operatorname{Ker} D_p)^G$.
- G-invariant Bergman kernel : $P_p^G(x, x')$, $(x, x' \in X)$ is the $\mathscr{C}^\infty$ kernel of P_p^G associated to $\frac{\omega^n}{n!}(x')$.

G-invariant Bergman kernel

- Spinors : $\Lambda^{\cdot} := \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^{\pm}) \longrightarrow \mathscr{C}^\infty(X, E_p^{\mp})$,

$$D_p = \sum_i c(e_i) \nabla^{E_p}_{e_i}.$$

- P_p^G orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $(\operatorname{Ker} D_p)^G$.
- G-invariant Bergman kernel : $P_p^G(x, x')$, $(x, x' \in X)$ is the $\mathscr{C}^\infty$ kernel of P_p^G associated to $\frac{\omega^n}{n!}(x')$.
- $P_p^G(x, x')$ "concentrates" to $P_{G,p}(x_0, x_0')$, the Bergman kernel on X_G when $p \to \infty$?

G-invariant Bergman kernel

- Spinors : $\Lambda^{\cdot} := \Lambda^{\text{even}}(T^{*(0,1)}X) \oplus \Lambda^{\text{odd}}(T^{*(0,1)}X)$.
  $E_p := \Lambda^{\cdot} \otimes L^p \otimes E$.
  Dirac operator $D_p : \mathscr{C}^\infty(X, E_p^{\pm}) \longrightarrow \mathscr{C}^\infty(X, E_p^{\mp})$,
  $$D_p = \sum_i c(e_i) \nabla^{E_p}_{e_i}.$$
- P_p^G orthogonal projection from $\mathscr{C}^\infty(X, E_p)$ onto $(\operatorname{Ker} D_p)^G$.
- G-invariant Bergman kernel : $P_p^G(x, x')$, $(x, x' \in X)$ is the $\mathscr{C}^\infty$ kernel of P_p^G associated to $\frac{\omega^n}{n!}(x')$.
- $P_p^G(x, x')$ "concentrates" to $P_{G,p}(x_0, x_0')$, the Bergman kernel on X_G when $p \to \infty$?
- Scalar curvature of X_G from $P_p^G(x, x')$?

Application of Dai-Liu-Ma's formula

- $$\left|\frac{1}{p^n}P_p(0,Z)-\sum_{r=0}^{k}p^{-r/2}J_r(\sqrt{p}Z)e^{-\frac{\pi}{2}p|Z|^2}\right| \leq Cp^{-(k+1)/2}(1+|\sqrt{p}Z|)^N e^{-\sqrt{C''p}|Z|}.$$

$J_0 = I_{\mathbb{C}\otimes E}$, projection from $\Lambda^{\cdot}(T^{*(0,1)}X)\otimes E$ to $\mathbb{C}\otimes E$.

Application of Dai-Liu-Ma's formula

- $$\left|\frac{1}{p^n}P_p(0,Z) - \sum_{r=0}^{k} p^{-r/2} J_r(\sqrt{p}Z) e^{-\frac{\pi}{2}p|Z|^2}\right| \leq Cp^{-(k+1)/2}(1+|\sqrt{p}Z|)^N e^{-\sqrt{C''p}|Z|}.$$

  $J_0 = I_{\mathbb{C}\otimes E}$, projection from $\Lambda^{\cdot}(T^{*(0,1)}X)\otimes E$ to $\mathbb{C}\otimes E$.
- $P_p^G(x,x') = \int_G (1,g)\cdot P_p(x, g^{-1}x')dg.$

Application of Dai-Liu-Ma's formula

- $\left|\frac{1}{p^n}P_p(0,Z) - \sum_{r=0}^{k} p^{-r/2} J_r(\sqrt{p}Z)e^{-\frac{\pi}{2}p|Z|^2}\right|$

$$\leq Cp^{-(k+1)/2}(1+|\sqrt{p}Z|)^N e^{-\sqrt{C''p}|Z|}.$$

  $J_0 = I_{\mathbb{C}\otimes E}$, projection from $\Lambda^{\cdot}(T^{*(0,1)}X)\otimes E$ to $\mathbb{C}\otimes E$.
- $P_p^G(x,x') = \int_G (1,g)\cdot P_p(x,g^{-1}x')dg$.
- For $x_0 \in \mu^{-1}(0)$, $h(x_0) := \sqrt{\mathrm{vol}(Gx_0)}$

$$\left|p^{-n+\frac{\dim G}{2}} h^2(x_0) P_p^G(x_0,x_0) - \sum_{r=0}^{k} \mathbf{b}_r(x_0)p^{-r}\right| \leq Cp^{-k-1}.$$

$\mathbf{b}_0 = 2^{\frac{\dim G}{2}} I_{\mathbb{C}\otimes E}$.
Paoletti (2005) Adv. Math. : If (X,J,ω) Kähler,
$E=\mathbb{C} \Rightarrow \mathbf{b}_0 = 1$. Different ! Wrong ?

Asymp. expansion of G-invariant Bergman kernel

Asymp. expansion of G-invariant Bergman kernel

- Theorem (Ma - Weiping Zhang (2005)). U open neighborhood of $\mu^{-1}(0)$,

$$P_p^G(x,x') = \mathscr{O}(p^{-\infty}) \quad \text{for any } x \text{ or } x' \in X \setminus U.$$

Asymp. expansion of G-invariant Bergman kernel

- Theorem (Ma - Weiping Zhang (2005)). U open neighborhood of $\mu^{-1}(0)$,

$$P_p^G(x,x') = \mathscr{O}(p^{-\infty}) \quad \text{for any } x \text{ or } x' \in X \setminus U.$$

- Theorem (Ma-Zhang (2005)). For $x \in U$, $h(x) := \sqrt{\mathrm{vol}(Gx)}$. Asymptotic expansion of

$$p^{-n+\frac{\dim G}{2}} h(x)h(x') P_p^G(x,x') \text{ uniformly on } x, x' \in U.$$

Asymp. expansion II

Asymp. expansion II

- $x_0 \in X_G$, $B = U/G$, $Z = (Z^0, Z^\perp) \in TX_G \oplus N_B = TB$,

$$P(Z, Z') = \exp\Big(-\frac{\pi}{2}|Z^0 - Z'^0|^2 - \pi\sqrt{-1}\left\langle J_{x_0} Z^0, Z'^0 \right\rangle \Big) \times 2^{\frac{n_0}{2}} \exp\Big(-\pi\big(|Z^\perp|^2 + |Z'^\perp|^2\big)\Big),$$

Asymp. expansion II

- $x_0 \in X_G$, $B = U/G$, $Z = (Z^0, Z^\perp) \in TX_G \oplus N_B = TB$,

$$P(Z, Z') = \exp\Big(-\frac{\pi}{2}|Z^0 - Z'^0|^2 - \pi\sqrt{-1}\,\langle J_{x_0} Z^0, Z'^0\rangle \Big) \times 2^{\frac{n_0}{2}} \exp\Big(-\pi\big(|Z^\perp|^2 + |Z'^\perp|^2\big)\Big),$$

- (Ma-Zhang, CRAS, 2005) $\mathcal{Q}_r$ polynomial on Z, Z', $\mathcal{Q}_0 = I_{\mathbb{C}\otimes E_B}$,

$$p^{-n+\frac{n_0}{2}} h(Z)h(Z')P_p^G(Z, Z') \approx \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} P)(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} + \mathcal{O}(p^{(k+1)/2}).$$

Asymp. expansion II

- $x_0 \in X_G$, $B = U/G$, $Z = (Z^0, Z^\perp) \in TX_G \oplus N_B = TB$,
$$P(Z,Z') = \exp\Big(-\frac{\pi}{2}|Z^0 - Z'^0|^2 - \pi\sqrt{-1}\,\big\langle J_{x_0} Z^0, Z'^0\big\rangle\Big) \\ \times 2^{\frac{n_0}{2}} \exp\Big(-\pi\big(|Z^\perp|^2 + |Z'^\perp|^2\big)\Big),$$
- (Ma-Zhang, CRAS, 2005) $\mathcal{Q}_r$ polynomial on Z, Z', $\mathcal{Q}_0 = I_{\mathbb{C}\otimes E_B}$,
$$p^{-n+\frac{n_0}{2}} h(Z)h(Z') P_p^G(Z,Z') \\ \approx \sum_{r=0}^{k} (\mathcal{Q}_{r,x_0} P)(\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} + \mathcal{O}(p^{(k+1)/2}).$$
- Paoletti (Dec. 2006), for $|Z|, |Z'| < c/\sqrt{p}$, does not correct when X_G is an orbifold.

Diagonal asymptotic expansion

Diagonal asymptotic expansion

- $B = U/G$, $\mathbb{J}_{r,x_0}(Z)$ polynomials on Z, for $x_0 \in X_G$, $Z \in N_{G,x_0}$, $|Z| \leq \varepsilon_0$,

$$\Big| p^{-n+\frac{\dim G}{2}} h^2(Z) P_p^G(Z,Z) - \sum_{r=0}^{k} \mathbb{J}_{r,x_0}(\sqrt{p}Z) e^{-2\pi p|Z|^2} p^{-\frac{r}{2}} \Big| \leq C p^{-(k+1)/2} e^{-\sqrt{C''}\sqrt{p}|Z|}.$$

Diagonal asymptotic expansion

- $B = U/G$, $\mathbb{J}_{r,x_0}(Z)$ polynomials on Z, for $x_0 \in X_G$, $Z \in N_{G,x_0}$, $|Z| \leq \varepsilon_0$,

$$\Big| p^{-n+\frac{\dim G}{2}} h^2(Z) P_p^G(Z,Z) - \sum_{r=0}^{k} \mathbb{J}_{r,x_0}(\sqrt{p}Z) e^{-2\pi p|Z|^2} p^{-\frac{r}{2}} \Big| \leq C p^{-(k+1)/2} e^{-\sqrt{C''}\sqrt{p}|Z|}.$$

- $\mathbb{J}_{0,x_0} = 2^{\frac{\dim G}{2}} I_{\mathbb{C}\otimes E}$.

Idea of the proof

Idea of the proof

- $\mathcal{L}_p := D_p^2 - p\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}.$

Idea of the proof

- $\mathscr{L}_p := D_p^2 - p\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}$.
- Theorem (Ma-Zhang (2005)).

$$\mathrm{Ker}(\mathscr{L}_p) = (\mathrm{Ker}\, D_p)^G,$$
$$\mathrm{Spec}(\mathscr{L}_\mathrm{p}) \subset \{0\} \cup [2\nu p - C_L, +\infty[.$$

Idea of the proof

- $\mathcal{L}_p := D_p^2 - p\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}$.
- Theorem (Ma-Zhang (2005)).

$$\mathrm{Ker}(\mathcal{L}_p) = (\mathrm{Ker}\, D_p)^G,$$
$$\mathrm{Spec}(\mathcal{L}_\mathrm{p}) \subset \{0\} \cup [2\nu p - C_L, +\infty[.$$

- Slogan : Spectral gap property $\Longrightarrow$ the problem is local. Work on $G \times \mathbb{R}^{2n-\dim G}$.

Idea of the proof

- $\mathcal{L}_p := D_p^2 - p\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}$.
- Theorem (Ma-Zhang (2005)).

$$\mathrm{Ker}(\mathcal{L}_p) = (\mathrm{Ker}\, D_p)^G,$$
$$\mathrm{Spec}(\mathcal{L}_{\mathrm{p}}) \subset \{0\} \cup [2\nu p - C_L, +\infty[.$$

- Slogan : Spectral gap property $\Longrightarrow$ the problem is local. Work on $G \times \mathbb{R}^{2n-\dim G}$.
Local index techniques $\Longrightarrow$ Asymptotic expansion and effective method to compute the coefficients.

Consequences

Consequences

- $B = U/G$. $dv_B(x_0, Z) = \kappa(x_0, Z) dv_{X_G}(x_0) dv_{N_{G,x_0}}$, $x_0 \in X_G$, $Z \in N_{G,x_0}$. $\mathscr{I}_p \in \mathrm{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_G$:

$$\mathscr{I}_p(x_0) = \int_{\substack{|Z| \leq \varepsilon_0, \\ Z \in N_G}} (\kappa h^2)(x_0, Z) P_p^G((x_0, Z), (x_0, Z)) dv_{N_G}(Z).$$

Consequences

- $B = U/G$. $dv_B(x_0, Z) = \kappa(x_0, Z) dv_{X_G}(x_0) dv_{N_{G,x_0}}$, $x_0 \in X_G$, $Z \in N_{G,x_0}$. $\mathscr{I}_p \in \mathrm{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_G$:

$$\mathscr{I}_p(x_0) = \int_{\substack{|Z| \leq \varepsilon_0, \\ Z \in N_G}} (\kappa h^2)(x_0, Z) P_p^G((x_0, Z), (x_0, Z)) dv_{N_G}(Z).$$

- Modulo $\mathscr{O}(p^{-\infty})$, $\mathscr{I}_p(x_0)$ does not depend on ε_0, and

$$\begin{aligned} \dim \mathrm{Ker}\, D_{G,p} &= \int_X \mathrm{Tr}[P_p^G(y, y)] dv_X(y) \\ &= \int_B h^2(y) \,\mathrm{Tr}[P_p^G(y, y)] dv_B(y) + \mathscr{O}(p^{-\infty}) \\ &= \int_{X_G} \mathrm{Tr}[\mathscr{I}_p(x_0)] dv_{X_G}(x_0) + \mathscr{O}(p^{-\infty}). \end{aligned}$$

Scalar curvature r^{X_G} of X_G

Scalar curvature r^{X_G} of X_G

- Theorem (Ma-Zhang (2005)). There exist $\Phi_r \in \mathrm{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_{G,x_0}$, for $x_0 \in X_G$, $p \in \mathbb{N}$,

$$\left| p^{-n+\dim G} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0) p^{-r} \right|_{\mathscr{C}^{m'}} \leq C_{k,m'} p^{-k-1}.$$

Scalar curvature r^{X_G} of X_G

- Theorem (Ma-Zhang (2005)). There exist $\Phi_r \in \mathrm{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_{G,x_0}$, for $x_0 \in X_G$, $p \in \mathbb{N}$,

$$\left| p^{-n+\dim G} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0) p^{-r} \right|_{\mathscr{C}^{m'}} \leq C_{k,m'} p^{-k-1}.$$

- Theorem (Ma-Zhang (2005)). X Kähler, L, E holomorphic, then $\mathscr{I}_p(x_0), \Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$, and $\Phi_0 = \mathrm{Id}_{E_G}$,

$$\Phi_1(x_0) = \frac{1}{2\pi} R^{E_G}_{x_0}(w^0_j, \overline{w}^0_j) + \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{3}{4\pi} \Delta_{X_G} \log\left(h|_{X_G}\right).$$

Scalar curvature r^{X_G} of X_G

- Theorem (Ma-Zhang (2005)). There exist $\Phi_r \in \mathrm{End}(\Lambda^{\cdot}(T^{*(0,1)}X) \otimes E)_{G,x_0}$, for $x_0 \in X_G$, $p \in \mathbb{N}$,

$$\left| p^{-n+\dim G} \mathscr{I}_p(x_0) - \sum_{r=0}^{k} \Phi_r(x_0) p^{-r} \right|_{\mathscr{C}^{m'}} \leq C_{k,m'} p^{-k-1}.$$

- Theorem (Ma-Zhang (2005)). X Kähler, L, E holomorphic, then $\mathscr{I}_p(x_0), \Phi_r(x_0) \in \mathrm{End}(E_G)_{x_0}$, and $\Phi_0 = \mathrm{Id}_{E_G}$,

$$\Phi_1(x_0) = \frac{1}{2\pi} R^{E_G}_{x_0}(w^0_j, \overline{w}^0_j) + \frac{1}{8\pi} r^{X_G}_{x_0} + \frac{3}{4\pi} \Delta_{X_G} \log\left(h|_{X_G}\right).$$

- Applications in Donaldson's program ?

Geometric quantization

- X, L, E holomorphic, G acts holomorphically on X, L, E.

Geometric quantization

- X, L, E holomorphic, G acts holomorphically on X, L, E.
- $i : \mu^{-1}(0) \hookrightarrow X$ natural injection.
  $\pi_G : \mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ natural identification.

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G).$$

Geometric quantization

- X, L, E holomorphic, G acts holomorphically on X, L, E.
- $i : \mu^{-1}(0) \hookrightarrow X$ natural injection.
  $\pi_G : \mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ natural identification.

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G).$$

- Guillemin-Sternberg (1982) : If $E = \mathbb{C}$, for $p > 0$, σ_p is an isomorphism.

Geometric quantization

- X, L, E holomorphic, G acts holomorphically on X, L, E.
- $i : \mu^{-1}(0) \hookrightarrow X$ natural injection.
  $\pi_G : \mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$ natural identification.

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L_G^p \otimes E_G).$$

- Guillemin-Sternberg (1982) : If $E = \mathbb{C}$, for $p > 0$, σ_p is an isomorphism.
- Zhang (1999) : E general, $p \gg 0$, σ_p is an isomorphism.

Metric aspect of quantization

- Hermitian product $\langle\ \rangle_h$ on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$:

$$\langle s_1, s_2\rangle_h = \int_{X_G} \langle s_1(x_0), s_2(x_0)\rangle h^2(x_0)\, \frac{\omega_G^{n-\dim G}}{(n-\dim G)!}(x_0).$$

Metric aspect of quantization

- Hermitian product $\langle\, \rangle_h$ on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$:

$$\langle s_1, s_2\rangle_h = \int_{X_G} \langle s_1(x_0), s_2(x_0)\rangle h^2(x_0)\, \frac{\omega_G^{n-\dim G}}{(n-\dim G)!}(x_0).$$

- π_G isometry :

$$(\mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G, \langle\,\rangle) \to (\mathscr{C}^\infty(X_G, L_G^p \otimes E_G), \langle\,\rangle_h).$$

Metric aspect of quantization

- Hermitian product $\langle\ \rangle_h$ on $\mathscr{C}^\infty(X_G, L_G^p \otimes E_G)$:
$$\langle s_1, s_2\rangle_h = \int_{X_G} \langle s_1(x_0), s_2(x_0)\rangle h^2(x_0)\, \frac{\omega_G^{n-\dim G}}{(n-\dim G)!}(x_0).$$
- π_G isometry :
$(\mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G, \langle\ \rangle) \to (\mathscr{C}^\infty(X_G, L_G^p \otimes E_G), \langle\ \rangle_h)$.
- Ma-Zhang (2005) : $(2p)^{-\dim G/4}\sigma_p$ asymptotic isometry.
$\{s_i^p\}_{i=1}^{d_p}$ orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\quad\rangle)$,
$$(2p)^{-\dim G/2}\langle \sigma_p s_i^p, \sigma_p s_j^p\rangle_h = \delta_{ij} + \mathscr{O}(\frac{1}{p}).$$
- Symplectic case :
$(2p)^{-\dim G/4}\sigma_p : \mathrm{Ker}(D_p)^G \to \mathrm{Ker}(D_{G,p})$ is an isomorphism and asymptotic isometry.

Some remarks

- Ma-Zhang, Bergman kernels and symplectic reduction, C. R. A.S. **341** (2005), 297-302.
  Full version : math/0607605, Astérisque 318 (2008).
  Announced also in : Nankai Tracts in Mathematics Vol. **10**, (2006), 343-349.

Some remarks

- Ma-Zhang, Bergman kernels and symplectic reduction, C. R. A.S. **341** (2005), 297-302.
  Full version : math/0607605, Astérisque 318 (2008).
  Announced also in : Nankai Tracts in Mathematics Vol. **10**, (2006), 343-349.
- Assume (X, ω) Kähler, L holomorphic, $E = \mathbb{C}$. Charles (JFA 2006) when G is abelian, and Paoletti (Adv. Math. 2005) studied some Toeplitz properties on X_G. Charles shows $\sigma_p : H^0(X, L^p)^G \to H^0(X_G, L^p_G)$ is not an asymptotic isometry when G is abelian.

Some remarks

- Ma-Zhang, Bergman kernels and symplectic reduction, C. R. A.S. **341** (2005), 297-302.
  Full version : math/0607605, Astérisque 318 (2008).
  Announced also in : Nankai Tracts in Mathematics Vol. **10**, (2006), 343-349.
- Assume (X, ω) Kähler, L holomorphic, $E = \mathbb{C}$. Charles (JFA 2006) when G is abelian, and Paoletti (Adv. Math. 2005) studied some Toeplitz properties on X_G. Charles shows $\sigma_p : H^0(X, L^p)^G \to H^0(X_G, L^p_G)$ is not an asymptotic isometry when G is abelian.
- Hall-Kirwin (math/0610005) reproved Ma-Zhang's asymptotic isometric result for L^p, and $L^p \otimes \sqrt{K}$ for general G in the Kähler case.

Thanks !