Formal groups arising from formal punctured ribbons

Denis Osipov

(this talk is based on joint works with H.Kurke and A.Zheglov: arXiv:0708.0985 and arXiv:0901.1607)

1 History: Generalization of the KP-hierarchy

We consider a $\mathbb{C}$-algebra R with a derivation $\partial : R \to R$

$$\partial(ab) = \partial(a)b + a\partial(b), \quad a, b \in R.$$

We construct a ring

$$R((\partial^{-1})) \quad : \quad \sum_{i \ll +\infty} a_i \partial^i, a_i \in R$$

$$[\partial, a] = \partial(a)$$

$$\partial^{-1} a = a\partial^{-1} + \binom{-1}{1}\partial(a)\partial^{-2} + \binom{-1}{2}\partial^2(a)\partial^{-3} + \ldots,$$

where

$$\binom{i}{k} = \frac{i(i-1)\ldots(i-k+1)}{k(k-1)\ldots 1}, \text{ if } \quad k > 0.$$

Now we consider $R = \mathbb{C}[[x]]$ with usual derivation $\partial(x) = 1$. We add infinite number of "formal times" : $t_1, t_2, \ldots$. There is a unique decomposition in the ring $R((\partial^{-1}))[[t_1, t_2, \ldots]]$:

$$\text{if} \quad A \in R((\partial^{-1}))[[t_1, t_2, \ldots]], \quad \text{then} \quad A = A_+ + A_-,$$

where $A_+ \in R[\partial][[t_1, t_2, \ldots]]$, $A_- \in R[[\partial^{-1}]] \cdot \partial^{-1}[[t_1, t_2, \ldots]]$.

Let $L \in R((\partial^{-1}))[[t_1, t_2, \ldots]]$ be of the following type:

$$L = \partial + a_1\partial^{-1} + a_2\partial^{-2} + \ldots, \qquad a_i \in R[[t_1, t_2, \ldots]].$$

The classical **KP**-hierarchy is the following infinite system of equations:

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \qquad n \in \mathbb{N}.$$

From this system it follows

the **KP** equation $\quad (4u_t - u''' - 12uu')' = 3u_{yy} \quad$ for $\quad u(t, x, y)$,

and the **KdV** equation $\quad 4u_t - 7u''' - 12uu' = 0 \quad$ for $\quad u(t, x)$.

Some solutions of **KP**-hierarchy are obtained from flows on Picard varieties of algebraic curves (for example, solitons).

This is the way, how to do this: we define for each $i \geq 1$

$$w_{-i}(t) = \frac{p_i(-\partial_t)\tau(t)}{\tau(t)}|_{t_1 \mapsto t_1 + x},$$

where $\tau(t)$ is a tau function of the hierarchy which can be defined from the action of the group $GL(\infty)$ on the determinant bundle of the infinite-dimensional Sato Grassmanian $Gr(\mathbb{C}((z)))$,

$$\partial_t = (\partial/\partial t_1, \partial/2\partial t_2, \partial/2\partial t_2, \ldots),$$

and p_i are the Schur polynomials:

$$\exp(\sum t_i D^i) = \sum p_i D^i.$$

Then

$$L(t) = W(t)\partial W(t)^{-1},$$

$$\text{where} \qquad W(t) = 1 + w_{-1}\partial^{-1} + w_{-2}\partial^{-2} + \ldots .$$

A.N. Parshin gave in 1999 the following generalization of **KP**-hierarchy. (A. Zheglov modified it later.)

Let a ring

$$R = \mathbb{C}[[x_1, x_2]]((\partial_1^{-1})),$$

where the derivation

$$\partial_1 \quad : \quad \mathbb{C}[[x_1, x_2]] \to \mathbb{C}[[x_1, x_2]], \qquad \partial_1(x_1) = 1, \quad \partial_1(x_2) = 0.$$

We consider a derivation

$$\partial_2 \ : \ R \to R, \qquad \partial_2(x_1) = 0, \qquad \partial_2(x_2) = 1, \qquad \partial_2(\partial_1) = 0.$$

As before, we construct a ring

$$E = R((\partial_2^{-1})) = \mathbb{C}[[x_1, x_2]]((\partial_1^{-1}))((\partial_2^{-1})).$$

We add "formal times" $\{t_k\}$, $k = (i, j) \in \mathbb{Z} \times \mathbb{Z}$. As before, there is a decomposition (with respect to ∂_2):

$$E[[\{t_k\}]] = E_+[[\{t_k\}]] \oplus E_-[[\{t_k\}]].$$

We consider $L, M \in E[[\{t_k\}]]$ such that

$$L = \partial_1 + u_1\partial_2^{-1} + \dots, \qquad M = \partial_2 + v_1\partial_2^{-1} + \dots,$$

$$\text{where} \quad u_i, v_i \in R[[\{t_k\}]].$$

Let $N = (L, M)$ and $[L, M] = 0$, then **hierarchy** is

$$\frac{\partial N}{\partial t_k} = V_N^k,$$

$$\text{where} \quad V_N^k = ([(L^i M^j)_+, L], [(L^i M^j)_+, M]),$$

$$k = (i, j) \in \mathbb{Z} \times \mathbb{Z}, \quad j \geq 0, \quad i \leq \alpha j, \quad \alpha > 0.$$

2 Generalized Fredholm subspaces and formal ribbons

There is the following property. Let

$$L,\, M \in E, \qquad [L, M] = 0$$

be initial conditions of the hierarchy or arbitrary monic operators with algebraically independent highest symbols. Then there is

$$S \in 1 + E_{-} \;\subset\; E$$

$$\text{such that} \qquad L = S^{-1}\partial_1 S, \qquad M = S^{-1}\partial_2 S.$$

Besides, the ring E acts $\mathbb{C}$-linearly on $\mathbb{C}((u))((t))$ (and on the set of $\mathbb{C}$-vector subspaces of $\mathbb{C}((u))((t))$) in the following way:

$$E/E \cdot (x_1, x_2) = \mathbb{C}((u))((t)), \qquad \partial_1^{-1} \mapsto u, \quad \partial_2^{-1} \mapsto t,$$

now E acts naturally on the left on $E/E \cdot (x_1, x_2)$.

We recall that a $\mathbb{C}$-subspace W in $\mathbb{C}((u))$ is called a **Fredholm** subspace if

$$\dim_{\mathbb{C}} W \cap \mathbb{C}[[u]] < \infty \qquad \text{and} \qquad \dim_{\mathbb{C}} \frac{\mathbb{C}((u))}{W + \mathbb{C}[[u]]} < \infty.$$

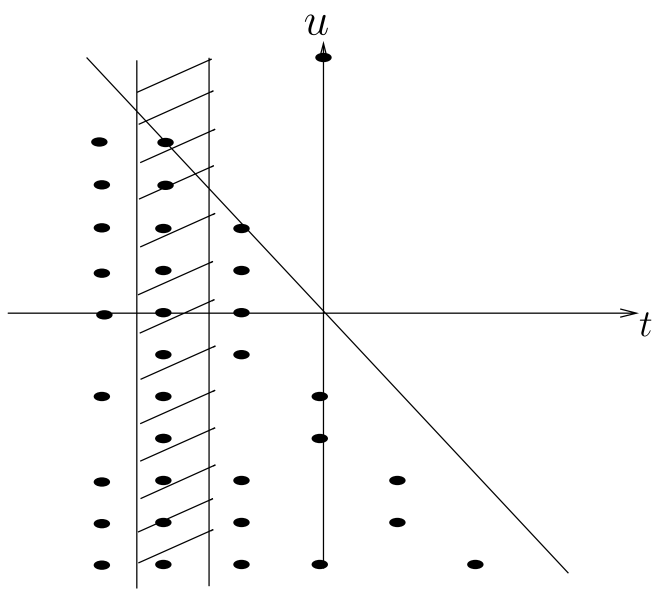
Remark 1. One the set of Fredholm subspaces of $\mathbb{C}((u))$ one can define the structure of infinite-dimensional projective algebraic variety, the Sato Grassmanian $Gr(\mathbb{C}((u)))$.

Definition 1. For a $\mathbb{C}$-subspace W in $\mathbb{C}((u))((t))$, for $n \in \mathbb{N}$ let

$$W(n) = \frac{W \cap t^n \mathbb{C}((u))[[t]]}{W \cap t^{n+1} \mathbb{C}((u))[[t]]}$$

be a $\mathbb{C}$-subspace in $\mathbb{C}((u)) = \frac{t^n \mathbb{C}((u))[[t]]}{t^{n+1}\mathbb{C}((u))[[t]]}$.

A $\mathbb{C}$-subspace W in $\mathbb{C}((u))((t))$ is called **a generalized Fredholm subspace** if for any $n \in \mathbb{Z}$ the $\mathbb{C}$-subspace $W(n)$ in $\mathbb{C}((u))$ is a Fredholm subspace.



Definition 2. Let a $\mathbb{C}$-subalgebra A in $\mathbb{C}((u))((t))$ be a generalized Fredholm subspace. Let a $\mathbb{C}$-subspace W in $\mathbb{C}((u))((t))$ be a generalized Fredholm subspace. We say that (A, W) is **a Schur pair** if $A \cdot W \subset W$.

Definition 3. A formal punctured **ribbon** $\overset{\circ}{X}_{\infty}$ (or $(C, \mathcal{A})$) is given by the following data.

1. An algebraic curve C.
2. A sheaf $\mathcal{A}$ of commutative $\mathbb{C}$-algebras on C.
3. A descending sheaf filtration $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ of $\mathcal{A}$ by $\mathbb{C}$-submodules which satisfies the following axioms:
   (a) $\mathcal{A}_i\mathcal{A}_j \subset \mathcal{A}_{i+j}$, $1 \in \mathcal{A}_0$ (thus $\mathcal{A}_0$ is a subring, and for any $i \in \mathbb{Z}$ the sheaf $\mathcal{A}_i$ is a $\mathcal{A}_0$-submodule);
   (b) $\mathcal{A}_0/\mathcal{A}_1$ is the structure sheaf $\mathcal{O}_C$ of C;
   (c) for each i the sheaf $\mathcal{A}_i/\mathcal{A}_{i+1}$ (which is a $\mathcal{A}_0/\mathcal{A}_1$-module by (3a)) is a torsion free of rank 1 coherent sheaf on C;
   (d) $\mathcal{A} = \varinjlim_{i\in\mathbb{Z}} \mathcal{A}_i$, and $\mathcal{A}_i = \varprojlim_{j>0} \mathcal{A}_i/\mathcal{A}_{i+j}$ for each i.

Sometimes we will call these objects as "ribbons".

Example 1. If X is a smooth algebraic surface, and $C \subset X$ is a curve, we obtain a ribbon $(C, \mathcal{A})$, where

$$\mathcal{A} := \mathcal{O}_{\hat{X}_C}(*C) = \varinjlim_{i\in\mathbb{Z}} \mathcal{O}_{\hat{X}_C}(-iC) = \varinjlim_{i\in\mathbb{Z}} \varprojlim_{j\geq 0} J^i/J^{i+j}$$

$$\mathcal{A}_i := \mathcal{O}_{\hat{X}_C}(-iC) = \varprojlim_{j\geq 0} J^i/J^{i+j}, \quad i \in \mathbb{Z},$$

where $\hat{X}_C$ is the formal scheme which is the completion of X at C, and J is the ideal sheaf of C on X.

In the similar way to the definition of ribbon, it is defined a notion of **torsion free sheaf** on a ribbon.

Definition 4. Let $(C,\mathcal{A})$ be a formal punctured ribbon. We say that $\mathcal{N}$ is a torsion free sheaf of rank r on $(C,\mathcal{A})$ if $\mathcal{N}$ is a sheaf of $\mathcal{A}$-modules on C with a descending filtration $(\mathcal{N}_i)_{i\in\mathbb{Z}}$ of $\mathcal{N}$ by $\mathcal{A}_0$-submodules which satisfies the following axioms.

1. $\mathcal{N}_i\mathcal{A}_j \subseteq \mathcal{N}_{i+j}$ for any i,j.
2. For each i the sheaf $\mathcal{N}_i/\mathcal{N}_{i+1}$ is a torsion free coherent sheaf of rank r on C.
3. $\mathcal{N} = \varinjlim_i \mathcal{N}_i$ and $\mathcal{N}_i = \varprojlim_{j>0} \mathcal{N}_i/\mathcal{N}_{i+j}$ for each i.

There is the following example of a torsion free sheaf of rank r on a ribbon $(C,\mathcal{A})$ which comes from an algebraic surface X and an algebraic curve C on X (see example 1). Let E be a locally free sheaf of rank r on the surface X. Then

$$\varinjlim_i \varprojlim_j E(iC)/E(jC)$$

is a torsion free sheaf of rank r on $(C,\mathcal{A})$.

It is possible to prove that for "good" ribbons $(C,\mathcal{A})$ any torsion free sheaf of rank r is just a **locally free sheaf** on a ribbon, i.e, an element of the set $\check{H}^1(C, GL_r(\mathcal{A}))$. For example, any ribbon which comes from an algebraic surface and a smooth curve on it is a "good" ribbon.

One can define the notion of a smooth point on a ribbon, and the notion of formal local parameters of the ribbon at this point. These notions generalize the usual notions of smooth points and formal local parameters on algebraic curves and surfaces when the ribbon comes from an algebraic surface. For example, if P is a smooth point of a ribbon $(C, \mathcal{A})$, then P is a smooth point on the curve C, and the local ring

$$\hat{\mathcal{A}}_{0,P} \simeq \mathbb{C}[[u,t]],$$

where u and t are formal local parameters. Similarly, one can define the notion of a smooth point of a torsion free sheaf on a ribbon at a smooth point.

Theorem 1. *Schur pairs* (A, W) *from*

$$\mathbb{C}((u))((t)) \, \oplus \, \mathbb{C}((u))((t))^{\oplus r}$$

are in one-to-one correspondence with data $(C, \mathcal{A}, \mathcal{N}, P, u, t, e_P)$ *up to an isomorphism, where* C *is a projective irreducible curve,* $(C, \mathcal{A})$ *is a ribbon,* $\mathcal{N}$ *is a torsion free sheaf of rank* r *on this ribbon,* P *is a point of* C *which is a smooth point of* $\mathcal{N}$, u, t *are formal local parameters of this ribbon at* P, e_P *is a formal local trivialization of* $\mathcal{N}$ *at* P.

3 Picard scheme of a ribbon

Due to the previous theorem and the connection of Schur pairs with generalizations of the KP-hierarchy, it is important to study the Picard variety of a ribbon.

We want to introduce and study the structure of algebraic scheme on the Picard group $Pic(X_\infty)$ of a ribbon $X_\infty = (C, \mathcal{A})$, i.e. on the group of linear bundles of the ribbon $X_\infty = (C, \mathcal{A})$.

By a ribbon $\overset{\circ}{X}_\infty = (C, \mathcal{A})$ we construct a locally ringed space $X_\infty = (C, \mathcal{A}_0)$.

Definition 5. Let $\mathcal{B}$ be a category of Noetherian $\mathbb{C}$-schemes. Then we define the following contravariant functors $\mathbf{Pic}_{X_\infty}$ and $\mathbf{Pic}_{\overset{\circ}{X}_\infty}$ from $\mathcal{B}$ to the category of Abelian groups. For any Noetherian $\mathbb{C}$-scheme S

$$\mathbf{Pic}_{X_\infty}(S) = H^0(S, R^1\pi_*\mathcal{A}^*_{S,0}), \qquad \mathbf{Pic}_{\overset{\circ}{X}_\infty}(S) = H^0(S, R^1\pi_*\mathcal{A}^*_S),$$

where $\pi : C \times_k S \to S$ is the projection morphism, and the sheaves of rings on $C \times_{\mathbb{C}} S$ are defined as

$$\mathcal{A}_S = \mathcal{A}\widehat{\boxtimes}_{\mathbb{C}}\mathcal{O}_S, \qquad \mathcal{A}_{S,0} = \mathcal{A}_0\widehat{\boxtimes}_{\mathbb{C}}\mathcal{O}_S.$$

The sheaves $\mathcal{A}^*_S$ and $\mathcal{A}^*_{S,0}$ are the sheaves of groups of invertible elements of the corresponding sheaves of rings.

We note that

$$\mathbf{Pic}_{X_\infty}(\operatorname{Spec}\mathbb{C}) = H^1(C, \mathcal{A}_0^*) = Pic(X_\infty),$$
$$\mathbf{Pic}_{\overset{\circ}{X}_\infty}(\operatorname{Spec}\mathbb{C}) = H^1(C, \mathcal{A}^*) = Pic(\overset{\circ}{X}_\infty).$$

Proposition 1. *Let* $\overset{\circ}{\mathrm{X}}_\infty = (C, \mathcal{A})$ *be a ribbon, and* C *be a projective irreducible curve. We have the following properties.*

1. *The functor* $\mathbf{Pic}_{X_\infty}$ *is representable by a* $\mathbb{C}$*-group scheme.*
2. *The following sequence of* $\mathbb{C}$*-group schemes is exact:*

$$0 \to \mathbb{V} \to \mathbf{Pic}_{X_\infty} \xrightarrow{\phi} \mathbf{Pic}_C \to 0, \qquad (1)$$

where $\mathbb{V}$ *is an affine* k*-group scheme, and* $\mathbf{Pic}_C$ *is the Picard variety of the curve* C*, whose connected component is the generalized Jacobian of the curve* C*.*

There is a splitting of the map ϕ *from sequence (1) over any affine subscheme* U *of the scheme* $\mathbf{Pic}_C$*.*

Let $\mathbf{Pic}^0_{X_\infty}$ be the connected component of zero in $\mathbf{Pic}_{X_\infty}$, which is a closed irreducible subgroup such that there is an exact sequence

$$0 \longrightarrow \mathbf{Pic}^0_{X_\infty} \longrightarrow \mathbf{Pic}_{X_\infty} \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the last map is induced by the degree map $\mathbf{Pic}_C \longrightarrow \mathbb{Z}$.

Theorem 2. *Let $\mathring{\mathrm{X}}_\infty$ be the ribbon constructed from a smooth projective algebraic surface X and a smooth projective irreducible curve $C \subset X$ (as in example 1 above). We assume that $H^1(X, \mathcal{O}_X) = 0$, and $(C \cdot C) \neq 0$. Then the functor $\mathbf{Pic}_{\mathring{\mathrm{X}}_\infty}$ is representable by a formal group scheme, which is non-canonically isomorphic to*

$$\mathbf{Pic}_{\mathring{\mathrm{X}}_\infty} \simeq \left(\coprod_{i=1}^{|(C\cdot C)|} \mathbf{Pic}^0_{X_\infty} \right) \times_{\mathbb{C}} \widehat{\mathbf{Br}}_X,$$

where the formal group scheme

$$\widehat{\mathbf{Br}}_X := \operatorname{Spf} \widehat{Sym}_{\mathbb{C}}(H^2(X, \mathcal{O}_X)^*)$$

is the formal Brauer group of the surface X.

Remark 2. The schemes $\mathbf{Pic}_{\mathring{X}_\infty}$ constructed above are analogs of the Sato Grassmanian $Gr(\mathbb{C}((u)))$ for the case of two-dimensional local field $\mathbb{C}((u))((t))$.

Remark 3. The results on ribbons and Picard schemes of ribbons can be formulated and proved not only for the based field $\mathbb{C}$, but for any algebraically closed field of characteristic zero.