Resurgence Analysis of Quantum Invariants

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Outline

This talk is based on two joint papers with J.E. Andersen. They concern asymptotic expansions of quantum invariants, and resurgence properties of the asymptotic series giving these expansions.

1. Quantum topology
2. Resurgence and Picard-Lefschetz theory
3. Resurgence in TQFT
4. Quantization of moduli spaces

1. Quantum topology
2. Resurgence and Picard-Lefschetz theory
3. Resurgence in TQFT
4. Quantization of moduli spaces

Knotted objects and quantum invariants

- **knotted objects:** a knotted object (M, K) is a closed oriented 3-manifold M with a framed oriented link $K \subset M$.

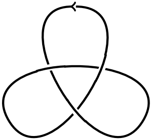

Figure: Example: the trefoil knot in S^3 .

- **Quantum invariants:** Let $r \in \mathbb{N}$, and $\kappa \in \{1, ..., r-1\}^{\pi_0(K)}$. The quantum invariant is a topological invariant

$$\tau_r(M, K, \kappa) \in \mathbb{C}.$$

We write $\mathrm{Col}(K, r) = \{1, ..., r-1\}^{\pi_0(K)}$.

Surgery presentations of knotted objects

- **Surgery:** Due to work of Kirby there exists a bijection Φ

$$\{\text{Pairs of framed links } (L,K) \subset \mathrm{S}^3 \times \mathrm{S}^3\}/\text{Kirby equivalence} \simeq \{\text{Knotted objects } (M,K)\}/\text{Diff}^+$$

$$(L,K) \overset{\Phi}{\mapsto} (M_L, K).$$

- **Construction:** The framing of $L = \{L_j\}_{j=1}^m$ induces $T \simeq \sqcup_{j=1}^m \mathrm{S}_j^1 \times \mathrm{B}_j^2$ where T is a tubular nbhd of L. We have

$$M_L = \left(\mathrm{S}^3 \setminus \text{Interior}\left(\sqcup_{j=1}^m \mathrm{S}_j^1 \times \mathrm{B}_j^2\right)\right) \cup_{\mathrm{S}_j^1 \times \mathrm{S}_j^1} \left(\sqcup_{j=1}^m \mathrm{B}_j^2 \times \mathrm{S}_j^1\right).$$

The framed unknot

A particularly important link, is the framed unknot $\mathbf{O}_m$ with m twists. Below we consider the example $m = 2$

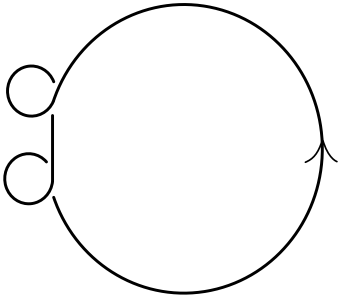

Figure: Example: the unknot $\mathbf{O}_2$ with framing 2.

The Jones polynomial and the Skein relation

Let $q = e^{\frac{2\pi i}{r}}, r \in \mathbb{N}$. The Jones polynomial $J(L, q) \in \mathbb{Z}[q^{\frac{\pm 1}{4}}]$ satisfy

$$J(\mathbf{O}_m, q) = q^{m\frac{3}{4}}(q^{\frac{1}{2}} + q^{-\frac{1}{2}}),$$

multiplicativity

$$J(\mathbf{L} \sqcup \mathbf{L}') = J(\mathbf{L})J(\mathbf{L}')$$

and the Skein-relation

$$q^{\frac{1}{4}} J(\mathbf{L}_+, q) - q^{-\frac{1}{4}} J(\mathbf{L}_-, q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) J(\mathbf{L}_0, q).$$

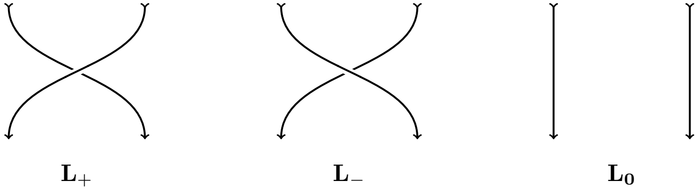

Figure: A Skein triple.

Cabling and the colored Jones polynomial

- **Cabling:** Consider a link $L = \{L_i\}_{i=1}^m$. Given $\kappa \in \mathrm{Col}(L, r)$ define L^κ by replacing each L_i by κ_i new components which are parallel push-offs of L_i

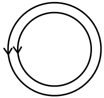

Figure: Example: the cabled unknot $\mathbf{O}^2$.

- **The colored Jones polynomial:** Given $\lambda \in \mathrm{Col}(L, r)$ let

$$J_\lambda(L, q) = \sum_{\kappa=0}^{\frac{\lambda-\mathbf{1}}{2}} (-1)^{\sum_{i=1}^m \kappa_i} \prod_{i=1}^m \binom{\lambda_i - 1 - \kappa_i}{\kappa_i} J(L^{\lambda - \mathbf{1} - 2\kappa}, q).$$

Atiyah's challenge

The Jones polynomial was mysterious to topologists. Atiyah posed the following challenges:

- Extend $J(K,q)$ to an invariant of (M_L, K).
- Give an intrinsic definition of $J(L,q)$ without link diagrams.

Witten's solution: Quantum Cherns-Simons theory

- **Classical theory:** Let $G = \mathrm{SU}(n)$. Let $\mathcal{A}/\mathcal{G}$ be the space of G-connections. For $[A] \in \mathcal{A}/\mathcal{G}$, we have the CS action

$$\mathrm{S}_{\mathrm{CS}}([A]) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(A\mathrm{d}A + \frac{2}{3}A^3) \mod \mathbb{Z}.$$

  The space of classical solutions $\mathrm{d}\,\mathrm{S}_{\mathrm{CS}[A]} = 0$ is equal to the moduli space $\mathcal{M}(G, M)$ of flat connections.

- **Quantum theory:** Set $k = r - n$. Witten considered the path integrals (which are mathematically ill-defined)

$$\mathrm{Z}_k^{\mathsf{phys}}(M, L) = \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k\,\mathrm{S}_{\mathrm{CS}}(A)} \prod_{L_i \in \pi_0(L)} \mathrm{tr}\left(\mathrm{Hol}_A(L_i)\right)\ \mathcal{D}A$$

  and showed $(n = 2)$ that $\mathrm{Z}_k^{\mathsf{phys}}(\mathrm{S}^3, L) = J(L, q)$.

The Reshetikhin-Turaev topological quantum field theory

Using modular categories Reshetikhin and Turaev defined a TQFT

$$\tau_r : (\mathrm{Cob}(3), \sqcup, \emptyset) \to (\mathrm{Vect}(\mathbb{C}), \otimes, \mathbb{C}) .$$

- To a surface Σ the TQFT assigns a vector space $\mathrm{V}_r(\Sigma)$.
- To a compact oriented 3-manifold M with $\partial M = (-\Sigma) \sqcup \Sigma'$ the TQFT assigns a linear map

$$\tau_r(M) : \mathrm{V}_r(\Sigma) \to \mathrm{V}_r(\Sigma').$$

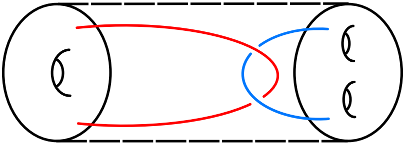

Figure: A cobordism $M : \Sigma_1 \to \Sigma_2$.

The $\mathrm{SU}(2)$ quantum invariant

Let $G = \mathrm{SU}(2)$. For $m \in \mathbb{Z}$ we introduce the quantum integer

$$[m] = (q^{\frac{m}{2}} - q^{-\frac{m}{2}})(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}$$

The quantum invariant of (a colored) knotted object (M_L, K, κ) is

$$\tau_r(M_L, K, \kappa) = \alpha_L \sum_{\lambda \in \mathrm{Col}(L,r)} \prod_{L_i \in \pi_0(L)} [\lambda_i] J_{(\lambda,\kappa)}(L \cup K, q)$$

where

$$\alpha_L = \exp\left(\frac{i\pi 3(2-r)}{4r}\right)^{-\sigma(L)} \left(\sqrt{\frac{2}{r}} \sin\left(\frac{\pi}{r}\right)\right)^{|\pi_0(L)|+1}$$

and $\sigma(L)$ is the signature of the linking matrix.

The RT-TQFT is a mathematical model for quantum Chern-Simons theory

The TQFT τ_r is considered to be a model for the path integrals $Z_k^{\text{phys}}(M)$ considered by Witten in quantum Chern-Simons theory

$$\tau_r(M) \text{``} = \text{``} \int_{\mathcal{A}/\mathcal{G}} e^{2\pi i k \, S_{\text{CS}}(A)} \, \mathcal{D}A.$$

Remark 1

The rest of the talk concerns the mathematically rigorously constructed quantum invariant $\tau_r(M)$ and their relation to Chern-Simons theory.

Classical solutions in Chern-Simons theory

Let (M, K, κ) be a knotted object. Let $\lambda \in \{1, ..., r-1\}$. We have a correspondence

$$\lambda \longleftrightarrow R_\lambda \longleftrightarrow C_\lambda$$

where

- R_λ is an irreducible G-representation and
- C_λ is a conjugacy class in G obtained by exponentiation of a highest weight v_λ of R_λ - here the Lie algebra $\mathfrak{g}$ is identified with $\mathfrak{g}^*$ through the Killing form.

Let

$$\mathcal{M}(G, M, K, C_\kappa)$$

be the moduli space of flat G-connections on $M \setminus K$ with holonomy C_λ around a component K_j colored with λ.

Semi-classical analysis: the asymptotic expansion conjecture

Let (M, K, κ) be a knotted object. Set

$$\mathrm{CS} = \mathrm{S}_{\mathrm{CS}}(\mathcal{M}(G, M, K, C_\kappa)).$$

Conjecture 1 (The asymptotic expansion conjecture)

There exists

$$\{(d_\theta, b_\theta)\}_{\theta \in \mathrm{CS}} \subset \mathbb{Q} \times \mathbb{C}^*$$

and formal power series

$$\{Z_\theta(k)\}_{\theta \in \mathrm{CS}} \subset k^{-\frac{1}{2}} \mathbb{C}[[k^{-\frac{1}{2}}]]$$

giving an asymptotic expansion in the Poincaré sense

$$\tau_k(M, K, \kappa) \sim_{k \to \infty} \sum_{\theta \in \mathrm{CS}} \exp(2\pi i k \theta) k^{d_\theta} b_\theta (1 + Z_\theta(k)).$$

Analytic continuation: from semi-classical analysis to resurgence and complexification

- **Complexification:** The Chern-Simons action S_{CS} can be holomorphically extended to the $\mathrm{SL}(n, \mathbb{C})$-connections.

- **Analytic continuation:** Witten has proposed an analytic continuation of

$$k \mapsto \mathrm{Z}_k^{\text{phys}}(M)$$

by formally applying Pham-Picard-Lefschetz theory to the holomorphic extension of the Chern-Simons action S_{CS}.

Saddle point analysis of Laplace integrals over Picard-Lefschetz thimbles

- **Laplace integrals:** Let Y be a complex manifold. We discuss resurgence and saddle point analysis of Laplace integrals

$$\mathrm{I}(\lambda) = \int_{\Delta} e^{-\lambda f(z)}\, \omega(z).$$

  Here $f \in \mathcal{O}(Y)$ will be a so-called resurgence phase, and $\Delta \subset Y$ will be a Picard-Lefschetz thimble.

- **Work of Malgrange, Pham and Howls:** We present some results which are natural generalizations of results due to Malgrange, Pham and Howls.

Resurgence phases

- **Resurgence phase:** Let $Y \in \mathcal{M}\mathrm{an}_d(\mathbb{C})$ be a complex manifold of complex dimension d. Let $f \in \mathcal{O}(Y)$. Let S be the set of saddle points of f. Let $\Omega = f(S)$. Let $C = f(Y) \setminus \Omega$. Then f is called a resurgence phase if S is discrete and

$$f : f^{-1}(C) \to C$$

is a fibre bundle.

- **Homological bundle:** Let

$$\mathrm{H} = \mathrm{H}_{d-1}(f^{-1}(\,\cdot\,)) \to C$$

be the associated homological bundle, associated with the Gauss-Manin connection.

Milnor fibrations, vanishing cycles and monodromy

- **Milnor fibration:** If B is a small ball centered at $z \in S \cap f^{-1}(\eta)$ then $f : \mathrm{B} \setminus f^{-1}(\eta) \to \mathrm{D} \setminus \{\eta\}$ is a Milnor fibration with fibres homotopy equivalent to $\vee_{j=1}^{\mu_z} \mathrm{S}_j^{d-1}$.
- **Vanishing cycles and monodromy:** A vanishing cycle σ is a flat section of the homological Milnor fibration

$$\mathrm{H}_{d-1}(B \cap f^{-1}(\,\cdot\,)) \to \mathrm{D} \setminus \{\eta\}.$$

Such cycles extends to flat sections of the homological bundle H associated with $f_{|f^{-1}(C)}$. We let M_z be the monodromy operator of the homological Milnor fibration.

Picard-Lefschetz thimble

Let $\lambda \in \mathbb{C}^*$. Let $\gamma : (\mathbb{R}_{\geq 0}, 0) \to (C \cup \{\eta\}, \eta)$ with $\mathrm{Re}(\lambda(\gamma - \eta))$ strictly increasing. The Picard-Lefschetz thimble $\Delta(\sigma, \gamma)$ is the formal sum of maps $\mathrm{S}^{d-1} \times \mathbb{R}_{\geq 0} \to Y$ with $\Delta(\sigma, \gamma)(t) = \sigma(\gamma(t))$.

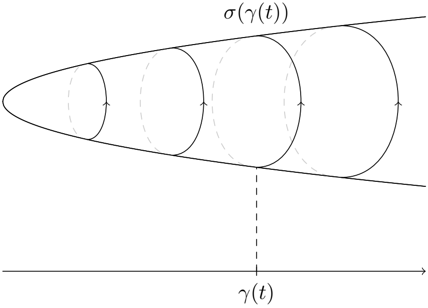

Figure: Thimble $\Delta(\sigma, \gamma)$ in $d = 2$.

Holomorphic saddle point analysis - Malgrange

Let ω be a holomorphic $(d,0)$-form on Y.

Theorem 1

There exists an unbounded set $\mathcal{A} \subset \mathbb{Q}_{0>}$, $\{d_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathbb{N}$ *and* $\{c^\omega_{\alpha,\beta}\}_{\alpha\in\mathcal{A},\ 0\leq\beta\leq d_\alpha} \subset \mathbb{C}$ *giving an asymptotic expansion*

$$\int_{\Delta(\sigma,\gamma)} e^{-\lambda f}\,\omega \sim_{\lambda\to\infty} e^{-\lambda\eta} \sum_{\alpha\in\mathcal{A}} \sum_{\beta=0}^{d_\alpha} c^\omega_{\alpha,\beta}\lambda^{-\alpha}\log(\lambda)^\beta.$$

The set $\exp(2\pi i\mathcal{A})$ *is a subset of the set of eigenvalues of* M_z *and for each* $\alpha \in \mathcal{A}$ *the number* $d_\alpha + 1$ *is less than or equal to the maximal dimension of any Jordan block associated with* $\exp(2\pi i\alpha)$.

Remark 2

There is no condition on the Hessian of f *at* z.

The Borel transform and the Laplace transform

- **The Borel transform:** Let $\{\alpha_j\}_{j=0}^{\infty} \subset \mathbb{R}_{>0}$ be an increasing sequence. Let $\{(\beta_j, c_j)\}_{j=0}^{\infty} \subset \mathbb{N} \times \mathbb{C}$. The Borel transform of the formal series $\tilde{\varphi}(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^{-\alpha_j} \log(\lambda)^{\beta_j}$ is the formal series

$$\mathcal{B}(\tilde{\varphi})(\zeta) = \sum_{j=0}^{\infty} c_j (-1)^{\beta_j} \frac{\partial^{\beta_j}}{\partial {\alpha_j}^{\beta_j}} \left(\frac{\zeta^{\alpha_j - 1}}{\Gamma(\alpha_j)} \right).$$

- **Inverse Laplace transform:** For a function g let $\mathcal{L}_{\mathbb{R}_+}(g)(\lambda) = \int_0^{\infty} e^{-\lambda t} g(t) \; \mathrm{d}\, t$ (provided the integral exsits). Let κ be a complex number with $\mathrm{Re}(\kappa) > 0$ and let $m \in \mathbb{N}$. We have that

$$\mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\lambda^{-\kappa} \log(\lambda)^m) = \lambda^{-\kappa} \log(\lambda)^m,$$
$$\mathcal{B} \circ \mathcal{L}_{\mathbb{R}_+}(\zeta^{\kappa-1} \log(\zeta)^m) = \zeta^{\kappa-1} \log(\zeta)^m.$$

Resurgence properties of the Borel transform

- **Algebra of resurgent functions:** The algebra of resurgent functions on the Riemann surface C is $\mathcal{R}(C) = \mathcal{O}(\tilde{C})$ where $\tilde{C} \to C$ is the universal covering space.

- **Borel transform:** The Borel transform $\mathcal{B}_{\sigma,\omega}$ is

$$\mathcal{B}_{\sigma,\omega}(\zeta) = \sum_{\alpha \in \mathcal{A}} \sum_{\beta=0}^{d_\alpha} c^{\omega}_{\alpha,\beta} (-1)^\beta \frac{\partial^\beta}{\partial \alpha^\beta} \left(\frac{\zeta^{\alpha-1}}{\Gamma(\alpha)} \right).$$

Theorem 2

We have that $\mathcal{B}_{\sigma,\omega} \in \mathcal{R}(C - \eta)$ and the following formula holds

$$\mathcal{B}_{\sigma,\omega}(\zeta) = \int_{\sigma(\zeta+\eta)} \frac{\omega}{\mathrm{d} f}$$

Resummation and analytic continuation

We can recover the original Laplace integral through the Laplace transform and the Laplace integral admits a multivalued analytic extension in λ.

Theorem 3

We have that

$$\int_{\Delta(\sigma,\gamma)} e^{-\lambda f}\ \omega = \oint_{\gamma} e^{-\lambda\zeta}\mathcal{B}_{\sigma,\omega}(\zeta-\eta)\ \mathrm{d}\,\zeta.$$

For every $\phi \in \Omega$, the cycle $\chi = \mathrm{var}_{\partial\,\mathrm{D}'(\phi)}(\sigma)$ is a sum of vanishing cycles above ϕ and we have that

$$\mathrm{Var}_{\partial\,\mathrm{D}'(\phi)-\eta}\left(\mathcal{B}_{\sigma,\omega}\right)(\zeta) = \mathcal{B}_{\chi,\omega}(\zeta+\eta-\phi).$$

The Borel transform of quantum invariants

Consider a 3-manifold M for which the AEC hold

$$\tau_k(M) \sim_{k\to\infty} \sum_{\theta\in \mathrm{CS}} \exp(2\pi i k\theta) k^{d_\theta} b_\theta (1 + Z_\theta(k)).$$

In two cases we prove resurgence properties of the Borel transform

$$\mathcal{B}(\mathrm{Z}_\theta)(\zeta) \in \zeta^{-\frac{1}{2}}\mathbb{C}[[\zeta^{\frac{1}{2}}]]$$

1. **Case one:** Seifert fibered integral homology three-spheres with at least three exceptional fibers (with CS replaced by $\mathrm{CS}_{\mathbb{C}}$).
2. **Case two:** Hyperbolic surgeries on the figure eight knot.

The Seifert fibered case X

Let $n \in \mathbb{N}$ and $p_j, q_j \in \mathbb{Z}, j = 1, \ldots, n$ with $(p_j, q_j) = 1$ and $(p_j, p_l) = 1$ for $l \neq j$. Consider the Seifert fibered three-manifold $X = \Sigma((p_1/q_1), \ldots, (p_n/q_n))$. Assume $\mathrm{H}_1(X, \mathbb{Z}) = 0$.

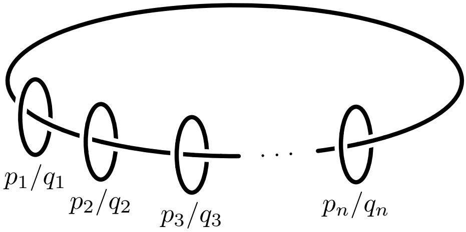

Figure: Surgery link for X.

Normalized invariant $\widetilde{Z}_k(X)$

Let $P = \prod_{i=1}^n p_i$, $H = P \sum_{j=1}^n \frac{q_j}{p_j}$. Let $S(\,\cdot\,,\,\cdot\,)$ be the Dedekind sum and set

$$C_k = \sqrt{P} \exp\left(\left(3 - \frac{H}{P} + 12 \sum_{j=1}^n S(q_j, p_j)\right) \frac{i\pi}{2k} - \frac{\pi i 3H}{4}\right).$$

Consider the normalized quantum invariant ($G = \mathrm{SU}(2)$)

$$\widetilde{Z}_k(X) = \frac{\tau_k(X)}{\tau_k(S^2 \times S^1)} C_k.$$

Set

$$\mathrm{CS}^*_{\mathbb{C}} = \mathrm{S}_{\mathrm{CS}}(\mathcal{M}^*(\mathrm{SL}(2,\mathbb{C}), X)).$$

The Borel transform and complex Chern-Simons

Theorem 4

There exists $\{\mathrm{Z}_\theta(x) \in \mathbb{C}[x]\}_{\theta \in \mathrm{CS}^*_{\mathbb{C}}}$ *and* $\mathrm{Z}_\infty(x) \in x^{-\frac{1}{2}}\mathbb{C}[[x^{-1}]]$:

$$\widetilde{\mathrm{Z}}_k(X) \sim_{k\to\infty} \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}} e^{2\pi i k\theta}\, \mathrm{Z}_\theta(k) + \mathrm{Z}_\infty(k).$$

We have that $\mathcal{B}(\mathrm{Z}_\infty)(\zeta)$ *is the resurgent function given by*

$$-\sqrt{\frac{P2}{i\zeta\pi H}}\left(\sinh\left(\sqrt{\frac{i2P\pi\zeta}{H}}\right)\right)^{2-n} \prod_{j=1}^{n} \sinh\left(\sqrt{\frac{i2P\pi\zeta}{H}}\frac{1}{p_j}\right).$$

Let Ω *be the set of poles of* $\mathcal{B}(\mathrm{Z}_\infty)$. *Then we have*

$$\mathrm{CS}^*_{\mathbb{C}} = \frac{i}{2\pi}\Omega \mod \mathbb{Z}.$$

Exact resummation

Introduce for $\mu \in \mathbb{Q}/\mathbb{Z}$ the set

$$\mathcal{T}(\mu) = \{m = 1, ..., 2P-1 : -m^2 H/4P = \mu \mod \mathbb{Z}\}.$$

Introduce the integral operators $\mathcal{L}_\mu$ defined by

$$\mathcal{L}_\mu(\hat{\varphi})(\xi) = \frac{1}{2\pi i} \sum_{x \in \mathcal{T}(\mu)} \oint_{y=2\pi i x} \frac{e^{\xi \frac{Hiy^2}{8\pi P}}}{(1-e^{-\xi y})} \frac{yH}{P4} \hat{\varphi}\left(\frac{y^2}{i8\pi P}\right) \, \mathrm{d}\, y.$$

Theorem 5

We have

$$\widetilde{Z}_k(X) = \int_0^\infty e^{-k\xi} \mathcal{B}(Z_\infty)(\xi) \, \mathrm{d}\, \xi + \sum_{\theta \in \frac{i}{2\pi}\Omega \mod \mathbb{Z}} \mathcal{L}_\theta \left(\mathcal{B}(Z_\infty)\right)(k).$$

Inspiration: work of Lawrence-Rozansky and work of Gukov-Marino-Putrov

- **Work of Lawrence-Rozansky:** The existence of an expansion

$$\widetilde{\mathrm{Z}}_k(X) \sim_{k\to\infty} \sum_{\theta\in R(X)} e^{2\pi i k\theta}\, \mathrm{Z}_\theta(k) + \mathrm{Z}_\infty(k)$$

  where $R(X) \subset \mathbb{Q}/\mathbb{Z}$ is a finite set was proven by Lawrence and Rozansky. Our contribution is to show $R(X) \subset \mathrm{CS}^*_{\mathbb{C}}$.

- **Work of Gukov-Marino-Putrov:** Previous to our work Gukov-Marino-Putrov have analysed $\tau_k(X)$ for some examples with 3 exceptional fibers.

The hyperbolic case $M_{p/s}$

We now turn to the hyperbolic three-manifolds $M_{p/s}$ with surgery link giving by the figure eight knot with framing p/s. Choose $c, d \in \mathbb{Z}$ with $pd - cs = 1$.

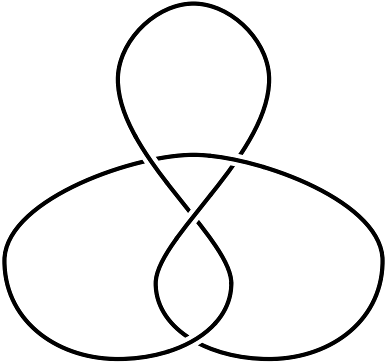

Figure: Figure eight knot

Quantum invariants and Fadeev's quantum dilogarithm

Andersen-Hansen have given an expression for $\tau_k(M_{p/s})$ involving Fadeev's quantum dilogarithm with parameter $\kappa = \pi/k \in (0,1)$

$$S_\kappa(z) = \exp\left(\frac{1}{4}\int_{\widetilde{C}} \frac{e^{zy}}{\sinh(\pi y)\sinh(\kappa y) y}\,\mathrm{d}\,y\right).$$

Here $|\mathrm{Re}(z)| < \kappa + \pi$, and $\widetilde{C} = (-\infty, -1/2) \cup \Delta \cup (1/2, \infty)$ where $\Delta = \mathrm{D}_{1/2}(0) \cap \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

Semi-classical approximation: The quantum dilogarithm can be semi-classically approximated by Euler's dilogarithm given by

$$\mathrm{Li}_2(z) = -\int_{\gamma_z} \frac{\log(1-u)}{u}\,\mathrm{d}\,u$$

where γ_z is the homotopy class of a path from 0 to z in $\mathbb{C} \setminus \{1\}$.

A conjecture due to Andersen-Hansen

Conjecture 2

Introduce for $\alpha, \beta \in \{0, 1\}$ *and* $n \in \mathbb{Z}/|s|\mathbb{Z}$ *the function*

$$\Phi^n_{\alpha,\beta}(x, y) = \frac{\mathrm{Li}_2(e^{2\pi i(x+y)}) - \mathrm{Li}_2(e^{2\pi i(x-y)})}{4\pi^2} - \frac{dn^2}{s} + (-\frac{p}{4s}x + \frac{n}{s} + y + \alpha + \beta)x + y(\alpha - \beta).$$

$\exists$ *2-dimensional chains* $\Gamma^n_{\alpha,\beta} \subset \mathbb{C}^2$ *meeting only non-degenerate stationary points of* $\Phi^n_{\alpha,\beta}$ *in* $\{(x, y) \in \mathbb{R} \times \mathbb{C} : e^{2\pi i y} \in]-\infty, 0[\}$, *and holomorphic 2-forms* $\chi^n_{\alpha,\beta}$: *for some* $m_0 \in \mathbb{N}$ *and* $\forall m \in \mathbb{N}$ *we have*

$$\tau_k(M_{p/s}) = k \sum_n \sum_{\alpha,\beta} \int_{\Gamma^n_{\alpha,\beta}} e^{2\pi i k \Phi^{\alpha,\beta}_n} \chi^n_{\alpha,\beta} + \mathcal{O}(k^{m_0 - m}).$$

A resurgence corollary

By using the framework of resurgence phases we obtain:

Theorem 6

Assume the conjecture of Andersen-Hansen is true. Then there exists $\{\mathrm{Z}_\theta(x) \in x^{-1}\mathbb{C}[[x^{-1}]]\}_{\theta \in \mathrm{CS}}$ *with*

$$\tau_k(M_{p/s}) \sim_{k\to\infty} k \sum_{\theta \in \mathrm{CS}} e^{2\pi i k\theta}\, \mathrm{Z}_\theta(k).$$

For each $\theta \in \mathrm{CS}$ *the Borel transform of* Z_θ *is a resurgent series*

$$\mathcal{B}(\mathrm{Z}_\theta) \in \mathcal{R}(\mathbb{C} \setminus \Omega(\theta))$$

where

$$\mathrm{CS}_{\mathbb{C}} - \theta \supset \frac{i}{2\pi}\Omega(\theta) \mod \mathbb{Z}.$$

Stokes phenomena

There are interesting resurgence relations between disctinct Chern-Simons values.

Corollary 7

Assume the conjecture of Andersen-Hansen is true. With notation as above, we have that each $\mathcal{B}(\mathrm{Z}_\theta)$ is a finite sum of resurgent functions

$$\mathcal{B}(\mathrm{Z}_\theta) = \sum_{\lambda \in \Lambda(\theta)} \check{\mathrm{Z}}_\lambda(\theta)$$

For $\theta, \theta' \in \mathrm{CS}_\mathbb{C}$ and $\lambda \in \Lambda(\theta)$, there exists $n_{\lambda,\mu}$:

$$\mathrm{Var}_{2\pi i(\theta - \theta')}(\check{\mathrm{Z}}_\lambda(\theta)) = \sum_{\mu \in \Lambda(\theta')} n_{\lambda,\mu} \check{\mathrm{Z}}_\mu(\theta').$$

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Quantum invariants of mapping tori

For a surface $\Sigma = (\Sigma, p)$ of genus $g \geq 2$ and a mapping class $\varphi \in \Gamma(\Sigma, p)$ consider the mapping torus

$$(T_\varphi, L) = (\Sigma \times \mathrm{I}\,/[(x,0) \sim (\varphi(x), 1)], [\{p\} \times \mathrm{I}]).$$

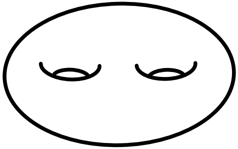

Figure: Surface Σ.

The TQFT τ_k induces $\mathrm{V}_k : \Gamma(\Sigma, p) \to \mathrm{PGL}(\mathrm{V}_k(\Sigma, p, \kappa))$ and:

$$\tau_k(T_\varphi, L, \kappa) = \mathrm{tr}(\mathrm{V}_k(\varphi)).$$

The quantum representations and the modular functor

- **The quantum representation** Z_k : Using moduli space techniques Hitchin have constructed a projective representation Z_k of $\Gamma(\Sigma, p)$ known as the quantum representation.

- **Projective equivalence:** There is a projective equivalence

$$\mathrm{Z}_k \simeq \mathrm{V}_k, \text{ projective equivalence.}$$

The proof relies on work by many authors: Andersen-Ueno, Tsuchiya-Ueno-Yamada, Axelrod-Della Pietra-Witten, Hitchin, Laszlo, and Blanchet-Habegger-Vogel-Masbaum.

Moduli spaces of flat connections: the coprime case

Let $C \in Z(G)$ be a generator. Let

$$\mathcal{M} = \mathcal{M}(G, \Sigma, p, C)$$

be the moduli space of flat G-connections on $\Sigma \setminus p$ with holonomy around p equal to C. Observe that $\Gamma(\Sigma, p)$ act on $\mathcal{M}$ and let

$$\mathcal{M}^{\varphi} = \{x \in \mathcal{M} : \varphi(x) = x\}.$$

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The moduli space $\mathcal{M}$ supports a symplectic form ω and a prequantum line bundle

$$\mathcal{L}_{\mathrm{CS}} \to \mathcal{M}$$

The mapping class group $\Gamma(\Sigma, p)$ act sympletically on $\mathcal{M}$.

The projective quantum representation Z_k

- **The Verlinde bundle:** Let $\mathcal{T}$ be Teichmüller space. Each $\sigma \in \mathcal{T}$ induces a Kähler structure on $\mathcal{M}$ by the Narasimhan-Seshadri theorem. The *Verlinde bundle* $\mathrm{H}_k \to \mathcal{T}$ is the bundle with fibre at σ given by the level k quantization

$$\mathrm{H}_k(\sigma) = \mathrm{H}^0(\mathcal{M}_\sigma, \mathcal{L}_{\mathrm{CS}}^{\otimes k}).$$

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- **The quantum action:** There exists a lift $\varphi_k^* : \mathrm{H}_k \to \varphi^*(\mathrm{H}_k)$ and a projectively flat connection ∇ on H_k that is preserved by φ_k^*. By composing φ_k^* with parallel transport of ∇ we obtain

$$\mathrm{Z}_k : \Gamma(\Sigma, p) \to \mathrm{PGL}\left(\mathrm{H}_k(\sigma)\right).$$

Quantization of symplectomorphisms

The construction of the quantum representation fits into a broader framework: given a symplectic manifold M with a prequantum bundle L and $\varphi \in \mathrm{Symp}(M)$, how does one quantize φ and compute its trace? This has been considered by several authors:

- Charles: in relation to Z_k .
- Zeldith: in relation to liftting a contactomorphism on the unit bundle of L^*,
- Ioos: in relation to non-Kähler polarizations and also in relation to quantum topology.

Our results relies on work by Karabegov-Schlichenmaier and Zeldith on Toeplitz operator theory and the Bergman kernel, and previous work of Andersen on the Hitchin connection.

Moduli space of flat connections on the mapping torus

Let $\mathcal{M}(G, T_\varphi, L, C)$ be the moduli space of flat G-connections on $T_\varphi \setminus L$ with holonomy C around L. The inclusion

$$\iota : \Sigma \hookrightarrow T_\varphi$$

induces a map

$$\iota^* : \mathcal{M}(G, T_\varphi, L, C) \to \mathcal{M}^\varphi.$$

Set

$$\mathrm{CS} = \mathrm{S}_{\mathrm{CS}}(\mathcal{M}(G, T_\varphi, L, C)).$$

For $\theta \in \mathrm{CS}$, let

$$2m_\theta = \mathsf{max}(\dim(\mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})) : {\iota^*}^{-1}(z) \subset {\mathrm{S}_{\mathrm{CS}}}^{-1}(\theta))$$

The case of a non-degenerate fixed point set $\mathcal{M}^\varphi$

We prove the following:

Theorem 8

If every component of $\mathcal{M}^\varphi$ is an integral manifold of

$$\operatorname{Ker}(\mathrm{d}\,\varphi - \mathrm{Id}) \subset T\mathcal{M}_{|\mathcal{M}^\varphi}$$

then there exists for each $\theta \in \mathrm{CS}$ smooth densities on $\mathcal{M}^\varphi$

$$\Omega_\alpha(\theta), \alpha = 0, 1, 2, 3, \ldots$$

giving an asymptotic expansion

$$\operatorname{tr}\left(\mathrm{Z}_k(\varphi)\right) \sim_{k\to\infty} \sum_{\theta\in\mathrm{CS}} e^{2\pi i r\theta} r^{m_\theta} \sum_{\alpha=0}^{\infty} r^{-\frac{\alpha}{2}} \int_{\mathcal{M}^\varphi} \Omega_\alpha(\theta).$$

Saddle point analysis

Let $\mathbb{H} = \{z \in \mathbb{C} : \Re(z) \leq 0\}$. There exists a smooth function

$$\widehat{\varphi} \in C^{\infty}(\mathcal{M}, \mathbb{H}/2\pi i\mathbb{Z})$$

and smooth top forms

$$\{\Omega_n^{\varphi}\}_{n=0}^{\infty} \subset \Omega^{2n_0}(\mathcal{M})$$

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with the following properties.We have that

$$\widehat{\varphi} \circ \iota^* = 2\pi i \, \mathrm{S}_{\mathrm{CS}} \, .$$

Furthermore, $\widehat{\varphi}$ is real analytic near $\mathcal{M}^\varphi$ and

$$\mathcal{M}^\varphi = \{\mathrm{d}\,\widehat{\varphi} = 0\} \cap \mathrm{Re}(\widehat{\varphi})^{-1}(0).$$

For every $\tilde{m} \in \mathbb{N}$ we have that

$$\mathrm{tr}\,(\mathrm{Z}_k(\varphi)) = r^{n_0} \sum_{n=0}^{\tilde{m}} r^{-n} \int_{\mathcal{M}} e^{r\widehat{\varphi}} \, \Omega_n^\varphi + O(k^{n_0 - \tilde{m} - 1}).$$

Complexification

- **Complexification:** For every $z \in \mathcal{M}^{\varphi}$ there exists a nbhd U of z and a holomorphic function

$$\widehat{\varphi}_{\mathbb{C}} \in \mathcal{O}(U + \sqrt{-1}U)$$

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- **Holomorphic saddle point analysis:** It follows that we can apply the holomorphic version of saddle point analysis to

$$\operatorname{tr}\left(\mathrm{Z}_k(\varphi)\right) = r^{n_0} \sum_{n=0}^{\tilde{m}} r^{-n} \int_{\mathcal{M}} e^{r\widehat{\varphi}}\, \Omega_n^{\varphi} + O(k^{n_0-\tilde{m}-1})$$

The advantage is that this imposes no condition on the Hessian of $\widehat{\varphi}_{\mathbb{C}}$.

The case of a degenerate fixed point set $\mathcal{M}^\varphi$

We prove the following result.

Theorem 9

Assume every $z \in \mathcal{M}^\varphi$ satisfy one of the following conditions:

- z *is a smooth point with* $T_z\mathcal{M}^\varphi = \mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})$,
- $\dim(\mathrm{Ker}(\mathrm{d}\,\varphi_z - \mathrm{Id})) \leq 1$, *or*
- z *is an isolated saddle point of the germ of* $\widehat{\varphi}_{\mathbb{C}}$ *at* z.

Then $\forall \theta \in \mathrm{CS}$ $\exists$ *an unbounded subset* $A_\theta \subset \mathbb{Q}_{\leq 0}$, $n_\theta \in \mathbb{Q}_{\geq 0}$, $d_\theta \in \mathbb{N}$ *and* $\{c_{\alpha,\beta}(\theta)\}_{\alpha \in A_\theta,\ 0 \leq \beta \leq d_\theta} \subset \mathbb{C}$ *giving an expansion*

$$\mathrm{tr}\,(\mathrm{Z}_k(\varphi)) \sim_{k \to \infty} \sum_{\theta \in \mathrm{CS}} e^{2\pi i r \theta} r^{n_\theta} \sum_{\alpha \in A_\theta} \sum_{\beta=0}^{d_\theta} c_{\alpha,\beta}(\theta) r^\alpha \log(r)^\beta.$$

Summary of results

- The AEC holds for a mapping torus T_φ of $\varphi \in \Gamma(\Sigma_{g,1}), g \geq 2$ for which $\mathcal{M}^\varphi$ is non-degenerate or at least not too singular.
- Let X be a Seifert fibered homology 3-sphere. The AEC holds over $\mathrm{CS}_\mathbb{C}$ and the Borel transform of the series Z_∞ (associated with the trivial connection) is resurgent with poles equal (modulo $\mathbb{Z}$) to $\mathrm{CS}^*_\mathbb{C}$. Moreover Z_∞ determines $\tau_r(X)$.
- If a conjecture of Andersen and Hansen holds, then the AEC holds for hyperbolic surgeries $M_{p/s}$ on the figure eight knot, and each $Z_\theta, \theta \in \mathrm{CS}$ will have a resurgent Borel transform.

Thank you for your attention!