Lax operator algebras and Hamiltonian integrable hierarchies

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GEOQUANT 2009

Outline

Geometric data

Riemann surface Σ. Classical Lie algebra $\mathfrak{g}$ over $\mathbb{C}$.

Fixed points $P_1, \ldots, P_N \in \Sigma$, $N \in \mathbb{Z}_+$

Points $\gamma_1, \ldots, \gamma_K \in \Sigma$, $K \in \mathbb{Z}_+$.

Vectors $\alpha_1, \ldots, \alpha_K \in \mathbb{C}^n$ associated with γ's.

γ's and α's are joined under the name Tyurin data, because of the

Theorem (*A.N.Tyurin):* Let $g = genus\ \Sigma$, $n \in \mathbb{Z}_+$. Then there is a 1-to-1 correspondence between the following data:
1) points $\gamma_1, \ldots, \gamma_{ng}$ of Σ ;
2) $\alpha_1, \ldots, \alpha_{ng} \in \mathbb{C}P^{n-1}$.
and the equivalence classes of the semi-stable holomorphic rank n vector bundles on Σ.

Lax operators on Riemann surfaces

$$\boxed{\{\text{fixed geometric data}\} \rightarrowtail L \in \Gamma(\Sigma \times \mathfrak{g} \to \Sigma)}$$

s.t. L has arbitrary poles at P_i's, simple or double poles at γ's, is holomorphic elsewhere, and at every γ is of the form

$$L(z) = \frac{L_{-2}}{(z-z_\gamma)^2} + \frac{L_{-1}}{(z-z_\gamma)} + L_0 + L_1(z-z_\gamma) + O(z-z_\gamma)$$

where z is a local coordinate at γ, $z_\gamma = z(\gamma)$, $\alpha, \beta \in \mathbb{C}^n$ (α associated with γ, β arbitrary), and the following relations hold:

$$\boxed{L_{-2} = \nu\alpha\alpha^t\sigma} \quad \boxed{L_{-1} = (\alpha\beta^t + \varepsilon\beta\alpha^t)\sigma} \quad \boxed{\beta^t\sigma\alpha = 0} \quad \boxed{L_0\alpha = k\alpha}$$

<u>L is called a Lax operator with a spectral parameter on the Riemann surface Σ.</u> ν, ε, σ depend on $\mathfrak{g}$ ▸ More

Lax operator algebras

THEOREM (*Krichever-Sh., 2007):* For a fixed Tyurin data the space of Lax operators is closed with respect to the point-wise commutator $[L, L'](P) = [L(P), L'(P)]$ $(P \in \Sigma)$ (in the case $\mathfrak{g} = \mathfrak{gl}(n)$) also with resp. to the point-wise multiplication).

Comment

It is called the Lax operator algebra and denoted by $\overline{\mathfrak{g}}$.

THEOREM (*Kr.-Sh., 2007):* (almost graded structure)

(1) $\overline{\mathfrak{g}} = \bigoplus_{m=-\infty}^{\infty} \mathfrak{g}_m$. (2) $\dim \mathfrak{g}_m = \dim \mathfrak{g}$. (3) $[\mathfrak{g}_k, \mathfrak{g}_l] \subseteq \bigoplus_{m=k+l}^{k+l+g} \mathfrak{g}_m$.

THEOREM: If $\mathfrak{g}$ is simple then $\overline{\mathfrak{g}}$ has only one almost graded central extension, up to equivalence (*Schlichenmaier-Sh., 2008*). It is given by a cocycle $\gamma(L, L') = -\operatorname{res}_{P_\infty} \operatorname{tr}(LdL' - [L, L']\theta)$ where θ is a certain 1-form (*Kr.-Sh., 2007*).

M-operators

$M = M(z, \alpha, \beta, \gamma, k, \dots)$ is defined by the same constrains as L, excluding $\beta^t \sigma \alpha = 0$ and $L_0 \alpha = k\alpha$, namely

$$M = \frac{M_{-2}}{(z - z_\gamma)^2} + \frac{M_{-1}}{z - z_\gamma} + M_0 + M_1(z - z_\gamma) + O(z - z_\gamma)$$

where

$$\boxed{M_{-2} = \lambda \alpha \alpha^t \sigma} \quad \boxed{M_{-1} = (\alpha \mu^t + \varepsilon \mu \alpha^t)\sigma}$$

M also takes values in $\mathfrak{g}$.

Lax equations

$L = L(z, \alpha, \beta, \gamma, k, \dots)$, $M = M(z, \alpha, \beta, \gamma, k, \dots)$.

For variative Tyurin data the collection of equations on α's, β's, γ's, k's ... equivalent to the relation

$$\dot{L} = [L, M]$$

is called a Lax equation.

Motion equations of Tyurin data assigned to a point γ:

$$\dot{z}_\gamma = -\mu^t \sigma \alpha, \quad \dot{\alpha} = -M_0 \alpha + k_a.$$

Besides, there are motion equations of main parts of the func. L at P_i's.

$D := \sum m_i P_i$ $(i = 1, \dots, N, \infty)$, s.t. supp $D \cap \{\gamma\} = \varnothing$.

$\mathcal{L}^D := \{(\alpha, \beta, \gamma, k, \dots) | (L) + D \geq 0$ outside γ's$\}$.

Under a certain (effective) condition a Lax equation defines the flow on $\mathcal{L}^D$.

Examples

1) $g = 0$, $\alpha = 0$ (i.e. $\Sigma = \mathbb{C}P^1$, the bundle is trivial), $P_1 = 0$, $P_2 = \infty$. Then $\bar{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ — loop algebra.

It yields a conventional Lax equation with a rational spectral parameter:

$$L_t = [L, M], \quad L, M \in \mathfrak{g} \otimes \mathbb{C}[\lambda^{-1}, \lambda), \quad \lambda \in \mathcal{D}^1$$

(I.Gelfand, L.Dikii, I.Dorfman, A.Reyman, M.Semenov-Tian - Shanskii, V.Drinfeld, V.Sokolov, V.Kac, P. van Moerbeke). All known integrable cases of motion and hydrodynamics of a solid body.

2) Elliptic curves: Calogero-Moser systems (to be discussed later).

3) Arbitrary genus: Hitchin systems

Hierarchy of commuting flows

THEOREM: Given a generic L, there is a family of M-operators $M_a = M_a(L)$ $(a = (P_i, n, m), n > 0,\ m > -m_i)$ uniquely defined up to normalization, such that outside the γ-points M_a has pole at the point P_i only, and in the neighborhood of P_i

$$M_a(w_i) = w_i^{-m} L^n(w_i) + O(1),$$

The equations

$$\partial_a L = [L, M_a],\ \partial_a = \partial/\partial t_a \qquad (1)$$

define a family of commuting flows on an open set of $\mathcal{L}^D$.

$(\mathfrak{g} = \mathfrak{gl}(n))$ — *Krichever, 2001; other classical Lie algebras — Sh., 2008*).

Krichever-Phong symplectic structure

We define an external 2-form on $\mathcal{L}^D$. For $L \in \mathcal{L}^D$ let Ψ be a matrix-valued function formed by the eigenvectors of L: $L\Psi = \Psi K$ (K — diagonal).

$$\Omega := \mathrm{tr}(\Psi^{-1}\delta L \wedge \delta\Psi - \Psi^{-1}\delta\Psi \wedge \delta K) = \delta \mathrm{tr}(\Psi^{-1} L \delta\Psi)$$

where $\delta\Psi$ is the differential of Ψ in $\alpha, \beta, \ldots$.

Let dz be a holomorphic 1-form on Σ and

$$\omega := \sum \mathrm{res}_{\gamma_s} \Omega dz + \sum \mathrm{res}_{P_i} \Omega dz$$

THEOREM: ω is a symplectic form on $\mathcal{L}^D$.

($\mathfrak{gl}(n)$ — Kr., 2001; other class. Lie algs. — Sh., 2008)

Hamiltonians

THEOREM ($\mathfrak{gl}(n)$ — *Krichever, 2001; other class. Lie algs — Sh., 2008*): The equations of the above commutative family are Hamiltonian with respect to the *Krichever-Phong symplectic structure* on $\mathcal{L}^D$, with the Hamiltonians given by

$$H_a = -\frac{1}{n+1}\,\mathrm{res}_{P_i}\, tr(w_i^{-m}L^{n+1})dw_i$$

Example. Let D be a divisor of a holomorphic 1-form. Then $\mathcal{L}^D \simeq T^*(\mathcal{M}_0)$ where $\mathcal{M}_0$ is an open subset of the moduli space of holomorphic vector bundles on Σ, H_a are Hitchin Hamiltonians.

Calogero-Moser systems

$\mathfrak{g}=\mathfrak{gl}(n)$: $L_{i,j}:=\dfrac{\sigma(z+q_i-q_j)\sigma(z-q_i)\sigma(q_j)}{\sigma(z)\sigma(z-q_j)\sigma(q_i-q_j)\sigma(q_i)}\ (i\neq j),\quad L_{ii}=p_i$

$$H=-\frac{1}{2}\operatorname{res}_{z=0}\operatorname{tr}(z^{-1}L^2)=-\frac{1}{2}\sum_{j=1}^{n}p_j^2+\sum_{i<j}\wp(q_i-q_j).$$

Tyurin parameters: (q_i,e_i), $e_i=(\ldots,\delta_{ij},\ldots)^t$, $\omega=\sum dp_i\wedge dq_i$.

$\mathfrak{g}=\mathfrak{so}(2n)$: $L=\begin{pmatrix}A & B\\ C & -A^t\end{pmatrix}\in\mathfrak{so}(2n),\quad B^t=-B,\quad C^t=-C.$

$A_{i,j}$ is the same as $L_{i,j}$ above.

$$B_{ij}=\frac{\sigma(z+q_j+q_i)\sigma(z-q_j)}{\sigma(z)\sigma(z+q_i)\sigma(q_i+q_j)},\quad C_{j\,i}=-\frac{\sigma(z-q_j-q_i)\sigma(z+q_i)}{\sigma(z)\sigma(z-q_j)\sigma(q_i+q_j)},\quad i<j.$$

$$H=-\frac{1}{2}\operatorname{res}\operatorname{tr}(z^{-1}L^2)=-\sum_{i=1}^{n}p_i^2+2\sum_{i<j}\wp(q_i-q_j)+2\sum_{i<j}\wp(q_i+q_j)$$

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Thank you

let deg $L = -m$, deg $M = -n$. Then $\deg[L, M] = -(m+n)$.
By Riemann-Roch theorem
<u>the number of unknowns</u> $= n^2(m+n-2g+1)$,
$(= \dim\{L\} + \dim\{M\})$, <u>but</u>
<u>the number of equations</u> $= n^2(m+n-g+1)$
$(= \dim\{[L, M]\})$.

Go back

$\boxed{L_{-2} = \nu\alpha\alpha^t\sigma}$ $\boxed{L_{-1} = (\alpha\beta^t + \varepsilon\beta\alpha^t)\sigma}$ $\boxed{\beta^t\sigma\alpha = 0}$ $\boxed{L_0\alpha = k\alpha}$

$\nu \in \mathbb{C}$, $\beta \in \mathbb{C}^n$, σ is a $n \times n$ matrix,

$$\begin{aligned} \nu \equiv 0, \varepsilon = 0, \quad \sigma = id \ \text{ for } \mathfrak{g} = \mathfrak{gl}(n), \mathfrak{sl}(n), \\ \nu \equiv 0, \varepsilon = -1, \ \sigma = id \ \text{ for } \mathfrak{g} = \mathfrak{so}(n), \\ \varepsilon = 1 \qquad\qquad \text{ for } \mathfrak{g} = \mathfrak{sp}(2n), \end{aligned} \tag{2}$$

and σ is a matrix of the symplectic form for $\mathfrak{g} = \mathfrak{sp}(2n)$.

In addition we assume that

$$\alpha^t\alpha = 0 \ \text{ for } \ \mathfrak{g} = \mathfrak{so}(n) \tag{3}$$

and

$$\alpha^t\sigma L_1\alpha = 0 \ \text{ for } \ \mathfrak{g} = \mathfrak{sp}(2n). \tag{4}$$

This means, in particular, that the commutator (the product) of Lax operators again has a simple or 2d order poles at γ's (depending on $\mathfrak{g}$), and the eigenvalue condition is preserved as well as the other relations.

Go back