Non-commutative structures and operators on a Hilbert space

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Motivation

Consider the operators of the coordinate and the momentum of a quantum particle

$$\hat{x}_j\psi = x_j\psi(x), \quad \hat{p}_j\psi = -i\hbar\,\frac{\partial}{\partial x_j}\psi(x), \qquad x \in \mathbb{R}^n.$$

They generate the Heisenberg algebra of polynomial (non-commutative) expressions in $\hat{x}_j, \hat{p}_j$ with the identities

$$\hat{x}_j\hat{x}_k - \hat{x}_k\hat{x}_j = 0, \quad \hat{p}_j\hat{p}_k - \hat{p}_k\hat{p}_j = 0, \quad \hat{p}_j\hat{x}_k - \hat{x}_k\hat{p}_j = -i\hbar\delta_{jk}.$$

Any element of the Heisenberg algebra corresponds to a differential operator with polynomial in x coefficients.

Natural problem. To extend the Heisenberg algebra so that its elements will still correspond to “good” operators on a reasonable function space.

We propose a certain approach to this problem. But first we develop some formalism, called *the non-commutative structures*.

Let $(\hat{\mathbb{O}}, +, \cdot)$ be the free associative algebra over $\mathbb{C}$ with m generators $\hat{z}_1, \ldots, \hat{z}_m$:

$$\hat{F} \in \hat{\mathbb{O}} \quad \Leftrightarrow \quad \hat{F} = \sum_{\hat{\zeta}} \hat{f}_{\hat{\zeta}}\, \hat{\zeta},$$

where $\hat{\zeta}$ is a word, composed from the letters $\hat{z}_1, \ldots, \hat{z}_m$, and $\hat{f}_{\hat{\zeta}} \in \mathbb{C}$.

Empty word is identified with 1. **Example**. $\hat{F} = \sum_{k=1}^{\infty} k!\, \hat{z}_1 \hat{z}_2^k \hat{z}_1^2$.

The corresponding commutative object, $(\mathbb{O}, +, \cdot)$, is a commutative associative algebra with generators $z_1, \ldots, z_m$.

$$F \in \mathbb{O} \quad \Leftrightarrow \quad F = \sum_{\alpha \in \mathbb{Z}_+^m} f_\alpha\, z^\alpha,$$

where α is a multi-index, an element of $\mathbb{Z}_+^m = \{0, 1, \ldots\}$, and $f_\alpha \in \mathbb{C}$.

Other structures are the involutions $\star : \hat{\mathbb{O}} \to \hat{\mathbb{O}}$ and $\star : \mathbb{O} \to \mathbb{O}$.
For any word $\hat{\zeta} = \hat{f}_\zeta \hat{z}_{j_1} \cdot \hat{z}_{j_k} \in \hat{\mathbb{O}}$ we have: $\hat{\zeta}^\star = \overline{\hat{f}}_\zeta \hat{z}_{j_k} \cdot \hat{z}_{j_1}$.
For any monomial $f_\alpha z^\alpha \in \mathbb{O}$ we have: $(f_\alpha z^\alpha)^\star = \overline{f}_\alpha z^\alpha$.

Then $\star$ is an anti-isomorphism and $\star^2 = \mathrm{id}$.

We define the homomorphism of "averaging" aver : $\hat{\mathbb{O}} \to \mathbb{O}$. For any word $\hat{\zeta}$

$$\operatorname{aver}(\hat{\zeta}) = z^{\alpha}, \quad \text{where } \hat{z}_j \text{ enters } \hat{\zeta} \text{ exactly } \alpha_j \text{ times.}$$

Example: $\operatorname{aver}(\hat{z}_2 \hat{z}_1^2 \hat{z}_2 \hat{z}_1^3) = z_1^5 z_2^2$.

Then aver is uniquely continued to $\hat{\mathbb{O}}$ by linearity.

Below we need the derivations

$$\partial_1, \dots, \partial_m \in \mathrm{Der}(\mathbb{O}), \quad \hat{\partial}_1, \dots, \hat{\partial}_m \in \mathrm{Der}(\hat{\mathbb{O}}).$$

By definition

$$\partial_j(z_k) = \hat{\partial}_j(\hat{z}_k) = \delta_{jk}, \qquad j, k = 1, \dots, m.$$

Hence, $\partial_j = \partial/\partial z_j$. Obviously,

$$\partial_j \ \mathrm{aver} = \mathrm{aver}\, \hat{\partial}_j, \quad \star\,\mathrm{aver} = \mathrm{aver}\,\star, \quad \partial_j \star = \star\partial_j, \quad \hat{\partial}_j \star = \star\hat{\partial}_j.$$

Let $D \subset \mathbb{R}^m$ be a domain. Consider the trivial bundles

$$\pi : D \times \mathbb{O} \to D, \quad \hat{\pi} : D \times \hat{\mathbb{O}} \to D.$$

Sections of these bundles have the form

$$F = F(\xi; z), \quad \hat{F} = \hat{F}(\xi; \hat{z}),$$

where for any $\xi \in D \;\; F \in \mathbb{O}$ and $\hat{F} \in \hat{\mathbb{O}}$.

The sections are said to be *horizontal* if

$$(\partial_{\xi_j} - \partial_j)F = 0, \quad (\partial_{\xi_j} - \hat{\partial}_j)\hat{F} = 0.$$

The algebras of horizontal sections are denoted $\mathbb{O}(D)$, $\hat{\mathbb{O}}(D)$.

For any $\xi_0 \in D$ there are obvious isomorphisms

$$\mathbb{O}(D)|_{\xi=\xi_0} \cong \mathbb{O} \quad \text{and} \quad \hat{\mathbb{O}}(D)|_{\xi=\xi_0} \cong \hat{\mathbb{O}}.$$

Therefore we have: aver : $\hat{\mathbb{O}}(D)|_{\xi=\xi_0} \to \mathbb{O}(D)|_{\xi=\xi_0}$. These maps generate the averaging homomorphism aver : $\hat{\mathbb{O}}(D) \to \mathbb{O}(D)$.

The involutions $\star$ are naturally extended to anti-isomorphisms of $\hat{\mathbb{O}}(D)$ and $\mathbb{O}(D)$.

Examples. **1**. The simplest examples (except constants) are

$$\mathbf{z}_j = \xi_j + z_j \in \mathbb{O}(\mathbb{R}^m), \quad \hat{\mathbf{z}}_j = \xi_j + \hat{z}_j \in \hat{\mathbb{O}}(\mathbb{R}^m).$$

2. Polynomials in $\mathbf{z}_1, \dots, \mathbf{z}_m$ lie in $\mathbb{O}(\mathbb{R}^m)$. Non-commutative polynomials in $\hat{\mathbf{z}}_1, \dots, \hat{\mathbf{z}}_m$ lie in $\hat{\mathbb{O}}(\mathbb{R}^m)$.

3. **Proposition**. $F \in \mathbb{O}(D)$ iff $F(\xi; z) = f(\mathbf{z})$, where $f : D \to \mathbb{C}$ is a smooth function.

Corollary. $\mathbb{O}(D) \cong C^\infty(D, \mathbb{C})$.

Informally speaking, any $\hat{F} \in \hat{\mathbb{O}}(D)$ is a "non-commutative smooth function of $\hat{\mathbf{z}}$". Sometimes we will use the notation $\hat{F} = \hat{F}(\hat{\mathbf{z}})$.

We extend $\hat{\mathbb{O}}(D)$ by adding the element $\hat{r}$ which commutes with everything,

$$\partial_{\xi_j}\hat{r} = \partial_j \hat{r} = \hat{\partial}_j \hat{r} = 0, \quad \hat{r}^\star = -\hat{r}.$$

Physical meaning of $\hat{r}$ is $-i\hbar$.

The extended algebra is denoted $\hat{\mathbb{O}}_{\hat{r}}(D)$. Averaging aver : $\hat{\mathbb{O}}_{\hat{r}}(D) \to \mathbb{O}(D)$ is the same as before, but first one should put $\hat{r} = 0$.

Consider the ideal $J \in \hat{\mathbb{O}}_{\hat{r}}(D)$ generated by the elements

$$\hat{\mathbf{z}}_j\hat{\mathbf{z}}_k - \hat{\mathbf{z}}_k\hat{\mathbf{z}}_j - \hat{r}\hat{\varphi}_{jk}, \qquad \hat{\varphi}_{jk} \in \hat{\mathbb{O}}_{\hat{r}}(D).$$

We put $\hat{\mathbb{O}}_J(D) = \hat{\mathbb{O}}_{\hat{r}}(D)/J$. Averaging (again denoted by aver) is defined on $\hat{\mathbb{O}}_J$ so that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathbb{O}}_{\hat{r}}(D) & \longrightarrow & \hat{\mathbb{O}}_J(D) \\ \text{aver} \searrow & & \swarrow \text{aver} \\ & \mathbb{O}(D) & \end{array}$$

Definition. If $J = J^\star$, then J is said to be Hermitian.

If the image of $\hat{r}$ under the projection $\hat{\mathbb{O}}_{\hat{r}}(D) \to \hat{\mathbb{O}}_J(D)$ is not a divisor of zero in $\hat{\mathbb{O}}_J(D)$ then J is said to be *Poissonian.*

Below for brevity we denote this image again by $\hat{r}$.

Proposition. J is Poissonian iff for all j, k, l

$$\hat{\varphi}_{jk} + \hat{\varphi}_{kj} \in J, \quad \hat{\mathbf{z}}_j \hat{\varphi}_{kl} + \hat{\mathbf{z}}_k \hat{\varphi}_{lj} + \hat{\mathbf{z}}_l \hat{\varphi}_{jk} \in J.$$

Main Example. Suppose that $m = 2n$. We denote

$$\hat{\mathbf{z}}_j = \hat{\mathbf{x}}_j, \quad \hat{\mathbf{z}}_{j+n} = \hat{\mathbf{p}}_j, \qquad j = 1, \ldots, n,$$

and generate J by the elements

$$\hat{\mathbf{x}}_j \hat{\mathbf{x}}_k - \hat{\mathbf{x}}_k \hat{\mathbf{x}}_j, \quad \hat{\mathbf{p}}_j \hat{\mathbf{p}}_k - \hat{\mathbf{p}}_k \hat{\mathbf{p}}_j, \quad \hat{\mathbf{p}}_j \hat{\mathbf{x}}_k - \hat{\mathbf{x}}_k \hat{\mathbf{p}}_j - \hat{r}\delta_{jk}, \quad \hat{\mathbf{x}}_j \hat{\mathbf{p}}_k - \hat{\mathbf{p}}_k \hat{\mathbf{x}}_j + \hat{r}\delta_{jk}.$$

J is Poissonian and Hermitian. The corresponding algebra $\hat{\mathbb{O}}_J(\mathbb{R}^{2n})$ is denoted $\hat{\mathbb{H}}(\mathbb{R}^{2n})$. Subalgebra of polynomial elements in $\hat{\mathbb{H}}(\mathbb{R}^{2n})$ is isomorphic to the Heisenberg algebra.

Any Poissonian ideal generates a quantum bracket $[\,,]$:

$$[\hat{F},\hat{G}] = \frac{1}{\hat{r}}\big(\hat{F}\hat{G} - \hat{G}\hat{F}\big) \qquad \text{for any } \hat{F},\hat{G} \in \hat{\mathbb{O}}_J.$$

Important: the right-hand side of this equation lies in $\hat{\mathbb{O}}_J$ because $\hat{F}\hat{G} - \hat{G}\hat{F}$ is divisible by $\hat{r}$.

Then the corresponding classical Poisson bracket $\{\,,\}$ is uniquely determined from the commutative diagram

$$\begin{array}{ccc} \hat{\mathbb{O}}_J(D)\times\hat{\mathbb{O}}_J(D) & \xrightarrow{[\,,]} & \hat{\mathbb{O}}_J(D) \\ {\scriptstyle \text{aver}\times\text{aver}}\big\downarrow & & \big\downarrow{\scriptstyle \text{aver}} \\ \mathbb{O}(D)\times\mathbb{O}(D) & \xrightarrow{\{\,,\}} & \mathbb{O}(D) \end{array}$$

If J is Hermitian, $\star$ is naturally defined as an anti-isomorphism of $\hat{\mathbb{O}}_J(D)$.

If we regard D as a coordinate domain, these algebras can be defined over a manifold. They are called *non-commutative structures*.

This language is analogous to the language of star-products in deformation quantization. There are strong relations between this approach to deformation quantization and the traditional one, but we will not discuss them now.

We concentrate on the algebra $\hat{\mathbb{H}}(\mathbb{R}^{2n})$ and try to interpret its elements as operators on $L^2(\mathbb{R}^n)$.

Recall the standard conventions:

$$\hat{\mathbf{x}}_j \psi = x_j \psi(x), \quad \hat{\mathbf{p}}_j \psi = -i\hbar \frac{\partial}{\partial x_j} \psi(x), \quad \hat{r}\psi = -i\hbar\psi, \qquad \psi : \mathbb{R}^n \to \mathbb{C}.$$

Consider $\hat{F} \in \mathbb{H}(\mathbb{R}^{2n})$. How it is possible to associate with $\hat{F}$ an operator on $L^2(\mathbb{R}^n)$? If $\hat{F}$ is not polynomial in $\hat{\mathbf{p}}$, power expansions do not help because we have to deal with a "differential operator of an infinite order".

Main idea. For any $\zeta \in \mathbb{R}^m$

$$e^{-i\zeta\hat{\mathbf{p}}}\psi(x) = e^{-\hbar\sum \zeta_j \partial/\partial x_j}\psi(x) = \psi(x - \hbar\zeta),$$

a quite innocent operator.

Expansion in exponents $e^{-i\zeta\hat{\mathbf{p}}}$ means the Fourier expansion in the variable $\hat{\mathbf{p}}$. The most convenient space for harmonic analysis on $\mathbb{R}^m$ is the Schwartz space $\mathcal{S}$. We define

$$\mathcal{S}_{\hat{r}} = \Big\{ f(\eta,\zeta,\hat{r}) = \sum_{k=0}^{\infty} f_k(\eta,\zeta)\hat{r}^k, \quad f_k \in \mathcal{S} \Big\}, \qquad \eta,\zeta \in \mathbb{R}^n,$$

the series is formal. The structure of a topological space is introduced in a standard way.

Fourier transform:

$$\mathcal{S}_{\hat{r}} \ni f \mapsto \hat{f} = \mathcal{F}(f) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\hat{\mathbf{x}}, \zeta, \hat{r}) e^{-i\zeta\hat{\mathbf{p}}}\, d\zeta.$$

We denote $\hat{\mathcal{S}}_{\hat{r}} = \mathcal{F}(\mathcal{S}_{\hat{r}})$. **Proposition**. $\hat{\mathcal{S}}_{\hat{r}} \subset \hat{\mathbb{H}}(\mathbb{R}^{2n})$ is a subalgebra.

Inverse Fourier transform:

$$\hat{\mathcal{S}}_{\hat{r}} \ni \hat{f} \mapsto f = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) e^{i\zeta\hat{\mathbf{p}}}\, dp,$$

where as usual, $\hat{\mathbf{p}} = p + \hat{p}$.

Proposition. $\mathcal{F}^{-1}\mathcal{F} = \mathrm{id}_{\mathcal{S}_{\hat{r}}}$ and $\mathcal{F}\mathcal{F}^{-1} = \mathrm{id}_{\hat{\mathcal{S}}_{\hat{r}}}$.

The derivations $\hat{\partial}_j$ still exist in $\mathbb{H}(\mathbb{R}^{2n})$. We use for them the notation $\hat{\partial}_{\hat{x}_j}$ and $\hat{\partial}_{\hat{p}_j}$.

Proposition. For any $f \in \mathcal{S}_{\hat{r}}$ and $\hat{f} = \mathcal{F}(f)$

$$\hat{\partial}_{\hat{p}_j} \hat{f} = \mathcal{F}(-i\zeta_j f), \quad i\hat{f}\hat{\mathbf{p}}_j = \mathcal{F}(\partial f / \partial \zeta_j).$$

For any $f = f(\eta,\zeta,\hat r)$, $g = g(\eta,\zeta,\hat r) \in \mathcal{S}_{\hat r}$ and $\hat f = \hat f(\hat{\mathbf{x}},\hat{\mathbf{p}})$, $\hat g = \hat g(\hat{\mathbf{x}},\hat{\mathbf{p}}) \in \hat{\mathcal{S}}_{\hat r}$ we define the convolutions

$$\begin{aligned} f * g &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\hat{\mathbf{x}}, \zeta - \rho, \hat r)\, g(\hat{\mathbf{x}} + i\hat r \zeta, \rho, \hat r)\, d\rho, \\ \hat f * \hat g &= \int_{\mathbb{R}^n} \hat g(\hat{\mathbf{x}}, P)\, \hat f(\hat{\mathbf{x}}, \hat{\mathbf{p}} - P)\, dP. \end{aligned}$$

Proposition. Suppose that $\hat f = \mathcal{F}(f)$ and $\hat g = \mathcal{F}(g)$. Then

$$\hat f * \hat g = \mathcal{F}(fg), \quad \hat f \hat g = \mathcal{F}(f * g).$$

Representation of $\hat{\mathcal{S}}_{\hat{r}}$ in $L^2(\mathbb{R}^n)$.

For any $\hat{f} \in \hat{\mathcal{S}}_{\hat{r}}$ we have:

$$\hat{f} = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(\hat{\mathbf{x}}, \zeta, \hat{r}) e^{-i\zeta\hat{\mathbf{p}}}\, d\zeta.$$

Therefore for $\psi \in L^2(\mathbb{R}^n)$

$$\hat{f}\psi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x, \zeta, -i\hbar)\psi(x - \hbar\zeta)\, d\zeta.$$

For any $f, g \in \mathcal{S}_{\hat{r}}$ and $\hat{f}, \hat{g} \in \hat{\mathcal{S}}_{\hat{r}}$ we define the scalar products

$$
\begin{aligned}
\langle f, g \rangle &= \int_{\mathbb{R}^n} f(\eta, \zeta, \hat{r})\, g^\star(\eta, \zeta, \hat{r})\, d\eta d\zeta, \\
\langle \hat{f}, \hat{g} \rangle &= \int_{\mathbb{R}^n} \hat{f}(\hat{\mathbf{x}}, \hat{\mathbf{p}})\, \hat{g}^\star(\hat{\mathbf{x}}, \hat{\mathbf{p}})\, dx dp.
\end{aligned}
$$

Proposition. Suppose that $\hat{f} = \mathcal{F}(f)$ and $\hat{g} = \mathcal{F}(g)$. Then

$$
\langle \hat{f}, \hat{g} \rangle = \langle f, g \rangle.
$$

By using these scalar products, we can consider the spaces of distributions $\mathcal{S}'_{\hat{r}}$ and $\hat{\mathcal{S}}'_{\hat{r}}$, corresponding to the spaces of "test functions" $\mathcal{S}_{\hat{r}}$ and $\hat{\mathcal{S}}_{\hat{r}}$.

The space $\hat{\mathcal{S}}'_{\hat{r}}$ contains the algebra of non-commutative polynomials in $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ (the Heisenberg algebra.)

Representation of $\hat{\mathcal{S}}'_{\hat{r}}$ in $L^2(\mathbb{R}^n)$ is given by the same formula.

Note that $\hat{\mathcal{S}}'_{\hat{r}} \not\subset \mathbb{H}(\mathbb{R}^{2n})$ and $\mathbb{H}(\mathbb{R}^{2n}) \not\subset \hat{\mathcal{S}}'_{\hat{r}}$.

A possible application

Suppose that a quantum system with Hamiltonian $\hat{H} \in \hat{\mathbb{H}}(\mathbb{R}^{2n})$ is integrable in the sense that there exist sufficiently regular independent $\hat{F}_1, \ldots, \hat{F}_n \in \hat{\mathbb{H}}(\mathbb{R}^{2n})$ such that

$$[\hat{F}_j, \hat{F}_k] = [\hat{F}_j, \hat{H}] = 0.$$

In particular, the corresponding classical Hamiltonian system, obtained by means of the averaging homomorphism, is Liouville integrable.

If the classical common integral levels are compact, there exist action-angle variables.

It turns out that there exist non-commutative action-angle variables $\hat{I}, \hat{\varphi}$, i.e.,

(a) $\hat{I} = \mathcal{M}(\hat{F}, i\hat{r})$, $\hat{F} = W(\hat{I}, i\hat{r})$.

(b) $\hat{\varphi}$ are angular (defined $\mathrm{mod} 2\pi$).

(c) $[\hat{I}_j, \hat{I}_k] = [\hat{\varphi}_j, \hat{\varphi}_k] = 0$, $[\hat{I}_j, \hat{\varphi}_k] = \delta_{jk}$.

Existence of such non-commutative action-angle variables has been proven so far locally [T2007]: $\hat{I}_j, \hat{\varphi}_j \in \hat{\mathbb{H}}(D)$, where D is a neighborhood of any classical Liouville torus. The corresponding global result is probably much more complicated, but under reasonable assumptions does not look hopeless.

Suppose that $\hat{I}_j, \hat{\varphi}_j \in \hat{\mathbb{H}}(\mathbb{R}^{2n})$.

Suppose moreover that $\hat{I}_j, e^{i\hat{\varphi}_j}$ correspond to some good (pseudo-differential) operators.

Then for any eigenfunction $\psi \not\equiv 0$, $\hat{F}_j\psi = \lambda_j\psi$ we have:

$$\hat{F}_j\psi_k = \Lambda_{jk}\psi_k, \qquad \psi_k = e^{ik\hat{\varphi}}\psi, \quad k \in \mathbb{Z}.$$

$\Lambda_{jk} = W_j\big(\mathcal{M}(\lambda, \hbar) + \hbar k, \hbar\big)$.

Hence, having one eigenfunction, we obtain many (infinitely many? all?) eigenfunctions and eigenvalues.