# Some naturally defined star products for Kähler manifolds

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- One mathematical aspect of quantization is the passage from the commutative world to the non-commutative world.
- one way: a deformation quantization (also called star product)
- can only be done (at least if one wants to quantize all smooth functions) on the level of formal power series over the algebra of functions
- was pinned down in a mathematically satisfactory manner by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer.
- second way: operator quantization
- for me also important: the relation between the two methods

# Outline

- give an overview of some naturally defined star products in the case that our “phase-space manifold” is a (compact) Kähler manifold
- here we have additional complex structure and search for star products respecting it
- yield star products of separation of variables type (Karabegov) resp. Wick or anti-Wick type (Bordemann and Waldmann)
- both constructions are quite different, but there is a 1:1 correspondence (Neumaier)
- still quite a lot of them

- single out certain naturally given ones.
- restrict to quantizable Kähler manifolds
- Berezin-Toeplitz star product, Berezin transform, Berezin star product
- related to the Berezin-Toeplitz operator quantization
- a side result: star product of geometric quantization
- all of the above are equivalent star product, but not the same
- give Deligne-Fedosov class and Karabegov forms
- give the equivalence transformations

# Geometric set-up

- $(M, \omega)$ a Kähler manifold.
  $M$ a complex manifold, and $\omega$, a non-degenerate closed $(1,1)$-form which is a positive form
- for pseudo-Kähler drop positive definite
- $C^\infty(M)$ the algebra of complex-valued differentiable functions with associative product given by point-wise multiplication
- define the Poisson bracket

$$\{ f, g \} := \omega(X_f, X_g) \qquad \omega(X_f, \cdot) = df(\cdot)$$

- $C^\infty(M)$ becomes a Poisson algebra.

# Star product

*star product* for $M$ is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, such

1. $f \star g = f \cdot g \mod \nu$,
2. $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

Also

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M),$$

differential (or local) if $C_k(\ ,\ )$ are bidifferential operators.
Usually: $1 \star f = f \star 1 = f$.

## Equivalence of star products

$\star$ and $\star'$ (the same Poisson structure) are *equivalent* means there exists
a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ and $B(f) \star' B(g) = B(f \star g)$.

to every equivalence class of a differential star product one assigns its Deligne-Fedosov class

$$cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M, \mathbb{C})[[\nu]]).$$

gives a 1:1 correspondence

Existence: by DeWilde-Lecomte, Omori-Maeda-Yoshioka, Fedosov, ...., Kontsevich.

# Separation of Variables type

- (pseudo-)Kähler case: we look for star products adapted to the complex structure
- separation of variables type (Karabegov)
- Wick and anti-Wick type (Bordemann - Waldmann)
- Karabegov convention: of separation of variables type if in $C_k(.,.)$ for $k \geq 1$ the first argument differentiated in anti-holomorphic and the second argument in holomorphic directions.
- we call this convention separation of variables (anti-Wick) type and call a star product of separation of variables (Wick) type if the role of the variables is switched
- we need both conventions

# Karabegov construction (sketch of a sketch)

- $(M, \omega_{-1})$ the pseudo-Kähler manifold
- a formal deformation of the form $(1/\nu)\omega_{-1}$ is a formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\,\omega_1 + \dots$$

$\omega_r,\ r \geq 0$, closed (1,1)-forms on $M$.

- Karabegov: to every such $\widehat{\omega}$ there exists a star product $\star$ of anti-Wick type
- and vice-versa
- Karabegov form of the star product $\star$ is $kf(\star) := \widehat{\omega}$,
- the star product $\star_K$ with classifying Karabegov form $(1/\nu)\omega_{-1}$ is Karabegov's standard star product.

- Formal Berezin transform
- for local antiholomorphic functions $a$ and holomorphic functions $b$ on $U \subset M$ we have the relation

$$a \star b = I_\star(b \star a) = I_\star(b \cdot a),$$

- can be written as

$$I_\star = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \quad I_0 = id, \quad I_1 = \Delta.$$

- the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

- Start with $\star$ separation of variables type (anti-Wick) $(M, \omega_{-1})$
- opposite of the dual

$$f \star' g = I^{-1}(I(f) \star I(g)).$$

on $(M, \omega_{-1})$, is of Wick type

- the formal Berezin transform $I_\star$ establishes an equivalence of the star products

$$(\mathcal{A}, \star) \text{ and } (\mathcal{A}, \star')$$

# Classifying forms

$\star$ star product of anti-Wick type with Karabegov form $kf(\star) = \widehat{\omega}$
Deligne-Fedosov class calculates as

$$cl(\star) = \frac{1}{\mathrm{i}}([\widehat{\omega}] - \frac{\delta}{2}).$$

[..] denotes the de-Rham class of the forms and $\delta$ is the canonical class of the manifold i.e. $\delta := c_1(K_M)$.

standard star product $\star_K$ (with Karabegov form $\widehat{\omega} = (1/\nu)\omega_{-1}$)

$$cl(\star_K) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega_{-1}] - \frac{\delta}{2}).$$

- For the Karabegov form to be in 1:1 correspondence, we need to fix a convention: Wick or anti-Wick for reference
- here we refer to the anti-Wick type product
- it $\star$ is of Wick type we set

$$kf(\star) := kf(\star^{op}),$$

- where

$$f \star^{op} g = g \star f$$

is obtained by switching the arguments. It is a star product of (anti-Wick) type for the pseudo-Kähler manifold $(M, -\omega)$

# Other general constructions

- Bordemann and Waldmann: modification of Fedosov's geometric existence proof.
- fibre-wise Wick product.
- by a modified Fedosov connection a star product $\star_{BW}$ of Wick type is obtained.
- Karabegov form is $-(1/\nu)\omega$
- Deligne class class

$$cl(\star_{BW}) = -cl(\star_{BW}^{op}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2}).$$

Neumaier: by adding a formal closed $(1,1)$ form as parameter each star product of separation of variables type can be obtained by the Bordemann-Waldmann construction

Reshetikhin and Takhtajan:
formal Laplace expansions of formal integrals related to the star product.
coefficients of the star product can be expressed (roughly) by Feynman diagrams

# BEREZIN-TOEPLITZ STAR PRODUCT

- compact and quantizable Kähler manifold $(M, \omega)$,
- quantum line bundle $(L, h, \nabla)$, $L$ is a holomorphic line bundle over $M$, $h$ a hermitian metric on $L$, $\nabla$ a compatible connection
- recall $(M, \omega)$ is quantizable, if there exists such $(L, h, \nabla)$, with
$$curv_{(L,\nabla)} = -\mathrm{i}\,\omega$$
- consider all positive tensor powers $(L^m, h^{(m)}, \nabla^{(m)})$,

scalar product

$$\langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi)\,\Omega, \qquad \Omega := \frac{1}{n!}\underbrace{\omega \wedge \omega \cdots \wedge \omega}_{n}$$

$$\Pi^{(m)} : L^2(M, L^m) \longrightarrow \Gamma_{hol}(M, L^m)$$

Take $f \in C^\infty(M)$, and $s \in \Gamma_{hol}(M, L^m)$

$$s \quad \mapsto \quad T_f^{(m)}(s) := \Pi^{(m)}(f \cdot s)$$

defines

$$T_f^{(m)} : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)$$

the Toeplitz operator of level $m$.

Berezin-Toeplitz operator quantization

$$f \mapsto \left(T_f^{(m)}\right)_{m\in\mathbb{N}_0}.$$

has the correct semi-classical behavior

Theorem (Bordemann, Meinrenken, and Schl.)

(a)

$$\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$$

(b)

$$||m\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| = O(1/m)$$

(c)

$$||T_f^{(m)}T_g^{(m)} - T_{f\cdot g}^{(m)}|| = O(1/m)$$

Theorem (BMS, Schl., Karabegov and Schl.)

$\exists$ a unique differential star product

$$f \star_{BT} g = \sum \nu^k C_k(f,g)$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}$$

Further properties: is of separation of variables type (Wick type)

classifying Deligne-Fedosov class $\frac{1}{i}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$ and Karabegov form $\frac{-1}{\nu}\omega + \omega_{can}$

possible: auxiliary hermitian line (or even vector) bundle can be added, meta-plectic correction.

Further result: The Toeplitz map of level $m$

$$T^{(m)} : C^\infty(M) \to End(\Gamma_{hol}(M, L^m))$$

is surjective

implies that the operator $Q_f^{(m)}$ of geometric quantization (with holomorphic polarization) can be written as Toeplitz operator of a function $f_m$ (maybe different for every $m$)

indeed Tuynman relation:

$$Q_f^{(m)} = \mathrm{i}\ T^{(m)}_{f-\frac{1}{2m}\Delta f}$$

- star product of geometric quantization
- set $B(f) := (id - \nu\frac{\Delta}{2})f$

$$f \star_{GQ} g = B^{-1}(B(f) \star_{BT} B(g))$$

defines an equivalent star product

- can also be given by the asymptotic expansion of product of geometric quantization operators
- it is not of separation of variable type
- but equivalent to $\star_{BT}$.

## Where is the Berezin star product ??

- It is an important star product: Berezin, Cahen-Gutt-Rawnsley, etc.
- The original definition is limited in applicability.
- We will give a definition for quantizable Kähler manifold.
- Clue: define it as the opposite of the dual of $\star_{BT}$.
- $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- Problem: How to determine $I$?
- describe the formal $I$ by asymptotic expansion of some geometrically defined $I^{(m)}$

- assume the bundle $L$ is very ample (i.e. has enough global sections)
- pass to its dual $(U,k) := (L^*, h^{-1})$ with dual metric $k$
- inside of the total space $U$, consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda,\lambda) = 1\},$$

- $\tau : Q \to M$ (or $\tau : U \to M$) the projection, coherent vectors/states in the sense of Berezin-Rawnsley-Cahen-Gutt:

Take $\alpha = \in U \setminus 0$, $m \in \mathbb{N}$

then the coherent vector $e_\alpha^{(m)} \in \Gamma_{hol}(M, L^m)$ is given

$$\alpha^{\otimes m}(s(\tau(\alpha))) = \langle e_\alpha^{(m)}, s\rangle$$

(for all $s \in \Gamma_{hol}(M, L^m)$).

As

$$e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \qquad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \ .$$

we obtain the coherent state

$$x \in M \mapsto \mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m))$$

with $\alpha = \tau^{-1}(x) \in U \setminus 0$

- Bergman projectors $\Pi^{(m)}$, Bergman kernels, ....
- Covariant Berezin symbol $\sigma^{(m)}(A)$ (of level $m$) of an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$

$$\sigma^{(m)}(A) : M \to \mathbb{C},$$

$$x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, Ae_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle} = \mathrm{Tr}(AP_x^{(m)})$$

# Importance of the covariant symbol

- Construction of the Berezin star product, only for limited classes of manifolds (see Berezin, Cahen-Gutt-Rawnsley)
- $\mathcal{A}^{(m)} \leq C^\infty(M)$, of level $m$ covariant symbols.
- symbol map is injective (follows from Toeplitz map surjective)
- for $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators $A$ and $B$ are uniquely fixed
- $$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$
- $\star_{(m)}$ on $\mathcal{A}^{(m)}$ is an associative and noncommutative product
- Crucial problem, how to obtain from $\star_{(m)}$ a star product for all functions (or symbols) independent from the level $m$ ?

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

Theorem: (Karabegov - Schl.)
$I^{(m)}(f)$ has a complete asymptotic expansion as $m \to \infty$

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

$I_i : C^\infty(M) \to C^\infty(M)$, $I_0(f) = f, \qquad I_1(f) = \Delta f.$

- $\Delta$ is the Laplacian with respect to the metric given by the Kähler form $\omega$

# INTEGRAL REPRESENTATION

$\tau(\alpha) = x$, $\tau(\beta) = y$ with $\alpha, \beta \in Q$

$$\left(I^{(m)}(f)\right)(x) = \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta)$$
$$= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \int_M \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle f(y)\Omega(y) \,.$$

Note that:

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle,$$
$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta)\cdot\mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle$$

are well-defined on $M$ and on $M \times M$ respectively.
need asymptotic expansion of Bergman kernel $\mathcal{B}_m$ outside (but near) the diagonal.
one crucial result joint with Karabegov (Crelle)

# Berezin star product

- from asymptotic expansion of the Berezin transform get formal expression

$$I = \sum_{i=0}^{\infty} I_i \, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M)$$

- set $f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g))$
- $\star_B$ is called the Berezin star product
- $I$ gives the equivalence to $\star_{BT}$ ($I_0 = id$). Hence, the same Deligne-Fedosov classes
- if the covariant symbol star product works, it will coincide with the star product $\star_B$.

- separation of variables type (but now of anti-Wick type).
- Karabegov form is $\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$
- $u_m$ is the Bergman kernel $\mathcal{B}_m(\alpha,\beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle$ evaluated along the diagonal
- $\mathbb{F}$ means: take asymptotic expansion in $1/m$ as formal series in $\nu$
- $I = I_{\star_B}$, the geometric Berezin transform equals the formal Berezin transform of Karabegov for $\star_B$
- both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other

# Summary of naturally defined star product

| | name | Karabegov form | Deligne Fedosov class |
|---|---|---|---|
| $\star_{BT}$ | Berezin-Toeplitz | $\frac{-1}{\nu}\omega + \omega_{can}$ (Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{B}$ | Berezin | $\frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m)$ (anti-Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{GQ}$ | geometric quantization | (—) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{K}$ | standard product | $(1/\nu)\,\omega$ (anti-Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2})$. |
| $\star_{BW}$ | Bordemann-Waldmann | $-(1/\nu)\,\omega$ (Wick) | $\frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$. |

$u_m$ Bergman kernel evalulated along the diagonal in $Q \times Q$

$\delta$ the canonical class of the manifold $M$

- Berezin transform is not only the equivalence relating $\star_{BT}$ with $\star_B$
- also it (resp. the Karabegov form) can be used to calculate the coefficients of these naturally defined star products (at least recursively)
- either directly
- or with the help of the certain type of graphs (see the very interesting works (independent) of Gammelgaard (uses the Karabegov form) and Hua Xu (uses the Berezin transform, resp. Bergman kernel).
- For the asymptotic expansion of $u_m$ and its relation to the asymptotics of the coherent state embedding (pull-back of Fubini–Study form) see Zelditch, Tian, Yau
- leads to questions about extremal metrics, balanced embeddings,....

- Schlichenmaier, M., Berezin-Toeplitz quantization and star products for compact Kähler manifolds, Contemp. Math. 583, 2012, http://dx.doi.org/10.1090/conm/583/11573
- Schlichenmaier, M., Berezin-Toeplitz Quantization for Compact Kähler Manifolds. A Review of Results, Advances in Mathematical Physics, Vol. 2010, 38 pages, doi:10.1155/2010/927280.
- Schlichenmaier, M., Some naturally defined star products for Kähler manifolds, Trav. math. 20(2012), 187–204.

- Statement of the previous theorem corresponds to the fact that we have a continuous field of $C^*$-algebras (with additionally Dirac condition on commutators).
- over $I = \{0\} \cup \{\frac{1}{m} \in \mathbb{N}\}$,
- over $\{0\}$ we set the algebra $C^\infty(M)$, over $\frac{1}{m}$ the algebra $\mathrm{End}(\Gamma_{hol}(M, L^m))$,
- section is given by $f \in C^\infty(M)$
-

$$f \quad \mapsto \quad (\, f\,,\ T_f^{(m)}, m \in \mathbb{N}).$$