Quantum elliptic Gaudin model

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$\mathfrak{sl}_2$ explicit form

Let us consider the $\mathfrak{sl}_2$ Gaudin model defined by commuting operators

$$H_i = \sum_{i \neq j} \frac{h_i h_j/2 + e_i f_j + e_j f_i}{z_i - z_j}$$

on the representation $V_\lambda = V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$ where V_{λ_i} is the λ_i-highest weight irreducible representation.

Bethe ansatz

$$\Omega = \prod_{j=1}^{M} C(\mu_j)|vac> \qquad C(z) = \sum \frac{f_i}{z - z_i}$$

is an eigenvector if the parameters μ_j satisfy the system of algebraic equation (Bethe system)

$$-\frac{1}{2} \sum_i \frac{\lambda_i}{\mu_j - z_i} + \sum_{k \neq j} \frac{1}{\mu_j - \mu_k} = 0 \quad j = 1, \ldots, M$$

If Ω is a Bethe vector with eigenvalues H_i^Ω then the equation

$$\left(\partial^2 - \frac{1}{4}\sum_i \frac{\lambda_i(\lambda_i+2)}{(z-z_i)^2} - \sum_i \frac{H_i^\Omega}{z-z_i}\right)\Psi(z) = 0$$

has a solution of the form

$$\Psi(z) = \prod_i (z-z_i)^{-\lambda_i/2}\prod_j (z-\mu_j)$$

where μ_j solve the system of Bethe equation. The eigenvalues of H_i on Ω are given by

$$H_i^\Omega = -\lambda_i\left(\sum_j \frac{1}{z_i-\mu_j} - \frac{1}{2}\sum_{j\neq i}\frac{\lambda_j}{z_i-z_j}\right)$$

Let us consider the Gaudin Lax operator for the $\mathfrak{sl}_2$ case

$$L = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} = \sum_i \frac{\Phi_i}{z - z_i}$$

where

$$\Phi_i = \begin{pmatrix} h_i/2 & e_i \\ f_i & -h_i/2 \end{pmatrix}$$

Let us define the quantum characteristic polynomial:

$$det(L(z) - \partial_z) = \partial_z^2 - \sum_i \frac{C_i^{(2)}}{(z - z_i)^2} - \sum_i \frac{H_i}{z - z_i}$$

Let $\mathcal{H}$ be the algebra generated by the coefficients of the quantum characteristic polinomial. Let χ be the character on it such that $\chi(C_i^{(2)}) = \frac{1}{4}(\lambda_i + 2)\lambda_i$

Theorem (MTV, math.AG/0512299)

There is a one-to-one correspondence between the set of characters χ such that the differential equation

$$\chi(det(L(z) - \partial_z))\Psi(z) = 0$$

has monodromy ± 1 and the set of common eigenvectors for the Gaudin model.

Hitchin system and generalities

Consider the moduli space of holomorphic bundles $\mathcal{M} = \mathcal{M}_{r,d}(z_1, \ldots, z_k)$ and its cotangent bundle $T^*\mathcal{M}$. The cotangent vector is an element of the space

$$\Phi \in H^0(End(E) \otimes \mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^{k} z_i))$$

Let π be the projection corresponding to the bundle $\mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^{k} z_i)$. The kernel of the map

$$0 \to \mathcal{L} \longrightarrow \pi^* E \stackrel{\Phi - \mu * Id}{\longrightarrow} \pi^*(E \otimes \mathcal{K} \otimes \mathcal{O}(\sum_{i=1}^{k} z_i))$$

defines a spectral curve Σ which is the support of the sheaf and a line bundle $\mathcal{L}$ on it. The spectral curve defines a Poisson-commuative subalgebra, the linear coordinates on $Jac(\Sigma)$ of the $\mathcal{L}$ provide the angle variables of the Hitchin system.

Lax operator

The Gaudin model can be considered as the generalized Hitchin system on curves with marked points. One should take the curve $\Sigma = \mathbb{C}P^1$ with N marked points $z_1, \ldots, z_N$. The Higgs field in this case is $\Phi = L(z)dz$ where

$$L(z) = \sum_{i=1\ldots N} \frac{\Phi_i}{z - z_i} = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}$$

$S(\mathfrak{gl}_n)^{\otimes N} \simeq \mathbb{C}[\mathfrak{gl}_n^* \oplus \ldots \oplus \mathfrak{gl}_n^*]$ is endowed with the Kirillov Poisson bracket whiten in coordinates as follows:

$$\{(\Phi_i)_{kl}, (\Phi_j)_{mn}\} = \delta_{ij}(\delta_{lm}(\Phi_i)_{kn} - \delta_{nk}(\Phi_i)_{ml}).$$

Characteristic polynomial

$$det(L(z) - \lambda) = \sum_{k=0}^{n} I_k(z)\lambda^{n-k}$$

Alternative basis of symmetric functions:

$$J_k(z) = TrL^k(z) \qquad k = 1, \ldots, n.$$

$$H_{2,k} = Res_{z=z_k} TrL^2(z) = \sum_{j \neq k} \frac{2Tr\Phi_k\Phi_j}{(z_k - z_j)} = 2\sum_{j \neq k} \frac{\sum_{lm} e_{lm}^{(k)} e_{ml}^{(j)}}{z_k - z_j}$$

Quantum spectral curve

Consider the quantum Lax operator:

$$L(z) = \sum_{ij} E_{ij} \otimes \sum_{s=1}^{N} \frac{e_{ij}^{(s)}}{z - z_s}.$$

Here $L(z)$ is a rational function in z with values in $End(\mathbb{C}^n) \otimes U(\mathfrak{gl}_n)^{\otimes N}$. We define the quantum characteristic polynomial of the quantum Lax operator

$$det(L(z) - \partial_z) = \sum_{k=0}^{n} QI_k(z)\partial_z^{n-k}$$

Theorem

The coefficients $QI_k(z)$ commute

$$[QI_k(z), QI_m(u)] = 0$$

they quantize the classical Hamiltonians for the Gaudin model.

Matrix monodromy condition

Let us consider the Fuchsian system

$$(\partial_z - A(z))\Psi(z) = 0 \qquad A(z) = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix} = \sum_{i=1}^{k} \frac{A_i}{z - z_i}$$

with additional conditions

$$Tr(A_i) = 0; \quad Det(A_i) = -d_i^2; \quad \sum_i A_i = \begin{pmatrix} \kappa & 0 \\ 0 & -\kappa \end{pmatrix}.$$

This system is related with the following Sturm-Liouville operator

$$\Phi'' + U\Phi = 0 \quad \Phi = \psi_1/\sqrt{a_{12}}$$

where the potential is given by the formula

$$U = \sum_{j=1}^{k-2} \frac{-3/4}{(z - w_j)^2} + \sum_{i=1}^{k} \frac{1/4 + detA_i}{(z - z_i)^2} + \sum_{j=1}^{k-2} \frac{H_{w_j}}{z - w_j} + \sum_{i=1}^{k} \frac{H_{z_i}}{z - z_i}$$

and the points w_j are defined by the condition

$$a_{12}(z) = \frac{c\prod_{j=1}^{k-2}(z - w_j)}{\prod_{i=1}^{k}(z - z_i)}$$

Theorem, math-ph 0802.0383

The equation

$$(\partial_z - A(z))\Psi = 0 \tag{1}$$

has solutions of the form

$$\Psi = \prod_{i=1}^{k}(z - z_i)^{-d_i} \begin{pmatrix} \phi_1(z) \\ \phi_2(z) \end{pmatrix} \tag{2}$$

$$\phi_1 = \prod_{j=1}^{M}(z - \gamma_j); \quad \phi_2/\phi_1 = \sum_{j=1}^{M} \frac{\alpha_j}{z - \gamma_j} \tag{3}$$

iff the set of numbers γ_i where $i = 1, \ldots, M$ satisfy the system of Bethe equations with parameters: the set of poles is $z_1, \ldots, z_k$ and $w_1, \ldots, w_{k-2}$ with the highest weights $2d_1 - 1, \ldots, 2d_k - 1$ and $1, \ldots, 1$ correspondingly.

Schlesinger transformation

Action on bundles

Consider C - a curve, F a holomorphic bundle on C, $\mathcal{F}$ - the corresponding sheaf, $x \in C$ and $l \in F_x^*$ then the lower Hecke transformation $T_{(x,l)}F$ is defined in the language of sheaves as the subsheaf $\mathcal{F}' = \{s \in \mathcal{F} : (s(x), l) = 0\}$

Action on connections

A connection is the map of sheaves

$$\Delta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1$$

The Schlesinger action can be extended to the space of connection preserving the $Ann_l = \{v \in \mathcal{F}_x :< l, v >= 0\}$

$$\Delta_x : Ann_l \to Ann_l \otimes \Omega^1_x$$

The local consideration shows that the eigenvalues of the residues A_i transform depending on the choice of the subspace and the type (upper or lower) of the Schlesinger transformations correspondingly to the following table

$$\begin{array}{rcl}(\ldots,\lambda_i,\ldots,\lambda_j,\ldots) & \longmapsto & (\ldots,\lambda_i+1,\ldots,\lambda_j-1,\ldots),\\ (\ldots,\lambda_i,\ldots,\lambda_j,\ldots) & \longmapsto & (\ldots,\lambda_i+1,\ldots,\lambda_j+1,\ldots),\\ (\ldots,\lambda_i,\ldots,\lambda_j,\ldots) & \longmapsto & (\ldots,\lambda_i-1,\ldots,\lambda_j-1,\ldots),\\ (\ldots,\lambda_i,\ldots,\lambda_j,\ldots) & \longmapsto & (\ldots,\lambda_i-1,\ldots,\lambda_j+1,\ldots).\end{array}$$

Let us define the odd θ-functions on $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ and $\tau \in \mathbb{C}$, $\Im\tau > 0$

$$\theta(u+1) = -\theta(u), \qquad \theta(u+\tau) = -e^{-2\pi i u - \pi i \tau}\theta(u), \qquad \theta'(0) = 1$$

Let $\{e_i\}$ be a standard basis of $\mathbb{C}^n$ and $\{E_{ij}\}$ be a standard basis of $End(\mathbb{C}^n)$, that is $E_{ij}e_k = \delta^j_k e_i$.

$e^{(s)}_{ij}$ are generators of the s-th copy of $\mathfrak{gl}_n \subset U(\mathfrak{gl}_n)^{\otimes N}$

$h_i = \sum_s e^{(s)}_{ii}$ are the Cartan elements of the diagonal action.

$$L(u;\lambda) = \sum_{ij=1}^{n} E_{ij} \otimes e_{ji}(u;\lambda)$$

where the coefficients are expressed via

$$e_{ii}(u;\lambda) = \sum_{s=1}^{N} \frac{\theta'(u-v_s)}{\theta(u-v_s)} e^{(s)}_{ii},$$

$$e_{ij}(u;\lambda) = \sum_{s=1}^{N} \frac{\theta(u-v_s+\lambda_i-\lambda_j)}{\theta(u-v_s)\theta(\lambda_i-\lambda_j)} e^{(s)}_{ij},$$

The quantum characteristic polinomial

QCP

$$Q(u,\partial_u) = \det\Big(\frac{\partial}{\partial u} - \hat{D}_\lambda + L(u;\lambda)\Big) = \sum_{m=0}^{n} s_m(u)\Big(\frac{\partial}{\partial u}\Big)^{n-m}$$

where $\hat{D}_\lambda = \sum_{l=1}^{n} E_{ll}\frac{\partial}{\partial \lambda_l}$

Theorem, A. Silantiev, V. Rubtsov, D. Talalaev, ITEP-TH-56/08

$$[s_m(u), s_l(v)] = 0 \mod \mathfrak{h}$$

R-matrix

The Felder *R*-matrix is given by the expression:

$$R(u;\lambda_k) = \frac{\theta(u+\hbar)}{\theta(u)} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} +$$

$$+ \sum_{i \neq j} \Big(\frac{\theta(\lambda_{ij}+\hbar)}{\theta(\lambda_{ij})} E_{ii} \otimes E_{jj} + \frac{\theta(u-\lambda_{ij})\theta(\hbar)}{\theta(u)\theta(-\lambda_{ij})} E_{ij} \otimes E_{ji} \Big),$$

where $\lambda_{ij} = \lambda_i - \lambda_j$.

$$R^{(12)}(u_1-u_2;\lambda_k) R^{(13)}(u_1-u_3;\lambda_k+\hbar E_{kk}^{(2)}) R^{(23)}(u_2-u_3;\lambda_k) =$$
$$= R^{(23)}(u_2-u_3;\lambda_k+\hbar E_{kk}^{(1)}) R^{(13)}(u_1-u_3;\lambda_k) R^{(12)}(u_1-u_2;\lambda_k+\hbar E_{kk}^{(3)})$$

Lax operator

The Lax operator takes the form:

$$\mathcal{L}(u;\lambda_k) = \begin{pmatrix} h(u)/2 & f(u,\lambda) \\ e(u,\lambda) & -h(u)/2 \end{pmatrix},$$

where the half-currents are

$$h(u) = \sum_{s=1}^{N} \frac{\theta'(u - v_s)}{\theta(u - v_s)} (e_{11}^{(s)} - e_{22}^{(s)}),$$

$$e(u,\lambda) = \sum_{s=1}^{N} \frac{\theta(u - v_s + \lambda)}{\theta(u - v_s)\theta(\lambda)} e_{12}^{(s)},$$

$$f(u,\lambda) = \sum_{s=1}^{N} \frac{\theta(u - v_s - \lambda)}{\theta(u - v_s)\theta(-\lambda)} e_{21}^{(s)}.$$

QCP

$$Q(u,\partial_u) \quad = \quad \det\left(\frac{\partial}{\partial u} \; - \; \hat{D}_\lambda \; + \; \mathcal{L}(u;\lambda)\right) \quad = \quad \left(\frac{\partial}{\partial u}\right)^2 \; - \; S_\lambda(u),$$

where

$$S_\lambda(u) = \big(\partial_\lambda - h(u)/2\big)^2 + \big(f(u,\lambda)e(u,\lambda) + e(u,\lambda)f(u,\lambda)\big)/2.$$

SoV

Let us consider the $\mathfrak{sl}_2$ Gaudin model and fix the representation $V = V_1 \otimes \ldots \otimes V_N$ where V_i is the Λ_i-highest weight module. Let us realize V_i as the subspace in the space of functions on a variable t_i such that the generators of $\mathfrak{sl}_2$ take the form:

$$h^{(s)} = -2t_s \frac{\partial}{\partial t_s} + \Lambda_s, \qquad e^{(s)} = -t_s \frac{\partial^2}{\partial t_s^2} + \Lambda_s \frac{\partial}{\partial t_s}, \qquad f^{(s)} = t_s.$$

We introduce the variables y_j defined by the formula:

$$\sum_{s=1}^{N} \frac{\theta(u - v_s - \lambda)}{\theta(u - v_s)\theta(-\lambda)} t_s = C \prod_{s=1}^{N} \frac{\theta(u - y_s)}{\theta(u - v_s)}.$$

Eigenproblem

Let us consider the eigenvector of the elliptic Gaudin model:

$$S_\lambda(u)\Omega(C, y_1, \dots, y_N) = s_\lambda(u)\Omega(C, y_1, \dots, y_N)$$

Substituting $u = y_j$ on the right one obtains:

$$\Big(\frac{\partial}{\partial y_j} - \frac{1}{2}\sum_{s=1}^{N} \frac{\theta'(y_j - v_s)}{\theta(y_j - v_s)}\Lambda_s\Big)^2 \Omega(C, y_1, \dots, y_N) = s_\lambda(y_j)\Omega(C, y_1, \dots, y_N)$$

This implies the product formula:

$$\Omega(C, y_1, \dots, y_N) = C^a \prod_j \omega(y_j)$$

Then $w(u) = \prod_{s=1}^{N} \theta(u - v_s)^{-\Lambda_s/2}\omega(u)$ solves the equation:

$$\big(\partial_u^2 - s_\lambda(u)\big)\, w(u) = 0$$

Isomonodromic formulation

The differential equation in separated variables is of the form

$$\left(\partial_u^2 - \sum c_i \wp(u - v_i) - \sum H_i \frac{\theta'(u - v_i)}{\theta(u - v_i)}\right) w(u) = 0$$

There is an equivalent matrix linear problem

$$(\partial_u - A(u))\Psi(u) = 0 \qquad A_i = res_{u=u_i} A(u)$$

where

$$A(u) = \begin{pmatrix} A_{11}(u) & A_{12}(u) \\ A_{21}(u) & A_{22}(u) \end{pmatrix} = \begin{pmatrix} \sum a_{11}^i \frac{\theta'(u-u_i)}{\theta(u-u_i)} & \sum a_{12}^i \frac{\theta(u-u_i-\lambda)}{\theta(u-u_i)\theta(-\lambda)} \\ \sum a_{21}^i \frac{\theta(u-u_i+\lambda)}{\theta(u-u_i)\theta(\lambda)} & \sum a_{11}^i \frac{\theta'(u-u_i)}{\theta(u-u_i)} \end{pmatrix}$$

Correspondence

$$w'' - Uw = 0 \quad w = \psi_1/\sqrt{a_{12}}$$

where

$$U(u) = -\sum(1/4 + det(A_i))\wp(u - u_i) + \sum 3/4\wp(u - w_i) + ...$$

and the points w_j are defined by the condition

$$a_{12}(u) = c\frac{\prod \theta(u - w_i)}{\prod \theta(u - u_i)}$$