HITCHIN SYSTEMS ON HYPERELLIPTIC CURVES

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$\Sigma$ – genus $g$ algebraic curve / $\mathbb{C}$, $G$ – complex s/s Lie group, $\mathfrak{g} = \mathcal{L}ie(G)$, $P_0$ – a fixed principle $C^\infty$ $G$-bundle on $\Sigma$.

**Holomorphic structure** = $(0, 1)$-connection on $P_0$ is a $(0, 1)$ $\mathfrak{g}$-valued form $\omega$ on $\Sigma$ with gauge transformation $\omega \rightarrow \gamma \omega \gamma^{-1} - (\bar{\partial} \gamma) \gamma^{-1}$ under gluing function $\gamma$.

$\mathcal{A} = \{(P_0, \omega)\} \rightarrow$ space of s/s holomorphic structures on $P_0$, $\mathcal{G}$ – group of global smooth gauge transformations, $\mathcal{N} = \mathcal{A}/\mathcal{G}$ – the moduli space of holomorphic structures on $P_0$.

$$\dim \mathcal{N} = \dim \mathfrak{g} \cdot (g - 1)$$

Any point in $\mathcal{N}$ is a gauge equivalence class of holomorphic principal $G$–bundles on $\Sigma$, denoted by $P$: $P = [(P_0, \omega)]$. 

**Hitchin systems - conventional set-up** (Hitchin’87)
Hitchin systems – construction (Hitchin’87)

Configuration space – $\mathcal{N}$, phase space – $T^*(\mathcal{N})$

By Kodaira–Spencer theory $T_P(\mathcal{N}) \cong H^1(\Sigma, \text{Ad } P)$.

Then by Serr duality $T^*_P(\mathcal{N}) \cong H^0(\Sigma, \text{Ad } P \otimes \mathcal{K})$,
and $(P, \Phi) \in T^*(\mathcal{N}) \iff \Phi \in H^0(\Sigma, \text{Ad } P \otimes \mathcal{K})$

Given a homogeneous degree $d$ invariant polynomial $\chi_d$ on $\mathfrak{g}$,

\[
\forall P \in \mathcal{N}, \text{ we obtain } \chi_d(P) : H^0(\Sigma, \text{Ad } P \otimes \mathcal{K}) \to H^0(\Sigma, \mathcal{K}^d)
\]

\[
\Phi \mapsto \chi_d(P, \Phi)
\]

Pick up a base $\{\Omega^d_j\} \subset H^0(\Sigma, \mathcal{K}^d)$.

Then $\chi_d(P, \Phi) = \sum H_{d,j}(P, \Phi)\Omega^d_j$, where $H_{d,j}(P, \Phi)$ is a scalar function on $T^*(\mathcal{N})$ called a Hitchin Hamiltonian.

**Theorem (Hitchin, ’87):** $\{H_{d,j}\}$ Poisson commute on $T^*(\mathcal{N})$.
Hitchin systems in terms of separated variables

Assume $g$ to be a complex simple Lie algebra of one of the types $A_n$, $B_n$, $C_n$, $g \in \mathbb{Z}_+$ and $P_{2g+1}(x) = x^{2g+1} + \ldots$ a given polynomial of degree $2g + 1$. Let $n = \text{rank } g$, and $d_1, \ldots, d_n$ be degrees of the basis invariants of $g$, $d$ be the dimension of the standard representation of $g$.

Phase space: tuples $\{(\lambda_1, x_1, y_1), \ldots, (\lambda_{\hat{g}}, x_{\hat{g}}, y_{\hat{g}})\}$, 
$\hat{g} = (\dim g)(g - 1)$, $\lambda_i, x_i, y_i \in \mathbb{C}$, $y_i^2 = P_{2g+1}(x_i)$ $(i = 1, \ldots, \hat{g})$

Poisson bracket is given by $\{\lambda_i, x_j\} = \delta_{ij}y_i$

Hamiltonians $H_{jk}^{(0)}$, $H_{js}^{(1)}$ are defined from the system of linear equations $(i = 1, \ldots, \hat{g})$:

$$\lambda_i^d + \sum_{j=1}^{n} \left( \sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x_i^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x_i^s y_i \right) \lambda_i^{d-d_j} = 0$$
Proof of equivalence (beginning)

The proof is based on the classification of spectral curves of hyperelliptic Hitchin systems.

Pick up a holomorphic differential $\omega$ on $\Sigma$, say $\omega = dx/y$.

By spectral curve we mean $\text{det}(\lambda E - \Phi(P)/\omega) = 0$, $P \in \Sigma$.

For $\Sigma$ hyperelliptic it descends to two equations in $\mathbb{C}^3$:

$$R(x, y, \lambda) = \lambda^d + \sum_{i=1}^{n} r_i(x, y)\lambda^{d-d_i} = 0,$$

and $y^2 = P_{2g+1}(x)$.

For $A_n$, $B_n$, $C_n$ every $r_i$ is a basis degree $d_i$ invariant of $g$:

- for $A_n$: $d = n + 1$, $d_i = i + 1$ ($G = SL(n + 1)$);
- for $B_n$: $d = 2n + 1$, $d_i = 2i$ ($G = SO(2n + 1)$);
- for $C_n$: $d = 2n$, $d_i = 2i$ ($G = Sp(2n)$).
Spectral curves of hyperelliptic $A_n, B_n, C_n$ Hitchin systems (Sh'2018)

Analytical properties of $\Phi$ determine $r_j$’s completely:

**Theorem:** Basis degree $d_j$ invariants of $\Phi/\omega$ run over $\mathcal{O}(-d_jD)$ where $D = (\omega) = 2(g - 1)\infty$. The functions $1, x, \ldots, x^{d_j(g-1)}$, and $y, yx, \ldots, yx^{(d_j-1)(g-1)-2}$ form a base in $\mathcal{O}(-d_jD)$.

Then

$$r_j(x, y) = \sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x^s y$$

where $H_{jk}^{(0)}, H_{js}^{(1)}$ are parameters (Hamiltonians).
Proof of equivalence (the end)

With the knowledge of a general form of the spectral curve we define the Hamiltonians from the requirement that the spectral curve passes through the points \((\lambda_1, x_1, y_1), \ldots, (\lambda_{\hat{g}}, x_{\hat{g}}, y_{\hat{g}})\), \(\lambda_i, x_i, y_i \in \mathbb{C}\), \(y_i^2 = P_{2g+1}(x_i)\) \((i = 1, \ldots, \hat{g})\), \(\hat{g} = (\dim g)(g - 1)\). This way we obtain the above equations on Hamiltonians:

\[
\lambda_i^d + \sum_{j=1}^{n} \left( \sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x_i^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x_i^s y_i \right) \lambda_i^{d-d_j} = 0
\]

By Krichever’02 (CMPh) the symplectic form is of the form \(\sigma = \sum_{s} d\lambda_s \wedge \omega(\gamma_s)\) for an appropriate set of points \(\{(\lambda_s, \gamma_s)\}\) on the spectral curve. Plugging the above points and \(\omega = dx/y\) we obtain

\[
\sigma = \sum_{i=1}^{\hat{g}} d\lambda_i \wedge \frac{dx_i}{y_i}.
\]
Example: $\mathfrak{g} = \mathfrak{sl}(2) \ (\sim A_1)$, genus 2 Hitchin system

(previous results E.Previato, 1994; Kz. Gawędzki, 1998)

Phase space: triplets \( \{ (\lambda_1, x_1, y_1), (\lambda_2, x_2, y_2), (\lambda_3, x_3, y_3) \} \) s.t.

\[
\lambda_i^2 = H_0 + H_1 x_i + H_2 x_i^2, \quad y_i^2 = P_5(x_i) \ (i = 1, 2, 3)
\]

Hamiltonians:

\[
H_i = \frac{\Delta_i}{\Delta}, \quad \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} \lambda_1^2 & x_1 & x_1^2 \\ \lambda_2^2 & x_2 & x_2^2 \\ \lambda_3^2 & x_3 & x_3^2 \end{vmatrix}, \quad \text{etc.}
\]

Symplectic form:

\[
\sigma = d\lambda_1 \wedge \frac{dx_1}{y_1} + d\lambda_2 \wedge \frac{dx_2}{y_2} + d\lambda_3 \wedge \frac{dx_3}{y_3}
\]
Example: $g = \mathfrak{sl}(2)$, genus 2: Hitchin equations

\[ H_2 = \frac{\Delta_2}{\Delta}, \quad \Delta_2 = \begin{vmatrix} 1 & x_1 & \lambda_1^2 \\ 1 & x_2 & \lambda_2^2 \\ 1 & x_3 & \lambda_3^2 \end{vmatrix}, \quad \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} \]

\[
\{x_1, H_2\} = \frac{2\lambda_1 y_1}{\Delta} (x_3 - x_2),
\]

\[
\{\lambda_1, H_2\} = \frac{y_1}{(x_1 - x_2)^2(x_1 - x_3)^2} \left( \Delta_2 \frac{2x_1 - x_2 - x_3}{x_2 - x_3} - 1 \right)
\]

+ cyclic permutations of indices for $(x_2, \lambda_2), (x_3, \lambda_3)$
Separation of variables (Liouville–Jacobi–Stäckel–Arnold–Sklyanin)

Assume Hamiltonians are given by separation relations

\[ R(x_i, \lambda_i, H) = 0 \] (\(H = (H_1, \ldots, H_{\hat{g}})\)), and \(\lambda_i, x_i\) are Darboux coordinates.

How to linearize flows?

Hurtubise’00, Talalaev’03: Let \(\phi = (\phi_1, \ldots, \phi_{\hat{g}})\) where

\[
\phi_j = -\sum_{i=1}^{\hat{g}} \int \frac{\partial R/\partial H_j}{\partial R/\partial \lambda} \, dx.
\]

Then \((H, \phi)\) are Darboux coordinates: \(\sigma = \sum dH_j \wedge d\phi_j\). Since \(\{H_i, H_j\} = 0\), we have \(\frac{d}{dt} H = 0\), \(\frac{d}{dt} \phi = H\), hence \(H = c_0\), \(\phi = c_0 t + c_1\) (\(c_0, c_1\) are constant vectors) – linearization of flows.
Darboux coordinates – cases $A_n, B_n, C_n$

Separation relations $R(\lambda_i, x_i, y_i, H) = 0$ (see slide 2) are nothing but giving the spectral curve by means points it passes through.

Plugging the precise form of separation relations we find Darboux coordinates $(H_{jk}^{(0)}, \phi_{jk}^{(0)})$, and $(H_{js}^{(1)}, \phi_{js}^{(1)})$ where $H_{jk}^{(0)}, H_{js}^{(1)}$ are to be found from the separation relations, and

\[
\phi_{jk}^{(0)} = \sum_{i=1}^{(\dim g)(g-1)} (x_i, y_i) \int x^k \lambda^{d-d_j} \frac{dx}{R'_{\lambda}(x, y, \lambda)} y, \ 0 \leq k \leq d_j(g - 1);
\]

\[
\phi_{js}^{(1)} = \sum_{i=1}^{(\dim g)(g-1)} (x_i, y_i) \int x^s \lambda^{d-d_j} \frac{dx}{R'_{\lambda}(x, y, \lambda)}, \ 0 \leq s \leq (d_j - 1)(g - 1) - 2
\]
**Theorem:** The differentials \( \frac{x^k \lambda^{d_j} dx}{R'_\lambda(x, y, \lambda)y} (0 \leq k \leq d_j(g - 1)) \) and \( \frac{x^s \lambda^{d_j} dx}{R'_\lambda(x, y, \lambda)} (0 \leq k \leq (d_j - 1)(g - 1) - 2), j = 1, \ldots, n \) form a base of holomorphic differentials on the spectral curve for \( A_n \) \((n > 1)\), and a base of holomorphic Pryme differentials for the systems \( A_1, B_n, C_n \) (w.r.t involution \( \lambda \to -\lambda \)).

For the case \( D_n \):

- Separation relations \( R(\lambda_i, x_i, y_i, H) = 0 \) are quadratic in \( H \) (because the last coefficient is \( \det(\Phi/\omega) = (\text{Pf} (\Phi/\omega))^2 \));
- Differentials of the angle coordinates are the same for \( j < n \), and are multiplied by \( \text{Pf} (\Phi/\omega) \) for \( j = n \);
- The differentials form a basis of holomorphic Prym differentials on the normalization of the spectral curve.
Hitchin foliation is the algebraic-geometrical analog of the Liouville foliation. The leaves of the Hitchin foliation are Jacobian varieties of the spectral curves for $A_n$, $n > 1$, and Prym varieties in cases $B_n$, $C_n$, or those for normalizations of the spectral curves in case of $D_n$ (Hitchin’87).

Algebraic-geometrical angle coordinates are coordinates on the leaves of the Hitchin foliation. To find them we must normalize the above differentials of the angle coordinates.

For the normalizing matrix $A$ we have $A^{-1} = \left(2 \int_{c_i} \omega_k\right)_{i,k=1,...,\hat{g}}$ where $\{c_i\}$ is the system of cuts between pairs of branching points. The problem of finding out of all branching points descends to the system of algebraic equations $R(\lambda, x, y) = 0$, $R'(\lambda, x, y) = 0$, and is normally unsolvable in radicals. But sometimes it is!
Spectral curve: $R(\lambda, x, y, H) = \lambda^4 + \lambda^2 p + q^2 = 0$

where $p = H_0 + xH_1 + x^2H_2$, $q = H_3 + xH_4 + x^2H_5$.

**Theorem (P. Borisova):** Separation equations and equations for branching points are solvable in radicals.

Normalized spectral curve has 16 branching points.
By Riemann–Hurwitz $\hat{g} = 13$.

Involution $\sigma: \lambda \rightarrow -\lambda$ is a rotation by $\pi$ around the center of the picture. No fixed points.

8 preimages of 4 singular points are located in the middles of horizontal lines (2 at each one).

Normalization map glues the points at the opposite horizontal lines.
\[ \mathfrak{sl}(2), \mathfrak{sp}(4), \mathfrak{so}(5), g = 2 \text{ cases} \]

Case \( \mathfrak{sl}(2) \) Spectral curve:
\[ \lambda^2 p + H_0 + xH_1 + x^2 H_2 = 0 \]
has 4 branching points.
By Riemann–Hurwitz \( \hat{g} = 5 \).
Involution \( \sigma : \lambda \to -\lambda \) is a rotation by \( \pi \) around the vertical axis of the picture.

Case \( \mathfrak{sp}(4) \) Spectral curve:
\[ \lambda^4 + \lambda^2 p + q = 0, \]
\[ p = H_0 + xH_1 + x^2 H_2, \quad q = H_3 + xH_4 + \ldots x^4 H_7 + yH_8 + xyH_9, \]
has 24 branching points.
By Riemann–Hurwitz \( \hat{g} = 17 \).
Involution \( \sigma : \lambda \to -\lambda \) is a reflection in the vertical axis.

Case \( \mathfrak{so}(5) \) Spectral curve is the same as for \( \mathfrak{sp}(4) \)

Fixed points=branching points with \( \lambda = 0 \)