Equivariant asymptotics of Szegö kernels

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H Hilbert space related to the geometric structure of M.

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Problem

Relate isotypical decomposition to the geometry of the action.

Geometric setting

A quantum line bundle for a given Kähler manifold (M, ω) is a triple $(A,\, h,\, \nabla)$ such that $\mathrm{curv}_{A,\,\nabla} = -2\,\imath\,\omega$.

$$X := \{a \in A^\vee \;:\; h^\vee(a,a) = 1\}$$

$$\downarrow$$

$(X,\, \alpha)$ is a contact manifold and CR manifold.

The Hardy space: $H(X) = \ker(\overline{\partial}_b) \cap L^2(X)$

Szegö projector: $\Pi : L^2(X) \to H(X)$

Contact lifts of Hamiltonian actions

Connected compact Lie group G ($\mathfrak{g} = \mathrm{Lie}(G)$),
Holomorphic and Hamiltonian action $\mu : G \times M \to M$, with moment map
$\Phi : M \to \mathfrak{g} \cong \mathfrak{g}^\vee$
Infinitesimal action $\mathrm{d}\mu : \mathfrak{g} \to \mathfrak{X}(M)$, $\xi \to \xi_M$.

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$$\mathrm{d}\mu \text{ lifts to } \widetilde{\mathrm{d}\mu} : \mathfrak{g} \to \mathfrak{X}_{cont}(X, \alpha),$$

$$\xi \mapsto \xi_X = \xi_M^\sharp - \langle \Phi, \xi \rangle \, \partial_\theta, \quad \xi_M = \mathrm{d}\pi(\xi_X).$$

Assumption

$\exists \tilde{\mu} : G \to \mathrm{Cont}(X, \alpha)$ s.t. $\mathrm{d}\tilde{\mu}(\xi) = \widetilde{\mathrm{d}\mu}(\xi)$. Thus $\tilde{\mu}$ is a **contact lift** of μ.

Equivariant Szegö projectors

$H(X)$ splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G (Peter-Weyl Theorem):

$$H(X) = \bigoplus_{\boldsymbol{\nu} \in \hat{G}} H_{\boldsymbol{\nu}}(X)$$

Equivariant Szegö Projector

$$\Pi_{\nu} \, : \, L^2(X) \rightarrow H_{\boldsymbol{\nu}}(X)$$

$$\text{If } \mathbf{0} \notin \Phi(M) \,\Rightarrow\, \Pi_{\nu}(\cdot,\cdot) \in C^{\infty}(X \times X),$$

Goal

Local asymptotics of $\Pi_{k\nu}$ and

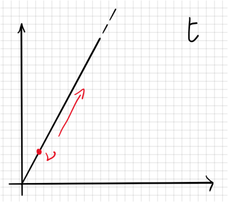
$$\dim\left(H_{k\boldsymbol{\nu}}(X)\right) = \int_X \Pi_{k\boldsymbol{\nu}}(x,x)\,\mathrm{dV}_X(x)\,.$$

Remark

In the standard picture of Berezin-Toeplitz quantization:

$$H^0(M,\, A^{\otimes k}) = H_k(X) := \bigoplus_{\nu \in \hat{G}} H_{k,\,\nu}(X) \quad \Rightarrow \quad \text{Asymptotics of } \Pi_{k,\,\nu}$$

Our approach is different and it is inspired by the article "Homogeneous quantization and multiplicities of group representation". Pictorially,



Remark

The spaces $H_{k\,\nu}(X)$ are not in general contained in a space of sections.

Standard case

Natural action of S^1 on X with trivial μ and $\Phi \equiv 1$

$$\Downarrow$$

$$H(X) = \bigoplus_{k \in \mathbb{Z}} H(X)_k.$$

Equivariant components $\Pi_k : L^2(X) \to H_k(X)$ of the Szegö projector.

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$$\Pi_k(x,x) \sim \left(\frac{k}{\pi}\right)^d \cdot \left[1 + \sum_{j \geq 1} k^{-j} a_j(m_x)\right] \qquad \text{[Catlin, Zelditch; 1998]},$$

where $a_j : M \to \mathbb{R}$ is $\mathcal{C}^\infty$, and we have set $m_x := \pi(x)$
a_1 is the scalar curvature, [Lu, 2000].

Circle actions

Contact Lifts of a Hamiltonian S^1-action μ.

$$H(X) = \bigoplus_{k\in\mathbb{Z}} H_k^{\mu}(X)$$

$$\Phi > 0 \Rightarrow H_k^{\mu}(X) \text{ is finite-dimensional}$$

$$\Downarrow$$

$$\Pi_k^{\mu}(x,x) \sim \left(\frac{k}{\pi}\right)^d \cdot \Phi(m)^{-(d+1)} \sum_{g\in T_x} g^k \cdot \left(1 + \sum_{j\geq 1} k^{-j} a_j(m_x)\right),$$

The term a_1 is explicitly computed in [Paoletti, 2015] and a recursive algorithm for computing the a_j's is given.

Torus actions

Contact Lifts of a Hamiltonian $\mathbb{T}^n$-action μ.

$$\Downarrow$$

$$H(X) = \bigoplus_{\boldsymbol{\nu} \in \mathbb{Z}^n} H_{\boldsymbol{\nu}}(X)$$

$$\Downarrow$$

$$\mathbf{0} \notin \Phi(M) \Rightarrow H_{\boldsymbol{\nu}}(X) \text{ is finite-dimensional.}$$

Theorem ([Paoletti, 2012])

If $\Phi(m) \notin \mathbb{R}_+ \cdot \boldsymbol{\nu}$, then $\Pi_{k\boldsymbol{\nu}}(x, x) = O(k^{-\infty})$ where $m = \pi(x)$.

On-diagonal asymptotics

Theorem ([Paoletti, 2012])

Assume that Φ *is transversal to* $\mathbb{R}_+ \cdot \boldsymbol{\nu}$. *Then for every* $m \in \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$,

$$\Pi_{k\boldsymbol{\nu}}(x,x) \sim \frac{1}{(\sqrt{2}\,\pi)^{n-1}} \left(||\boldsymbol{\nu}|| \cdot \frac{k}{\pi} \right)^{d+(1-n)/2} \cdot \sum_{g \in T_x} \chi_{\boldsymbol{\nu}}(g)^k \cdot \frac{1}{\mathcal{D}(m)} \left(\frac{1}{||\Phi(m)||} \right)^{d+1+(1-n)/2} \cdot \left(1 + \sum_{l \geq 1} B_l(m_x)\, k^{-l} \right);$$

where T_x *is the stabilizer of* x, $\chi_{\boldsymbol{\nu}}$ *is the character pertaining to* $\boldsymbol{\nu}$, $\mathcal{D} : M \to \mathbb{R}$ *is a distortion function and* B_l*'s are smooth functions.*

The dimension of the isotypes

Given that $\mathbf{0} \notin \Phi(M)$, if in addition Φ is transverse to $\mathbb{R}_+\boldsymbol{\nu}$ then $M_\nu =: \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\nu})$ (if non-empty) is a connected compact submanifold of M, of real codimension $n-1$.

Theorem ([Paoletti, 2012])

Under some suitable hypothesis, assume in addition that the action is generically free,

$$\lim_{k\to+\infty} \left(||\boldsymbol{\nu}|| \, \frac{k}{\pi} \right)^{-(d+1-n)} \dim\left(H_{k\nu}(X)\right) = \frac{1}{(2\,\pi)^{n-1}} \int_{M_\nu} ||\Phi(m)||^{-(d+2-n)} \cdot \frac{1}{\mathcal{D}(m)} \, \mathrm{dV}_{M_\nu}(m) \, .$$

Irreducible representations of SU(2)

$$G = SU(2), \qquad \mathfrak{g} = \imath\, h_2^0, \quad T = \left\{ e^{\vartheta\,\beta}, \quad \beta = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix} \right\}.$$

Unitary Irreducible Representations $\leftrightarrow \nu \in \mathbb{Z}_{>0}$

Weyl Character Formula $\chi_\nu(e^{\vartheta\,\beta}) = \dfrac{e^{\imath\,\nu\,\vartheta} - e^{-\imath\,\nu\,\vartheta}}{e^{\imath\,\vartheta} - e^{-\imath\,\vartheta}}.$

$$\dim(V_\nu) = \nu$$

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$$\dim\left(V_\nu\right) = \nu$$

$$\Downarrow$$

$$H(X) \cong \bigoplus_{\nu > 0} H(X)_\nu\,.$$

Diagonal expansions

If $\mathbf{0} \notin \Phi(M) \Rightarrow \exists!\, h_m T \in G/T$ such that

$$h_m \Phi(m) h_m^{-1} = \imath \begin{pmatrix} \lambda(m) & 0 \\ 0 & -\lambda(m) \end{pmatrix},$$

If $l \in \mathbb{Z}$, let us define $f_l : T \to \mathbb{C}$ by letting

$$f_l : e^{\vartheta \beta} \in T \mapsto e^{l \vartheta \beta} \in \mathbb{C}^*.$$

Theorem ([Galasso & Paoletti, 2018])

Le us assume that $\mathbf{0} \notin \Phi(M)$ and that the stabilizer G_x is contained in the center of G. Then $\Pi_{k\nu}(x, x)$ has asymptotic expansion

$$\left(\frac{1}{2\lambda(m_x)}\right)^{d+1} \cdot \left(\frac{\nu k}{\pi}\right)^d \cdot \sum_{g \in G_x} f_{1-k\cdot\nu}(g) \cdot \left[1 + \sum_{j=1}^{+\infty} k^{-j} B_{g\,j}(x)\right]$$

Dimensions of the isotypes

We can apply previous theorems to estimate the dimension of $H(X)_{k\,\boldsymbol{\nu}}$ when $k \to +\infty$. Let us make this explicit in the case where $\tilde{\mu}$ is generically free.

Theorem ([Galasso & Paoletti, 2018])

Le us assume that $\mathbf{0} \notin \Phi(M)$ and that $\tilde{\mu}$ is generically free on X, we have

$$\lim_{k\to+\infty}\left[\left(\frac{\pi}{k\nu}\right)^{d}\cdot \dim\left(H_{k\boldsymbol{\nu}}(X)\right)\right] = \int_M \left(\frac{1}{2\,\lambda(m_x)}\right)^{d+1}\,\mathrm{dV}_M(m)\,.$$

Proof.

For $k = 1, 2, \ldots$ let us define $f_k \in \mathcal{C}^\infty(M)$ by setting $f_k(m) := k^{-d}\,\Pi_{k\boldsymbol{\nu}}(x, x)$ if $m = \pi(x)$. By the previous theorems, $f_k \leq C$ for some constant $C > 0$ and $f_k \to (\nu/2\pi)^d\,\lambda^{-(d+1)}/2$ for $k \to +\infty$. By the dominate convergence theorem we can conclude. □

Equivariant Toeplitz operators

Assume that $G = S^1$ and $\Phi > 0$. For any $f \in \mathcal{C}^\infty(M)^\mu$ we have

$$T_k^\mu[f] := \Pi_k^\mu \, M_f \, \Pi_k^\mu \,.$$

Theorem ([Paoletti, 2015])

Under some hypothesis, let $f, g \in \mathcal{C}^\infty(M)^\mu$, as $k \to +\infty$, we have

$$T_k^\mu[f](x, x) \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j} \, S_j^\mu[f](m) \,,$$

$$\mathrm{Ber}_k^\mu[f](m) := \frac{T_k^\mu[f](x, x)}{\Pi_k^\mu(x, x)} \sim \sum_{j \geq 0} k^{-j} \, B_j^\mu(f) \,,$$

$$T_k^\mu[f] \circ T_k^\mu[g](x, x) \sim \left(\frac{k}{\pi}\right)^d \sum_{j \geq 0} k^{-j} \, A_j[f, g](x) \,.$$

Equivariant Toeplitz operators

In the previous expansion

$$A_0[f,g](x) = \Phi^{-(d+1)}(m) \cdot f\, g$$

and one can compute

$$T_k^{\mu}[f] \circ T_k^{\mu}[g](x,x) - T_k^{\mu}[g] \circ T_k^{\mu}[f](x,x)$$
$$= \left(\frac{k}{\pi}\right)^d \left[-\frac{\imath}{k}\, \Phi(m)^{-d} \{f,\, g\}_M(m) + O(k^{-2}) \right] .$$

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