# Torus fibrations and localization of index

## Computation of local index and application

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# In the previous talk

$W$: $\mathbb{Z}_2$-graded $Cl(TM)$-module bundle
$\downarrow$
$M$: Riemannian manifold (not necessarily compact)
$\cup$
$V$: open set s.t.

- $M \setminus V$: compact
- $\exists \{\pi_\alpha : V_\alpha \to U_\alpha, D_\alpha\}_{\alpha \in A}$: acyclic compatible fibration on $V$

**Theorem (Fujita-Furuta-Y '09)**

*Let $D$ be a Dirac-type operator on $W$. Then, $\exists\, \mathrm{Ind}(M, V) \in \mathbb{Z}$ satisfying*

1. $\mathrm{Ind}(M, V)$ *is invariant under continuous deformation of the data.*
2. $\mathrm{Ind}(M, V) = \mathrm{Ind}(D)$ *for closed $M$*
3. $\mathrm{Ind}(M, V) = 0$ *if $V = M$ (vanishing)*
4. *Suppose $\exists \cup_{i=1}^{k} O_i \supset M \setminus V$: mutually disjoint open covering. Then,*

$$\mathrm{Ind}(M, V) = \sum_{i=1}^{k} \mathrm{Ind}(O_i, O_i \cap V) \text{ (localization)}$$

## In the this talk

1. Computation of local index for four-dimensional elliptic singularities
2. Application to locally toric Lagrangian fibrations

**Joint work with Hajime Fujita and Mikio Furuta**

1. H. Fujita, M. Furuta, Y, *Torus fibrations and localization of index I*, arXiv:0804.3258.
2. H. Fujita, M. Furuta, Y, *Torus fibrations and localization of index II*, in preparation. Coming soon!

## 1. Computation

$\mu\colon (M,\omega) \to B$: $2n$-$\dim_{\mathbb{R}}$. Lagrangian fibration with singular fibers

**Definition**

A critical point of $\mu\colon (M,\omega) \to B$ is a *nondegenerate elliptic singular point of rank* $k$ ($\leq n$) if $\exists$symplectic coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ s. t. in these coordinates, $\mu$ is written as

$$\mu = (x_1, \ldots, x_k, x_{k+1}^2 + y_{k+1}^2, \ldots, x_n^2 + y_n^2).$$

In this part we define local indices $\mathrm{Ind}_0(a,b)$ and $\mathrm{Ind}_1(a,b)$ for two types of elliptic singularities in four-dimensional case and compute them.

## 1.1. A BS fiber in the product of discs.

$$\begin{array}{l} (L_0, \nabla^{L_0}) = \left(M_0 \times \mathbb{C}, d + \frac{1}{2}\sum_i (z_i d\bar{z}_i - \bar{z}_i dz_i)\right) \\ \quad \downarrow \\ (M_0, \omega_0) = \left(\{z \in \mathbb{C}^2 \mid |z_1| < 1, |z_2| < 1\}, \omega_{\mathbb{C}^2}\right) \\ \quad \Big\downarrow \mu_0(z) := (|z_1|^2, |z_2|^2) \\ \quad B_0 = [0,1) \times [0,1) \end{array}$$

$\longleftarrow$ • every fiber of $\mu_0$ is smooth.

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**Definition ($RR_0(a,b)$)**

$$\mathrm{Ind}_0(a,b) := \mathrm{Ind}(M_0, V_0) \quad (a, b \in \mathbb{Z})$$

## 1.2. A singular non BS fiber in the product of a cylinder and a disc

$$(L_1, \nabla^{L_1}) = \left(M_1 \times \mathbb{C}, d - 2\pi\sqrt{-1}rd\theta + \tfrac{1}{2}(zd\bar{z} - \bar{z}dz)\right)$$

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## 1.3. Computation

**Proposition (Fujita-Furuta-Y '09)**

$$\mathrm{Ind}_0(a,b) = 1, \ \mathrm{Ind}_1(a,b) = 0 \ \forall a,b \in \mathbb{Z}.$$

## Lemma 1

$$\forall a, b, c \in \mathbb{Z}, \quad \mathrm{Ind}_0(a, b) = \mathrm{Ind}_0(b, a) \tag{1}$$

$$\mathrm{Ind}_1(a, b) = \mathrm{Ind}_1(a + c, b + c) \tag{2}$$

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$$\forall a, b, c \in \mathbb{Z}, \quad \mathrm{Ind}_0(a, b) = \mathrm{Ind}_0(c, b) + \mathrm{Ind}_1(c, a) \tag{3}$$

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## Proof of Proposition.

$$a < b.\ \mathrm{Ind}_1(0, 1) \overset{3}{=} \mathrm{Ind}_0(1, 0) - \mathrm{Ind}_0(0, 0) \overset{1}{=} \mathrm{Ind}_0(0, 1) - \mathrm{Ind}_0(0, 0) \overset{5}{=} 0 \tag{6}$$

$$\therefore\ \mathrm{Ind}_1(a, b) \overset{4}{=} \sum_{i=0}^{b-a-1} \mathrm{Ind}_1(a + i, a + i + 1) \overset{2}{=} (b - a)\,\mathrm{Ind}_1(0, 1) \overset{6}{=} 0 \tag{7}$$

$$a = b.\ \mathrm{Ind}_1(a, a) \overset{3}{=} \mathrm{Ind}_0(a, a) - \mathrm{Ind}_0(a, a) = 0 \tag{8}$$

$$a > b.\ \mathrm{Ind}_1(a, b) \overset{4}{=} \mathrm{Ind}_1(a, a) - \mathrm{Ind}_1(b, a) \overset{8}{=} -\mathrm{Ind}_1(b, a) \overset{7}{=} 0$$

$$\therefore\ \mathrm{Ind}_0(a, b) \overset{3+\mathrm{Ind}_1=0}{=} \mathrm{Ind}_0(0, b) \overset{1}{=} \mathrm{Ind}_0(b, 0) \overset{3+\mathrm{Ind}_1=0}{=} \mathrm{Ind}_0(0, 0) = 1$$

□

**Remarks**

- In the above computation we used the properties of local index to reduce the computation to Lemma 3. There also exist some examples of local indices that are directly computable.
- In general case we have no systematic method to compute the local index.

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  (Ex. moment map of a nonsingular projective toric variety)
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**Theorem (Fujita-Furuta-Y '09)**

*In four-dimensional case,*

$$RR(M,\omega) = \#\text{(both singular and nonsingular) BS-fibers.}$$

**Point of Proof.**

In general, it is difficult to construct an acyclic compatible fibration on $V$ so that the contribution of $M \setminus V$ to $RR(M, \omega)$ is computable. But, in this case,

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### Key Point

*$\exists b_1, \ldots, b_k \in \mathcal{S}^{(1)}B$: non BS points and $\exists$acyclic compatible fibration on $V := M \setminus \{BS\text{-fibers}, \mu^{-1}(b_1), \ldots, \mu^{-1}(b_k)\}$ s. t.*

- *$\forall i$, the contribution of $\mu^{-1}(b_i) = \mathrm{Ind}_1(a, b)$,*
- *the contribution of each one-dim. BS-fiber $= 1$.*

$\because$ Use the local model: $\mu_{\mathbb{C}^n} : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \to \mathbb{R}^n_{\geq 0}$, $z \mapsto (|z_i|^2)$.

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Moreover,

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Moreover,

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- the contribution of each zero-dim. BS-fiber $= \mathrm{Ind}_0(a, b) = 1$. ($\because$ Proposition)

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*$\exists b_1, \ldots, b_k \in \mathcal{S}^{(1)}B$: non BS points and $\exists$acyclic compatible fibration on $V := M \setminus \{BS\text{-fibers}, \mu^{-1}(b_1), \ldots, \mu^{-1}(b_k)\}$ s. t.*

- *$\forall i$, the contribution of $\mu^{-1}(b_i) = \mathrm{Ind}_1(a, b)$,*
- *the contribution of each one-dim. BS-fiber $= 1$.*

$\because$ Use the local model: $\mu_{\mathbb{C}^n} : (\mathbb{C}^n, \omega_{\mathbb{C}^n}) \to \mathbb{R}^n_{\geq 0}$, $z \mapsto (|z_i|^2)$.

Moreover,

- $\mathrm{Ind}_1(a, b) = 0$. ($\because$ Proposition)
- the contribution of each zero-dim. BS-fiber $= \mathrm{Ind}_0(a, b) = 1$. ($\because$ Proposition)
- the contribution of each two-dim. BS-fiber $= 1$. ($\because$ nonsingular BS-fiber)

**Point of Proof.**

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$\Rightarrow$ $RR(M, \omega) = \#$(both singular and nonsingular) BS-fibers.

**Example (Non toric example)**

$$(L,\nabla^L) := \mathrm{p}^*_{\mathbb{R}\times S^1}(\underline{\mathbb{C}}, d - 2\pi\sqrt{-1}rd\theta) \otimes \mathrm{p}^*_{\mathbb{C}P^1}(\mathcal{O}(1), d + \tfrac{1}{2}\textstyle\sum_i(z_i d\bar{z}_i - \bar{z}_i dz_i))/\mathbb{Z}$$

$$\downarrow$$

$$(M,\omega) := \left(\mathbb{R}\times S^1 \times \mathbb{C}P^1, \omega_{\mathbb{R}\times S^1} \oplus \omega_{FS}\right)/\mathbb{Z}$$

$$\Big\downarrow \mu\colon (r,u,[z_0:z_1]) \mapsto (r,|z_1|^2)$$

$$B := (\mathbb{R}\times[0,1])/\mathbb{Z}$$

$\mathbb{Z} \curvearrowright \mathrm{p}^*_{\mathbb{R}\times S^1}\underline{\mathbb{C}} \otimes \mathrm{p}^*_{\mathbb{C}P^1}\mathcal{O}(1)$ :

$$n(r,u,z,w) := \left(r + n(|z_1|^2+1), u, [z_0 : u^{-n}z_1], u^n w\right)$$

$\mathbb{Z} \curvearrowright \mathbb{R}\times S^1 \times \mathbb{C}P^1$:

$$n(r,u,[z_0:z_1]) := \left(r + n(|z_1|^2+1), u, [z_0 : u^{-n}z_1]\right)$$

$\mathbb{Z} \curvearrowright \mathbb{R}\times[0,1]$:

$$n(r_1,r_2) := (r_1 + n(r_2+1), r_2).$$

Fundamental domain of $\mathbb{Z} \curvearrowright \mathbb{R} \times [0,1]$: $n(r_1, r_2) := (r_1 + n(r_2+1), r_2)$

$$F := \left\{ (r_1, r_2) \in \mathbb{R} \times [0,1] \mid -\frac{1}{2} \leq r_1 < r_2 + \frac{1}{2} \right\}$$

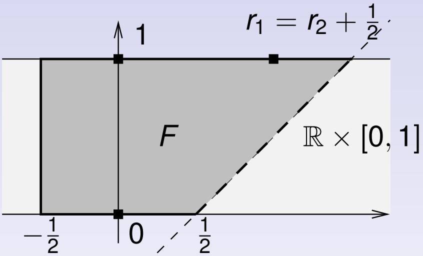


Figure: Fundamental domain and Bohr-Sommerfeld points

**In this example**

$$BS\ points \overset{1:1}{\Longleftrightarrow} F \cap \mathbb{Z}^2$$

$$\Rightarrow \quad RR(M, \omega) = \#BS\text{-}fibers = 3.$$