An asymptotic Euler-Maclaurin formula for Delzant polytopes

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Plan of the Talk

(1) Background and History

  (a) classical asymptotic Euler-Maclaurin formula

  (b) The Riemann sums over lattice polytopes

  (c) Buckground from spectral analysis on toric varieties

  (d) related works

(2) Results

  (a) Main Theorem

  (b) The 3^{rd} term

  (c) Relation to the work of Berline-Vergne

(3) One-dimensional computation (along with the method of proof of the main theorem)

Background and History

◊ Classical asymptotic Euler-Maclaurin formula

φ: a smooth function on $[0, 1]$

$$\frac{1}{N}\sum_{k=1}^{N}\varphi(k/N) \sim \int_0^1 \varphi(x)\,dx + \frac{1}{2N}(\varphi(1)-\varphi(0)) +$$

$$+\sum_{n\geq 1}\frac{(-1)^{n-1}B_n}{(2n)!}\left(\varphi^{(2n-1)}(1)-\varphi^{(2n-1)}(0)\right)N^{-2n},$$

where

$$\mathrm{Todd}(z) = \frac{z}{1-e^{-z}} = \sum_{n=0}^{\infty}(-1)^n\frac{b_n}{n!}z^n,$$

$$b_0 = 1, \quad b_1 = -\frac{1}{2}, \quad b_{2n+1} = 0 (n \geq 1),$$

$$b_{2n} = (-1)^{n-1}B_n \quad (B_n\text{: Bernoulli number})$$

◊ The Riemann sum over lattice polytopes

$P \subset \mathbb{R}^m$: A lattice polytope $\overset{\text{def}}{\Longleftrightarrow}$ vertices of $P \subset \mathbb{Z}^m$.

Definition: Define a Riemann sum of $\varphi \in C^\infty(P)$ over a lattice polytope $P \subset \mathbb{R}^m$ by:

$$R_N(P;\varphi) := \frac{1}{N^{\dim(P)}} \sum_{\gamma \in (NP)\cap \mathbb{Z}^m} \varphi(\gamma/N), \quad N \in \mathbb{Z}_{>0}$$

Asymptotic Euler-Maclaurin formula

$=$ Asymptotic expansion formula for $R_N(P;\varphi)$.

◊ Purpose of the talk: To give asymptotic Euler-Maclaurin formula which is effective for computation (for Delzant polytopes).

$\diamondsuit$ Terminology for lattice polytopes

$P \subset \mathbb{R}^m$: a lattice polytope.

Definition:

(1) P is **simple** $\overset{\text{def}}{\Longleftrightarrow}$ For each vertex v of P,

$$\sharp \text{ of edges (1-dim. faces) incident to } v = m.$$

(2) P is **Delzant** $\overset{\text{def}}{\Longleftrightarrow}$ P is simple and for each vertex v of P, there is a $\mathbb{Z}$-basis $w_1, \dots, w_m$ of $\mathbb{Z}^m$ such that

$$\text{each edge incident to } v \subset \{v + tw_j \,;\, t \geq 0\}$$
$$\text{for some } j = 1, \dots, m.$$

Remark: Roughly speaking:

Simple polytopes corresponds to toric varieties with quotient singularities.

Delzant polytopes corresponds to smooth projective toric varieties.

◊ Background from spectral analysis on toric varieties

(X, ω): a smooth toric Kähler manifold with Kähler form ω obtained by a GIT (symplectic) quotient $X = \mathbb{C}^d /\!\!/ T^r$ with respect to a suitable subtorus $T^r \subset T^d \curvearrowright (\mathbb{C}^d, \omega_{\mathrm{std}})$.

$\Longrightarrow$

$T^m \curvearrowright X$: Hamiltonian, $m = d - r$.

The moment map $\mu : X \to \mathbb{R}^m$ satisfy $\mu(X) = P$, where P is an m-dimensional Delzant polytope, which is realized as

$$P \cong \{x \in \mathbb{R}^d_+ \, ; \sum x_j \alpha_j = \alpha\}$$

with some $\alpha \in \mathbb{Z}^d$, where α_j $(j = 1, \ldots, d)$ are the weights of $T^r_{\mathbb{C}} \curvearrowright \mathbb{C}^d$.

$L \longrightarrow X$: a Hermitian line bundle obtained from the trivial bundle over $\mathbb{C}^d$.

$$H^0(X, L^{\otimes N}) \cong \bigoplus_{\gamma \in NP \cap \mathbb{Z}^d} \mathbb{C} \cdot \chi_\gamma$$

$$\subset L^2(\mathbb{C}^d, e^{-N|z|^2} dz d\bar{z}),$$

χ_γ monomial on $\mathbb{C}^d$ with weight γ.

Spectral measure:

$\pi_N : L^2(\mathbb{C}^d, e^{-N|z|^2} dz d\bar{z}) \longrightarrow H^0(X, L^{\otimes N})$: orthog. proj.

$f \in C^\infty(\mathbb{C}^d)$: T^d-invariant bounded function.

M_f: multiplication by f.

$$\nu_N(f) := \mathrm{trace}(\pi_N M_f \pi_N) : \text{ a measure on } \mathbb{R}^d.$$

Then, we have (Guillemin-Wang)

$$\nu_N(f) = \sum_{\gamma \in NP \cap \mathbb{Z}^d} (A_N f)(\gamma/N),$$

$A_N f$: 'twisted Mellin transform' defined by Wang

$(NP) \cap \mathbb{Z}^d$ can be replaced by $NP \cap \mathbb{Z}^m$ (P is m-dimensional).

It is known that $A_N f$ admits asymptotic expansion in N (Wang).

$\Longrightarrow$

$\nu_N(f)$ reduces to the Riemann sums $R_N(P; \varphi)$

where φ are functions appeaing the expansion of $A_N f$.

◊ Related works Suppose that $P \subset \mathbb{R}^m$: Delzant polytope.

(1) Khovanskii-Pukhlikov (1993), Brion-Vergne (1997):

$$R_N(P;\varphi) = \mathrm{Todd}(P;\partial/N\partial h) \int_{P_h} \varphi(x)\,dx \Big|_{h=0}$$

$h = (h_1, \dots, h_d) \in \mathbb{R}^d$: small parameter, φ: polynomial on $\mathbb{R}^m$,

If $P = \{x \in \mathbb{R}^m \,;\, \langle\, x, u_j \,\rangle \geq c_j \ (j = 1, \dots, d)\}$ then
$P_h = \{x \in \mathbb{R}^m \,;\, \langle\, x, u_j \,\rangle \geq c_j + h_j \ (j = 1, \dots, d)\}$,

$$\mathrm{Todd}(P;\partial/N\partial h) = \prod_{i=1}^{d} \mathrm{Todd}(\partial/N\partial h_i)$$

Remark:

- Brion-Vergne obtained similar formula for simple P with a modification of $\mathrm{Todd}(P;\partial/N\partial h)$.
- Brion-Vergne proved the above without using toric geometry.

(2) Berline-Vergne (2007): Set $N = 1$ for simlicity. For each face f of P, there exists a differential operator $\mu(P; f)$ of infinite order such that

$$R_1(P;\varphi) = \sum_f \int_f \mu(P;f)\varphi$$

where φ is a polynomial on $\mathbb{R}^m$.

Remark:

- Berline-Vergne obtained the same formula for arbitrary lattice polytope P for polynomial functions φ.
- They proved above without toric geometry, and give application to toric geometry.

An asymptotic Euler-Maclaurin formula is first obtained by:

(3) Guillemin-Sternberg (2007):

$$R_N(P;\varphi) \sim \mathrm{Todd}(P;\partial/N\partial h) \int_{P_h} \varphi(x)\,dx\Big|_{h=0}$$

for arbitrary smooth function φ on P.

Remark:

- Guillemin-Sternberg obtained a similar formula for simple P with some modification of $\mathrm{Todd}(P;\partial/N\partial h)$.
- The operator $\mathrm{Todd}(P;\partial/N\partial h)$ is infinite order. But the asymptotic sum has a meaning.
- They obtained the above without toric geometry. Application to spectral analysis on toric geometry.

(4) Zelditch (2007): For each $n \in \mathbb{Z}_+$, there exists a differential operator $\mathcal{E}_n(P)$ (of finite order) such that

$$R_N(P;\varphi) \sim \int_P \varphi\, dx + \frac{1}{2N}\int_{\partial P} \varphi\, d\sigma + \sum_{n\geq 2} N^{-n}\int_P \mathcal{E}_n(P)\varphi\, dx$$

where $d\sigma$ on ∂P is defined, for each facet f, by the lattice $\mathbb{Z}^m \cap L(f)$ with a subspace f parallel to f.

Remark:

- The differential operators $\mathcal{E}_n(P)$ depend on the choice of metric on a line bundle over the toric variety X.
- The formula for the second term is stated by Szekelyhidi.
- He computed the second term by using Donaldson's integration by parts identity.

Results

◊ Main theorem

Theorem $P \subset \mathbb{R}^m$: a Delzant polytope. $\mathcal{F}(P)$: the set of faces of P.

For each $n \in \mathbb{Z}_+$ and $f \in \mathcal{F}(P)$ such that $n - m + \dim(f) \geq 0$, there exists a homogeneous differential operator $D_n(P; f)$ of order $n - m + \dim(f)$ with rational constant coefficients which involves derivatives perpendicular to f such that, for $\varphi \in C^\infty(P)$,

$$R_N(P; \varphi) \sim \sum_{n \geq 0} A_n(P; \varphi) N^{-n},$$

$$A_n(P; \varphi) = \sum_{f \in \mathcal{F}(P)\,;\, \dim(f) \geq m-n} \int_f D_n(P; f)\varphi$$

Remark:

- The formula in the main theorem is rather similar to the formula of Berline-Vergne. But, the construction of $D_n(P; f)$ is independent of Berline-Vergne.
- There is an algorithm of computing $D_n(P; f)$.
- In the following, some of corollaries are presented. But, we just mention that we have a concrete formula for $A_n(P; \varphi)$ for any $n \in \mathbb{Z}_+$ when $\dim(P) = m = 2$.
- Proof uses:

(1) 'Szasz function' to obtain expansion of Riemann sum over 'unimodular cones' C. (use Hörmander's idea.)

(2) an integration by parts procedure (this makes the final formula complicated).

(3) An Euler's formula due to Brion-Vergne.

◊ The 3^{rd} term

Corollary $P \subset \mathbb{R}^m$: Delzant polytope. $\mathcal{F}(P)_k$: faces of dim. k.

$$A_2(P;\varphi) = -\frac{1}{12} \sum_{f \in \mathcal{F}(P)_{m-1}} \frac{1}{|\alpha_f|^2} \int_f \nabla_{\alpha_f} \varphi + \sum_{g \in \mathcal{F}(P)_{m-2}} c(P;g) \int_g \varphi,$$

$$c(P;g) = \frac{1}{4} - \frac{1}{12} \langle \alpha_1(g), \alpha_2(g) \rangle \left(\frac{1}{|\alpha_1(g)|^2} + \frac{1}{|\alpha_2(g)|^2} \right)$$

where, for $f \in \mathcal{F}(P)_{m-1}$, $\alpha_f \in \mathbb{Z}^m$: inward primitive normal to f,

for $g \in \mathcal{F}(P)_{m-2}$, take $f_1, f_2 \in \mathcal{F}(P)_{m-1}$ such that $g = f_1 \cap f_2$. $\alpha_i(g) \in \mathbb{Z}^m$ is the inward primitive normal to f_i $(i = 1, 2)$.

Remark:

- By our 'algorithm' for computing $D_n(P; f)$ shows easily

$$A_0(P;\varphi) = \int_P \varphi,$$

$$A_1(P;\varphi) = \frac{1}{2} \sum_{f \in \mathcal{F}(P)_{m-1}} \int_f \varphi = \frac{1}{2} \int_{\partial P} \varphi.$$

- The formula for $A_2(P;\varphi)$ seems new. Furthermore, one has

$$\int_P \mathcal{E}_2(P)\varphi = A_2(P;\varphi),$$

where $\mathcal{E}_2(P)$ is Zelditch's operator.

Question: Can one compute $\mathcal{E}_2(P)$ in terms of curvatures of the toric Kähler manifold? If yes, the above formula gives an integration by parts identities for curvatures.

◊ Relation to the work of Berline-Vergne Recall

Berline-Vergne formula:

$$R_1(P;\varphi) = \sum_f \int_f \mu(P;f)\varphi$$

where φ is a polynomial on $\mathbb{R}^m$. The differential operators $\mu(P;f)$ is of infinite order.

But its symbol is real analytic around the origin; Taylor expansion gives

$$\mu(P;f) = \sum_{k=0}^{\infty} \mu_k(P;f),$$

where $\mu_k(P;f)$ is a homogeneous diff. op. of order k.

Theorem $P \subset \mathbb{R}^m$: Delzant polytope. We have

$$\boxed{D_n(P; f) = \mu_{n-\dim(P)+\dim(f)}(P; f)}$$

for each $f \in \mathcal{F}(P)$.

Remark:

- From this, the operators $D_n(P; f)$ has a nice property; 'valuation property'
- For the proof, we use one of results of Berline-Vergne on the relation among 'polytope characters', 'exponential integral over faces' and the symbol of $\mu(P; f)$.

Computation in one dimension

Let us show, for $P = [0,1] \subset \mathbb{R}$, how our formula is proved.

(1) Extend $\varphi \in C^\infty([0,1])$ to $\varphi \in C_0^\infty(\mathbb{R})$.

(It is clear.)

(2) Compute the asymptotics of

$$R_N(\varphi) := \frac{1}{N}\sum_{k=0}^{\infty} \varphi(k/N),$$

($[0,+\infty)$ corresponds to the so-called 'feasible direction' of a face f in a polytope P.)

We use Szasz functions to compute $R_N(\varphi)$.

(3) Use the formula:

$$\begin{aligned}&R_N([0,1];\varphi) = \\ &\quad R_N(\varphi) + R_N(\psi) - R_N(\mathbb{R};\varphi),\end{aligned}$$

$$R_N(\mathbb{R};\varphi) = \frac{1}{N}\sum_{k\in\mathbb{Z}} \varphi(k/N)$$

with $\psi(x) = \varphi(1-x)$.

(This corresponds to a variant of Euler's formula due to Brion-Vergne.)

(4) Sum the results of the above.

(For general P, the 'naturality' of the differential operators $D_n(P;f)$ is important in this step. There are NO such a step for $[0,1]$.)

It is enough to consider $R_N(\varphi)$.

For one dimension, one can do more; Consider the twisted Riemann sum:

$$R_N^\omega(\varphi) := \frac{1}{N}\sum_{k=0}^{\infty}\omega^k\varphi(k/N),$$

where $\omega \in U(1)$ (q-th root of unity). One has

$$R_N(\varphi) = R_N^1(\varphi) \sim \int_0^\infty \varphi\, dx - \sum_{n\geq 1}\frac{b_n}{n!}\varphi^{(n-1)}(0)N^{-n},$$

and, for $\omega \neq 1$, by Guillemin-Sternberg;

$$R_N^\omega(\varphi) \sim \sum_{n\geq 1}(-1)^{n-1}b_n(\omega)\varphi^{(n-1)}(0)N^{-n},$$

$$\frac{s}{1-\omega e^{-s}} = \sum_{n\geq 1} b_n(\omega)s^n, \quad b_1(\omega) = \frac{1}{1-\omega}.$$

◊ Szasz functions

To obtain an expansion of $R_N^\omega(\varphi)$, we use the (twisted) Szasz function:

$$S_N^\omega(\varphi)(x) := e^{-Nx} \sum_{k=0}^{\infty} \omega^k \varphi(k/N) \frac{(Nx)^k}{k!}$$

We have

$$\int_0^\infty S_N^\omega(\varphi)(x)\, dx = R_N^\omega(\varphi)$$

Therefore:

It is enough to obtain the expansion of $S_N^\omega(\varphi)$ as $N \longrightarrow \infty$ with a suitable reminder.

To state the asymptotics of $S_N^\omega(\varphi)$; we use Stirling numbers of the 2^{nd} kind and related polynomials.

◇ Stirling ♯ of 2^{nd} kind $S(n,k)$ are defined by the recursion formula:

$$S(0,0)=1, \quad S(n,0)=0, \quad S(n,n)=1 \quad (n \geq 1)$$
$$S(n+1,k)=kS(n,k)+S(n,k-1) \quad (1 \leq k \leq n)$$

◇ Polynomials we use here are defined, for $0 \leq k \leq n$, by

$$p(n,k;z)=\sum_{t=0}^{k}\binom{n}{t}(-1)^t S(n-t,k-t)z^{k-t}, \quad z \in \mathbb{C}$$

Remark:

- Set $p(n,k) := p(n,k;1)$. Then;

$$\boxed{p(n,k)=0 \quad \text{for } [n/2]+1 \leq k \leq n.}$$

- Question: Are there any combinatorial meaning of $p(n,k;z)$?

Proposition Let $\varphi \in \mathcal{S}(\mathbb{R})$. Let $\omega \in U(1)$. Then for any $n \in \mathbb{Z}_+$ and $K > 0$ with $n < K < 2n$, $\exists C_{K,n} > 0$ such that

$$S_N^{\omega}(\varphi)(x) = \sum_{\mu=0}^{2n-1} \frac{\varphi^{(\mu)}(x)}{\mu!} J_{\mu}^{\omega}(Nx) N^{-\mu} + S_{2n,N}(x),$$

where

$$|S_{2n,N}(x)| \leq C_{K,n} N^{-n} (1+x)^{n-K}, \quad x > 0,$$

and the function $J_{\mu}^{\omega}(x)$ is given by

$$J_{\mu}^{\omega}(x) = e^{-(1-\omega)x} \sum_{k=0}^{\mu} p(\mu, k; \omega) x^k.$$

In particular, when $\omega = 1$, $J_{\mu}^{1}(x)$ is a polynomial in x of degree at most $[\mu/2]$.

By the above proposition, we have the following.

Proposition When $\omega \neq 1$, we have

$$R_N^{\omega}(\varphi) \sim \sum_{n \geq 1} c_n(\omega) \varphi^{(n-1)}(0) N^{-n},$$

$$c_n(\omega) = \sum_{\mu=0}^{n-1} \sum_{k=0}^{\mu} \frac{(n-k-1)!}{\mu!(n-\mu-1)!} \frac{p(\mu, \mu-k; \omega)}{(1-\omega)^{n-k}}.$$

For $\omega = 1$, we have

$$R_N^{1}(\varphi) \sim \int_0^{\infty} \varphi(x)\, dx + \sum_{n \geq 1} c_n \varphi^{(n-1)}(0) N^{-n},$$

$$c_n = \sum_{\mu=n}^{2n} \frac{(\mu-n)!}{\mu!} (-1)^{\mu-n+1} p(\alpha, \alpha-n).$$

Remark:

- A direct computation and a well-known formula among the number b_n (Bernoulli numbers), Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ and the Stirling numbers $S(n,k)$ shows

$$c_n = -(n+1)\binom{2n}{n}^{-1}\sum_{l=0}^{n}\frac{(-1)^l}{l+1}\binom{2n}{n+l}S(n+l,l)$$
$$= -\frac{b_n}{n!}.$$

- One should have

$$c_n(\omega) = (-1)^{n-1}b_n(\omega).$$

Question: Are there any combinatorial (or number theoretical) meaning of this formula ?

◊ Further problems

(1) Find asymptotic EM for simple (or more general lattice) polytopes in a similar form discussed as above.

(One could use the 'valuation property' of the operators $D_n(P; f)$.)

(2) Find an effective formula for Zelditch's operators $\mathcal{E}_n(P)$, and perform integration by parts to obtain a formula for $A_n(P; \varphi)$ (it might be possible to handle in the case where $n = 2$).

(We have an effective formula for $A_n(P; \varphi)$. Thus, this will give $A_n(P; \varphi)$ a geometrical meaning.)