Toric Degeneration of Gelfand-Cetlin Systems and Potential Functions

Yuichi Nohara
Mathematical Institute, Tohoku University

joint work with Takeo Nishinou and Kazushi Ueda

Third International Conference on Geometry and Quantization
University of Luxembourg, Sep. 10, 2009

§1 Introduction

Polarized toric varieties and moment polytopes:
Let $\mathcal{L} \to X$ be a polarized toric variety of $\dim_{\mathbb{C}} = N$ and fix a T^N-invariant Kähler form $\omega \in c_1(\mathcal{L})$. Then the moment polytope Δ of X appears in two different stories:

- Monomial basis of $H^0(X, \mathcal{L})$:

$$H^0(X, \mathcal{L}) = \bigoplus_{I \in \Delta \cap \mathbb{Z}^N} \mathbb{C} z^I \quad \text{(weight decomposition)}.$$

- Moment map image:
  $\Phi : (X, \omega) \longrightarrow \mathbb{R}^N$ moment map of T^N-action, $\Delta = \Phi(X)$.
  $\Phi^{-1}(u)$ Bohr-Sommerfeld iff $u \in \Delta \cap \mathbb{Z}^N$.

"Real quantization $\cong$ Kähler quantization"

Flag manifolds.

$$
\begin{aligned}
Fl_n &:= \{0 \subset V_1 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n \mid \dim V_i = i\} \\
&= U(n)/T = GL(n, \mathbb{C})/B,
\end{aligned}
$$

where $T \subset U(n)$ is a maximal torus and $B \subset GL(n, \mathbb{C})$ is a Borel subgroup. Note that

$$
N := \dim_{\mathbb{C}} Fl_n = \frac{1}{2}n(n-1).
$$

For

$$
\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \quad \lambda_1 > \lambda_2 > \cdots > \lambda_n,
$$

we can associate

- ω_λ Kostant-Kirillov form (a $U(n)$-invariant Kähler form),
- $\mathcal{L}_\lambda \to Fl_n$ $U(n)$-equivariant line bundle, $c_1(\mathcal{L}_\lambda) = [\omega_\lambda]$ (if $\lambda_i \in \mathbb{Z}$),
- $\Delta_\lambda \subset \mathbb{R}^N$ Gelfand-Cetlin polytope.

Flag manifolds and Gelfand-Cetlin polytopes Δ_λ:

(i) Gelfand-Cetlin basis(Gelfand-Cetlin):
a basis of an irreducible representation $H^0(Fl_n, \mathcal{L}_\lambda)$ of $U(n)$ of highest weight λ, indexed by $\Delta_\lambda \cap \mathbb{Z}^N$.

(ii) Gelfand-Cetlin system(Guillemin-Sternberg):
a completely integrable system (a set of Poisson-commuting independent functions)

$$\Phi_\lambda : (Fl_n, \omega_\lambda) \longrightarrow \mathbb{R}^N, \quad \Phi_\lambda(Fl_n) = \Delta_\lambda.$$

$\Phi_\lambda^{-1}(u)$ Bohr-Sommerfeld iff $u \in \Delta_\lambda \cap \mathbb{Z}^N$.

"Real quantization $\cong$ Kähler quantization"

The common idea is to consider

$$U(1) \subset U(2) \subset \cdots \subset U(n-1) \subset U(n).$$

In the case of flag manifolds, we have one more relation:
(iii) Toric degeneration(Gonciulea-Lakshmibai, etc.):

Fl_n degenerate into a toric variety X_0 corresponding to Δ_λ.

We call X_0 the Gelfand-Cetlin toric variety.

Kogan-Miller: The Gelfand-Cetlin basis can be deformed into the monomial basis on X_0 under the toric degeneration. In particular, Kähler quantizations for Fl_n and X_0 are "isomorphic".

This talk: The Gelfand-Cetlin system can be deformed into the toric moment map on the Gelfand-Cetlin toric variety.

Corollary: $\exists$ isomorphism between real quantizations for Fl_n and X_0.

Application to symplectic geometry/ mirror symmetry: Computation of the potential function for Gelfand-Cetlin torus fibers.

§2 Gelfand-Cetlin systems

Identify Fl_n with the adjoint orbit $\mathcal{O}_\lambda$ of $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$:

$$U(n)/T \cong \mathcal{O}_\lambda = \left\{ x \in M_n(\mathbb{C}) \;\middle|\; x^* = x, \text{ eigenvalues} = \lambda_1, \ldots, \lambda_n \right\}$$
$$gT \leftrightarrow g\lambda g^*$$

For each $k = 1, \ldots, n-1$ and $x \in \mathcal{O}_\lambda$, set

$$x^{(k)} = \text{upper-left } k \times k \text{ submatrix of } x,$$
$$\lambda_1^{(k)}(x) \geq \cdots \geq \lambda_k^{(k)}(x) \ : \text{ eigenvalues of } x^{(k)}.$$

Theorem (Guillemin-Sternberg).

$$\Phi_\lambda : \mathcal{O}_\lambda \longrightarrow \mathbb{R}^N, \quad x \longmapsto \left(\lambda_i^{(k)}(x)\right)_{\substack{k=1,\ldots,n-1,\\ i=1,\ldots,k}}$$

is a completely integrable system on (Fl_n, ω_λ) *and* $\Phi_\lambda(\mathcal{O}_\lambda) = \Delta_\lambda$.

Φ_λ is called the Gelfand-Cetlin system.

The Gelfand-Cetlin polytope $\Delta_\lambda \subset \mathbb{R}^N = \{(\lambda_i^{(k)}); 1 \leq i \leq k \leq n-1\}$ is a convex polytope given by

$$\begin{array}{ccccccccccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 & & \cdots & & \lambda_{n-1} & & & & \lambda_n \\
 & \geq & & \leq & & \geq & & \leq & & & & & & \geq & & \leq & \\
 & & \lambda_1^{(n-1)} & & & & \lambda_2^{(n-1)} & & & & & & & & \lambda_{n-1}^{(n-1)} & & \\
 & & & \geq & & \leq & & & & & & & & \leq & & & \\
 & & & & \lambda_1^{(n-2)} & & & & & & & & \lambda_{n-2}^{(n-2)} & & & & \\
 & & & & & \geq & & & & & & \leq & & & & & \\
 & & & & & & \ddots & & & & \iddots & & & & & & \\
 & & & & & & & \geq & & \leq & & & & & & & \\
 & & & & & & & & \lambda_1^{(1)} & & & & & & & &
\end{array}$$

Remark. (i) For $k = 1, \ldots, n-1$, we embed $U(k)$ in $U(n)$ by

$$U(k) \cong \left(\begin{array}{c|c} U(k) & 0 \\ \hline 0 & 1_{n-k} \end{array} \right) \subset U(n).$$

$x \mapsto x^{(k)} \in \sqrt{-1}\mathfrak{u}(k) \cong \mathfrak{u}(k)^*$ is a moment map of the $U(k)$-action.

(ii) The moment map of the action of maximal torus T is given by

$$x \in \mathcal{O}_\lambda \longmapsto \mathrm{diag}(x_{11}, x_{22}, \ldots, x_{nn}).$$

Since

$$x_{kk} = \mathrm{tr} x^{(k)} - \mathrm{tr} x^{(k-1)} = \sum_i \lambda_i^{(k)} - \sum_i \lambda_i^{(k-1)},$$

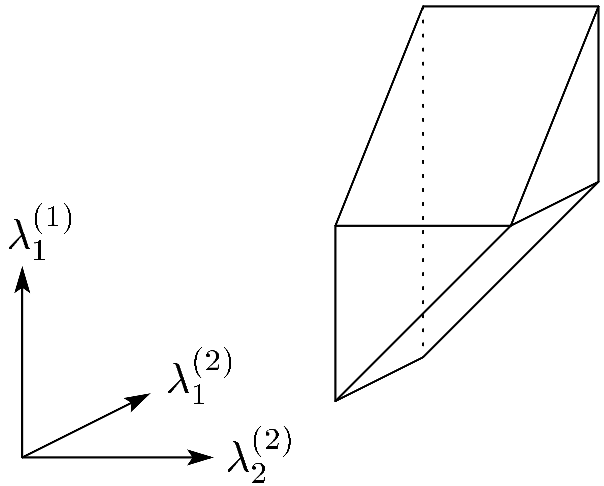
the T-action is contained in the Hamiltonian torus action of the Gelfand-Cetlin system.

(iii) The Hamiltonian torus action of G-C system is not holomorphic. Hence inverse image of a face of Δ_λ is not a subvariety in general.

Example (the case of Fl_3).

$$\Phi_\lambda = (\lambda_1^{(2)}, \lambda_2^{(2)}, \lambda_1^{(1)}) : Fl_3 \longrightarrow \mathbb{R}^3.$$

Gelfand-Cetlin polytope Δ_λ:



For every $u \in \operatorname{Int}\Delta_\lambda$, $L(u) := \Phi_\lambda^{-1}(u)$ is a Lagrangian T^3.
The fiber of the vertex emanating four edges is a Lagrangian S^3.

§3 Gelfand-Cetlin basis

Borel-Weil: $H^0(Fl_n, \mathcal{L}_\lambda)$ is an irred. rep. of $U(n)$ of h.w. λ.

$$H^0(Fl_n, \mathcal{L}_\lambda) = \bigoplus_{\lambda^{(n-1)}} V_{\lambda^{(n-1)}} \quad \text{irred. decomp. as a } U(n-1)\text{-rep.}$$

Fact:

- Each $V_{\lambda^{(n-1)}}$ has multiplicity at most 1.
- multiplicity $= 1$ iff

$$\lambda_1 \geq \lambda_1^{(n-1)} \geq \lambda_2 \geq \lambda_2^{(n-1)} \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \lambda_{n-1}^{(n-1)} \geq \lambda_n.$$

Repeating this process we obtain Gelfand-Cetlin decomposition:

$$H^0(Fl_n, \mathcal{L}_\lambda) = \bigoplus_{\Lambda \in \Delta_\lambda \cap \mathbb{Z}^N} V_\Lambda, \quad \dim V_\Lambda = 1.$$

Taking $v_\Lambda (\neq 0) \in V_\Lambda$ for each Λ, we have Gelfand-Cetlin basis.

§4 Toric degeneration of flag manifolds

Toric degeneration is given by deforming the Plücker embedding

$$Fl_n \hookrightarrow \prod_{i=1}^{n-1} \mathbb{P}\Big(\bigwedge^i \mathbb{C}^n\Big), \quad (V_1 \subset \cdots \subset V_{n-1}) \mapsto (\wedge^1 V_1, \ldots, \wedge^{n-1} V_{n-1}).$$

Theorem (Gonciulea-Lakshmibai, ...). *There exists a flat family*

$$\begin{array}{ccccc} X_t & \subset & \mathfrak{X} & \subset & \prod_i \mathbb{P}(\wedge^i \mathbb{C}^n) \times \mathbb{C} \\ \downarrow & & \downarrow & & \\ t & \in & \mathbb{C} & & \end{array}$$

of projective varieties such that

$$X_1 = Fl_n,$$
$$X_0 = \textit{Gelfand-Cetlin toric variety.}$$

Example. Plücker embedding of Fl_3 is given by

$$Fl_3 = \left\{ \left([z_0 : z_1 : z_2], [w_0 : w_1 : w_2] \right) \in \mathbb{P}^2 \times \mathbb{P}^2 \;\middle|\; z_0 w_0 = z_1 w_1 + z_2 w_2 \right\}.$$

Its toric degeneration:

$$\mathfrak{X} = \left\{ \left([z_0 : z_1 : z_2], [w_0 : w_1 : w_2], t \right) \;\middle|\; t z_0 w_0 = z_1 w_1 + z_2 w_2 \right\} \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{C}$$

$$X_1 = \left\{ z_1 w_1 + z_2 w_2 = z_0 w_0 \right\}$$ Flag manifold,

$$X_0 = \left\{ z_1 w_1 + z_2 w_2 = 0 \right\}$$ Gelfand-Cetlin toric variety.

Remark. General X_t does not have $U(k)$-actions.

Multi-parameter family (Kogan-Miller):

There exists an $(n-1)$-parameter family

$$\begin{array}{ccccc} X_{(t_2,\dots,t_n)} & \subset & \widetilde{\mathfrak{X}} & \subset & \prod_i \mathbb{P}(\Lambda^i \mathbb{C}^n) \times \mathbb{C}^{n-1} \\ \downarrow & & \downarrow & & \\ (t_2,\dots,t_n) & \in & \mathbb{C}^{n-1} & & \end{array}$$

such that

- $\widetilde{\mathfrak{X}}|_{t_2=\cdots=t_n} = \mathfrak{X}$,
- $X_{(1,\dots,1)} =$ Flag manifold,
- $X_{(0,\dots,0)} =$ Gelfand-Cetlin toric variety,
- $U(k-1)$ acts on $X_{(1,\dots,1,t_k,\dots,t_n)}$,
- $T^{n-1} \times \cdots \times T^k$ acts holomorphically on $X_{(t_2,\dots,t_k,0,\dots,0)}$, where T^k is a k-torus given by $(\lambda_i^{(k)})_{i=1,\dots,k}$.
  ($T^{n-1} \times \cdots \times T^1$ is the torus acting on X_0.)

Degeneration is stages (Kogan-Miller):
Restrict $\widetilde{f} : \widetilde{\mathfrak{X}} \to \mathbb{C}^{n-1}$ to the following piecewise linear path

$$(t_2, \ldots, t_n) = (1, \ldots, 1) \rightsquigarrow (1, \ldots, 1, 0) \rightsquigarrow \cdots \rightsquigarrow (1, 0, \ldots, 0) \rightsquigarrow (0, \ldots, 0)$$

The $(n-k+1)$-th stage is given by

$$\begin{array}{rccc} f_k : & \mathfrak{X}_k = \widetilde{\mathfrak{X}}|_{\substack{t_2=\cdots=t_{k-1}=1 \\ t_{k+1}=\cdots=t_n=0}} & \longrightarrow & \mathbb{C} \\ & \cup & & \Psi \\ & X_{k,t} = X_{(1,\ldots,1,t,0,\ldots,0)} & \longrightarrow & t. \end{array}$$

Then $T^{n-1} \times \cdots \times T^k$ and $U(k-1)$ acts on $X_{k,t}$ for each t.

§5 Toric degeneration of Gelfand-Cetlin systems

Theorem. *The Gelfand-Cetlin system can be deformed into the moment map on X_0 in the following sense:*

(i) *For each stage* $f_k : \mathfrak{X}_k \to \mathbb{C}$, *there exists* $\Phi_k : \mathfrak{X}_k \to \mathbb{R}^N$ *s.t.*

- $\Phi_k|_{X_{k,t}} : X_{k,t} \to \mathbb{R}^N$ *is a completely integrable system,*
- $\Phi_n|_{X_{n,1}}$ *is the Gelfand-Cetlin system on* $X_{n,1} = Fl_n$,
- $\Phi_2|_{X_{2,0}}$ *is the moment map on* $X_{2,0} = X_0$,
- $\Phi_k|_{X_{k,0}} = \Phi_{k-1}|_{X_{k-1,1}}$ *on* $X_{k,0} = X_{k-1,1}$.

(ii) *There exists a vector field* ξ_k *on* $\mathfrak{X}_k$ *such that*

$$\begin{array}{ccc} X_{k,1} & \xrightarrow{\exp(1-t)\xi_k} & X_{k,t} \\ & {\scriptstyle \Phi_k} \searrow \quad \swarrow {\scriptstyle \Phi_k} & \\ & \Delta_\lambda & \end{array}$$

Constructions: Using $U(k-1)$ and $T^{n-1} \times \cdots \times T^k$-actions, we have

$\widetilde{\lambda}_i^{(l)} : \mathfrak{X}_k \longrightarrow \mathbb{R}$ eigenvalues for moment map of $U(l)$-action,

$\left(\widetilde{\nu}_i^{(j)}\right)_{i=1,\ldots,j} : \mathfrak{X}_k \longrightarrow \mathbb{R}^N$ moment map of T^j-action.

Then Φ_k is given by

$$\Phi_k = \left(\widetilde{\nu}_i^{(n-1)}, \ldots, \widetilde{\nu}_j^{(k)}, \widetilde{\lambda}_l^{(k-1)}, \ldots, \widetilde{\lambda}_1^{(1)}\right) : \mathfrak{X}_k \longrightarrow \mathbb{R}^N.$$

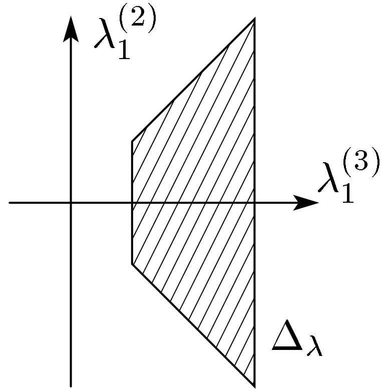
ξ_k = gradient-Hamiltonian vector field introduced by W.-D. Ruan.

Remark. Theorem is true for

- partial flag manifolds of type A,
- orthogonal flag manifolds.

Example. The Gelfand-Cetlin system on $Fl_{SO(4)} = SO(4)/T = \mathbb{P}^1 \times \mathbb{P}^1$ is not the standard moment map. In fact the G-C polytope is given by

$$\begin{array}{ccccc} \lambda_1 & & & & |\lambda_2| \\ & \geq & & \geq & \\ & & \lambda_1^{(3)} & & \\ & & & \geq & \\ & & & & |\lambda_1^{(2)}| \end{array} \quad ,$$



which is moment polytope of $F_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$. The G-C system on $Fl_{SO(4)}$ is the pull-back of the moment map on F_2 under a diffeomorphism $Fl_{SO(4)} \cong F_2 = X_0$ given by the gradient-Hamiltonian flow.

Remark. In the case of $SO(n)/T$, T-action is not contained in the Hamiltonian torus action of the G-C system.

§6 Application

Mirror symmetry is a duality between symplectic geometry on X and complex geometry on Y, and vice versa.

Mirror of a Fano manifold X: Landau-Ginzburg model $(Y, \mathcal{F})$

- Y is a non-compact complex manifold,
- $\mathcal{F} : Y \longrightarrow \mathbb{C}$ is a holomorphic function (superpotential).

Example. Mirror of $\mathbb{P}^1$ is given by

$$Y \cong \mathbb{C}^*, \quad \mathcal{F}(y) = y + \frac{Q}{y},$$

where Q is a parameter.

Potential functions (Fukaya-Oh-Ohta-Ono):

Let X be a toric variety (or flag manifold) with a completely integrable system $\Phi : X \to \Delta$,

$$\Lambda_0 = \left\{ \sum_{i=1}^{\infty} a_i T^{r_i} \,\middle|\, a_i \in \mathbb{C},\ r_i \geq 0,\ \lim_{i\to\infty} r_i = \infty \right\}$$

be the Novikov ring and Λ_+ its maximal ideal. We can consider potential function $\mathfrak{PO}$ as a function on

$$\bigcup_{u \in \mathrm{Int}\,\Delta} H^1(L(u); \Lambda_+) \cong \mathrm{Int}\,\Delta \times (\Lambda_+)^N, \quad L(u) := \Phi^{-1}(u).$$

Roughly, $\mathfrak{PO}$ is given by counting holomorphic disks:

$$\mathfrak{PO}(L) \text{``}=\text{''} \sum_{\substack{\phi: D^2 \to X \text{ holo.,} \\ \partial\phi(D^2) \subset L}} T^{\mathrm{Area}(\phi(D^2))}.$$

Toric Fano case.

Theorem (Cho-Oh, Fukaya-Oh-Ohta-Ono). *Let X be a smooth toric Fano manifold, and suppose that Δ is given by $\ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0$, $i = 1, \ldots, m$. Then the potential function is given by*

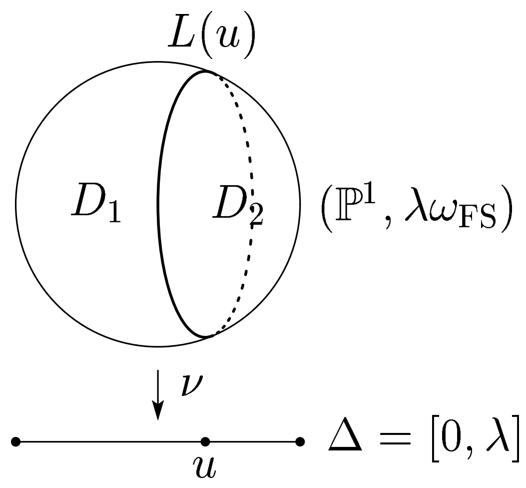
$$\mathfrak{PO}(u,x) = \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}, \quad u \in \mathrm{Int}\,\Delta,\ x \in (\Lambda_+)^N.$$

Moreover, $\mathfrak{PO}$ gives the superpotential of the L-G mirror of X.

Example. $X = \mathbb{P}^1$ with $x = 0$,

$$\begin{aligned}\mathfrak{PO}(L(u)) &= T^{\mathrm{Area}(D_1)} + T^{\mathrm{Area}(D_2)} \\ &= T^u + T^{\lambda - u} \\ &= y + \frac{Q}{y},\end{aligned}$$

where $y = T^u$, $Q = T^\lambda$.



The case of flag manifolds of type A.

Theorem. *Suppose that* Δ_λ *is given by* $\ell_i(u) = \langle v_i, u\rangle - \tau_i \geq 0$, $i = 1, \ldots, m$. *Then the potential function for G-C torus fibers is given by*

$$\mathfrak{PO}(u,x) = \sum_{i=1}^{m} e^{\langle v_i, x\rangle} T^{\ell_i(u)}, \quad u \in \mathrm{Int}\,\Delta,\ x \in (\Lambda_+)^N.$$

Moreover, $\mathfrak{PO}$ *gives the Givental's superpotential of the mirror of* Fl_n.

Example (The case of Fl_3).

$$\begin{aligned}\mathfrak{PO} &= e^{-x_1}T^{-u_1+\lambda_1} + e^{x_1}T^{u_1-\lambda_2} + e^{-x_2}T^{-u_2+\lambda_2} \\ &\quad + e^{x_2}T^{u_2-\lambda_3} + e^{x_1-x_3}T^{u_1-u_3} + e^{-x_2+x_3}T^{-u_2+u_3} \\ &= \frac{Q_1}{y_1} + \frac{y_1}{Q_2} + \frac{Q_2}{y_2} + \frac{y_2}{Q_3} + \frac{y_1}{y_3} + \frac{y_3}{y_2},\end{aligned}$$

where $y_k = e^{u_k}T^{x_k}$ and $Q_j = T^{\lambda_j}$.

Idea of the proof.
By comparing holomorphic disks in Fl_n and X_0, we show the following, which is the same as in the toric Fano case.

Lemma. *Only holomorphic disks of Maslov index* 2 *contribute to* $\mathfrak{PO}$*, and such disks are Fredholm regular.*

Note: X_0 is singular in general.

Key facts:

(i) X_0 is singular Fano toric variety,

(ii) X_0 has a small resolution $p : \widetilde{X}_0 \to X_0$, i.e.,

$$\operatorname{codim}_{\mathbb{C}} p^{-1}(\operatorname{Sing}(X_0)) \geq 2.$$

Remark: These are not true in the $SO(n)$-case in general.

Non-displaceable Lagrangian submanifolds.

From the fact that

$$HF((L(u), x); \Lambda_0) \cong H^*(L(u); \Lambda_0)$$

if $(u, x) \in \mathrm{Int}\,\Delta \times (\Lambda_+)^N$ is a critical point of $\mathfrak{PO}$, we have:

Theorem. *The Gelfand-Cetlin system* $\Phi_\lambda : Fl_n \to \Delta_\lambda$ *has a non-displaceable Lagrangian torus fiber* $L(u) = \Phi_\lambda^{-1}(u)$, $u \in \mathrm{Int}\,\Delta_\lambda$:

$$\psi(L(u)) \cap L(u) \neq \emptyset$$

for any Hamiltonian diffeomorphism $\psi : Fl_n \to Fl_n$. *Moreover, if* $\psi(L(u))$ *is transverse to* $L(u)$, *then*

$$\#\Big(\psi(L(u)) \cap L(u)\Big) \geq 2^N (= \dim H^*(L(u))).$$