HITCHIN SYSTEMS ON HYPERELLIPTIC CURVES

Oleg Sheinman

Steklov Mathematical Institute
Moscow

GEOQUANT-2019
Department of Mathematics
National Tsing Hua University
September, 1 – 13, 2019

Hitchin systems - conventional set-up (Hitchin'87)

Σ – genus g algebraic curve $/\mathbb{C}$, G – complex s/s Lie group, $\mathfrak{g} = \mathcal{L}ie(G)$, P_0 – a fixed principle C^∞ G-bundle on Σ.

Holomorphic structure $=$ $(0,1)$-connection on P_0 is a $(0,1)$ $\mathfrak{g}$-valued form ω on Σ with gauge transformation $\omega \to \gamma\omega\gamma^{-1} - (\bar{\partial}\gamma)\gamma^{-1}$ under gluing function γ.

$\mathcal{A} = \{(P_0, \omega)\}$ – space of s/s holomorphic structures on P_0, $\mathcal{G}$ – group of global smooth gauge transformations, $\mathcal{N} = \mathcal{A}/\mathcal{G}$ – the moduli space of holomorphic structures on P_0.

$$\boxed{\dim \mathcal{N} = \dim \mathfrak{g} \cdot (g-1)}$$

Any point in $\mathcal{N}$ is a gauge equivalence class of holomorphic principal G–bundles on Σ, denoted by P: $P = [(P_0, \omega)]$.

Hitchin systems – construction (Hitchin'87)

Configuration space – $\mathcal{N}$, phase space – $T^*(\mathcal{N})$

By Kodaira–Spencer theory $T_P(\mathcal{N}) \simeq H^1(\Sigma, \mathrm{Ad}\, P)$.

Then by Serr duality $T_P^*(\mathcal{N}) \simeq H^0(\Sigma, \mathrm{Ad}\, P \otimes \mathcal{K})$,

and $(P, \Phi) \in T^*(\mathcal{N}) \Longleftrightarrow \Phi \in H^0(\Sigma, \mathrm{Ad}\, P \otimes \mathcal{K})$

Given a homogeneous degree d invariant polynomial χ_d on $\mathfrak{g}$,

$$\forall P \in \mathcal{N},\ \text{we obtain } \chi_d(P) : H^0(\Sigma, \mathrm{Ad}\, P \otimes \mathcal{K}) \to H^0(\Sigma, \mathcal{K}^d)$$
$$\Phi \longmapsto \chi_d(P, \Phi)$$

Pick up a base $\{\Omega_j^d\} \subset H^0(\Sigma, \mathcal{K}^d)$.

Then $\chi_d(P, \Phi) = \sum H_{d,j}(P, \Phi)\Omega_j^d$, where $H_{d,j}(P, \Phi)$ is a scalar function on $T^*(\mathcal{N})$ called a Hitchin Hamiltonian.

<u>Theorem</u> (Hitchin, '87): $\{H_{d,j}\}$ Poisson commute on $T^*(\mathcal{N})$

Hitchin systems in terms of separated variables

Assume $\mathfrak{g}$ to be a complex simple Lie algebra of one of the types A_n, B_n, C_n, $g \in \mathbb{Z}_+$ and $P_{2g+1}(x) = x^{2g+1} + \ldots$ a given polynomial of degree $2g+1$. Let $n = \operatorname{rank} \mathfrak{g}$, and $d_1, \ldots, d_n$ be degrees of the basis invariants of $\mathfrak{g}$, d be the dimension of the standard representation of $\mathfrak{g}$.

Phase space: tuples $\{(\lambda_1, x_1, y_1), \ldots, (\lambda_{\hat{g}}, x_{\hat{g}}, y_{\hat{g}})\}$, $\hat{g} = (\dim \mathfrak{g})(g-1)$, $\lambda_i, x_i, y_i \in \mathbb{C}$, $y_i^2 = P_{2g+1}(x_i)$ $(i = 1, \ldots, \hat{g})$

Poisson bracket is given by $\{\lambda_i, x_j\} = \delta_{ij} y_i$

Hamiltonians $H_{jk}^{(0)}$, $H_{js}^{(1)}$ are defined from the system of linear equations $(i = 1, \ldots, \hat{g})$:

$$\lambda_i^d + \sum_{j=1}^{n} \left(\sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x_i^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x_i^s y_i \right) \lambda_i^{d-d_j} = 0$$

Proof of equivalence (beginning)

The proof is based on the classification of spectral curves of hyperelliptic Hitchin systems.

Pick up a holomorphic differential ω on Σ, say $\omega = dx/y$.

By spectral curve we mean $\det(\lambda E - \Phi(P)/\omega) = 0$, $P \in \Sigma$.

For Σ hyperelliptic it descends to two equations in $\mathbb{C}^3$:

$$R(x, y, \lambda) = \lambda^d + \sum_{i=1}^{n} r_i(x, y)\lambda^{d-d_i} = 0, \text{ and } y^2 = P_{2g+1}(x).$$

For A_n, B_n, C_n every r_i is a basis degree d_i invariant of $\mathfrak{g}$:

- for A_n: $d = n + 1$, $d_i = i + 1$ ($G = SL(n + 1)$);
- for B_n: $d = 2n + 1$, $d_i = 2i$ ($G = SO(2n + 1)$);
- for C_n: $d = 2n$, $d_i = 2i$ ($G = Sp(2n)$).

Spectral curves of hyperelliptic A_n, B_n, C_n Hitchin systems (Sh'2018)

Analytical properties of Φ determine r_j's completely:

THEOREM: Basis degree d_j invariants of Φ/ω run over $\mathcal{O}(-d_j D)$ where $D = (\omega) = 2(g-1)\infty$. The functions $1, x, \ldots, x^{d_j(g-1)}$, and $y, yx, \ldots, yx^{(d_j-1)(g-1)-2}$ form a base in $\mathcal{O}(-d_j D)$.

Then

$$r_j(x,y) = \sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x^s y$$

where $H_{jk}^{(0)}$, $H_{js}^{(1)}$ are parameters (Hamiltonians).

Proof of equivalence (the end)

With the knowledge of a general form of the spectral curve we define the Hamiltonians from the requirement that the spectral curve passes through the points $(\lambda_1, x_1, y_1), \ldots, (\lambda_{\hat{g}}, x_{\hat{g}}, y_{\hat{g}})$, $\lambda_i, x_i, y_i \in \mathbb{C}$, $y_i^2 = P_{2g+1}(x_i)$ $(i = 1, \ldots, \hat{g})$, $\hat{g} = (\dim \mathfrak{g})(g-1)$. This way we obtain the above equations on Hamiltonians:

$$\lambda_i^d + \sum_{j=1}^{n} \left(\sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x_i^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x_i^s y_i \right) \lambda_i^{d-d_j} = 0$$

By Krichever'02 (CMPh) the symplectic form is of the form $\sigma = \sum_s d\lambda_s \wedge \omega(\gamma_s)$ for an appropriate set of points $\{(\lambda_s, \gamma_s)\}$ on the spectral curve. Plugging the above points and $\omega = dx/y$ we obtain

$$\sigma = \sum_{i=1}^{\hat{g}} d\lambda_i \wedge \frac{dx_i}{y_j}.$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$ ($\sim A_1$), genus 2 Hitchin system

(previous results E.Previato, 1994; Kz. Gawędzki, 1998)

Phase space:
triples $\{(\lambda_1, x_1, y_1), (\lambda_2, x_2, y_2), (\lambda_3, x_3, y_3)\}$ s.t.

$$\lambda_i^2 = H_0 + H_1 x_i + H_2 x_i^2, \quad y_i^2 = P_5(x_i) \; (i = 1, 2, 3)$$

Hamiltonians:

$$H_i = \frac{\Delta_i}{\Delta}, \quad \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} \lambda_1^2 & x_1 & x_1^2 \\ \lambda_2^2 & x_2 & x_2^2 \\ \lambda_3^2 & x_3 & x_3^2 \end{vmatrix}, \quad \text{etc.}$$

Symplectic form:

$$\sigma = d\lambda_1 \wedge \frac{dx_1}{y_1} + d\lambda_2 \wedge \frac{dx_2}{y_2} + d\lambda_3 \wedge \frac{dx_3}{y_3}$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$, genus 2: Hitchin equations

$$H_2 = \frac{\Delta_2}{\Delta}, \quad \Delta_2 = \begin{vmatrix} 1 & x_1 & \lambda_1^2 \\ 1 & x_2 & \lambda_2^2 \\ 1 & x_3 & \lambda_3^2 \end{vmatrix}, \ \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

$$\{x_1, H_2\} = \frac{2\lambda_1 y_1}{\Delta}(x_3 - x_2),$$
$$\{\lambda_1, H_2\} = \frac{y_1}{(x_1 - x_2)^2 (x_1 - x_3)^2} \left(\Delta_2 \frac{2x_1 - x_2 - x_3}{x_2 - x_3} - 1 \right)$$

$+$ cyclic permutations of indices for (x_2, λ_2), (x_3, λ_3)

Separation of variables (Liouville–Jacobi–Stäckel–Arnold–Sklyanin)

Assume Hamiltonians are given by separation relations $R(x_i, \lambda_i, H) = 0$ ($H = (H_1, \ldots, H_{\hat{g}})$), and λ_i, x_i are Darboux coordinates.

How to linearize flows?

Hurtubise'00, Talalaev'03: Let $\phi = (\phi_1, \ldots, \phi_{\hat{g}})$ where

$$\phi_j = -\sum_{i=1}^{\hat{g}} \int^{(x_i, y_i)} \frac{\partial R / \partial H_j}{\partial R / \partial \lambda} dx.$$

Then (H, ϕ) are Darboux coordinates: $\sigma = \sum dH_j \wedge d\phi_j$. Since $\{H_i, H_j\} = 0$, we have $\frac{d}{dt} H = 0$, $\frac{d}{dt}\phi = H$, hence $H = c_0$, $\phi = c_0 t + c_1$ (c_0, c_1 are constant vectors) – linearization of flows.

Darboux coordinates – cases A_n, B_n, C_n

Separation relations $R(\lambda_i, x_i, y_i, H) = 0$ (see slide 2) are nothing but giving the spectral curve by means points it passes through.

Plugging the precise form of separation relations we find Darboux coordinates $(H_{jk}^{(0)}, \phi_{jk}^{(0)})$, and $(H_{js}^{(1)}, \phi_{js}^{(1)})$ where $H_{jk}^{(0)}, H_{js}^{(1)}$ are to be found from the separation relations, and

$$\phi_{jk}^{(0)} = \sum_{i=1}^{(\dim \mathfrak{g})(g-1)} \int^{(x_i, y_i)} \frac{x^k \lambda^{d-d_j} dx}{R'_\lambda(x, y, \lambda)\, y}, \; 0 \le k \le d_j(g-1);$$

$$\phi_{js}^{(1)} = \sum_{i=1}^{(\dim \mathfrak{g})(g-1)} \int^{(x_i, y_i)} \frac{x^s \lambda^{d-d_j} dx}{R'_\lambda(x, y, \lambda)}, \; 0 \le s \le (d_j - 1)(g-1) - 2$$

Darboux coordinates: differentials of angles, peculiarities of the case D_n

Theorem: The differentials $\frac{x^k \lambda^{d-d_j} dx}{R'_\lambda(x,y,\lambda)\, y}$ $(0 \le k \le d_j(g-1))$ and $\frac{x^s \lambda^{d-d_j} dx}{R'_\lambda(x,y,\lambda)}$ $(0 \le k \le (d_j-1)(g-1)-2)$, $j = 1, \ldots, n$ form a base of holomorphic differentials on the spectral curve for A_n $(n > 1)$, and a base of holomorphic Pryme differentials for the systems A_1, B_n, C_n (w.r.t involution $\lambda \to -\lambda$).

For the case D_n:

- Separation relations $R(\lambda_i, x_i, y_i, H) = 0$ are quadratic in H (because the last coefficient is $\det(\Phi/\omega) = (\mathrm{Pf}\,(\Phi/\omega))^2$);
- Differentials of the angle coordinates are the same for $j < n$, and are multiplied by $\mathrm{Pf}\,(\Phi/\omega)$ for $j = n$;
- The differentials form a basis of holomorphic Prym differentials on the normalization of the spectral curve.

Action–angle coordinates

Hitchin foliation is the algebraic-geometrical analog of the Liouville foliation

The leaves of the Hitchin foliation are Jacobian varieties of the spectral curves for A_n, $n > 1$, and Prym varieties in cases B_n, C_n, or those for normalizations of the spectral curves in case of D_n (Hitchin'87).

Algebraic-geometrical angle coordinates are coordinates on the leaves of the Hitchin foliation. To find them we must normalize the above differentials of the angle coordinates.

For the normalizing matrix A we have $A^{-1} = \left(2\int_{c_i}\omega_k\right)_{i,k=1,\ldots,\widehat{g}}$ where $\{c_i\}$ is the system of cuts between pairs of branching points. The problem of finding out of all branching points descends to the system of algebraic equations $R(\lambda, x, y) = 0$, $R'(\lambda, x, y) = 0$, and is normally unsolvable in radicals. But sometimes it is !

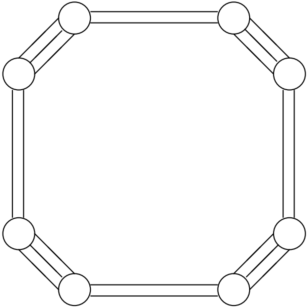
$\mathfrak{so}(4)$, $g = 2$ case (P.Borisova, Sh'19)

Spectral curve: $R(\lambda, x, y, H) = \lambda^4 + \lambda^2 p + q^2 = 0$

where $p = H_0 + xH_1 + x^2 H_2$, $q = H_3 + xH_4 + x^2 H_5$.

THEOREM (P.BORISOVA): Separation equations and equations for branching points are solvable in radicals.

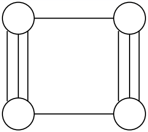
Normalized spectral curve has 16 branching points. By Riemann–Hurwitz $\hat{g} = 13$. Involution $\sigma : \lambda \to -\lambda$ is a rotation by π around the center of the picture. No fixed points. 8 preimages of 4 singular points are located in the middles of horizontal lines (2 at each one). Normalization map glues



the points at the opposite horizontal lines.

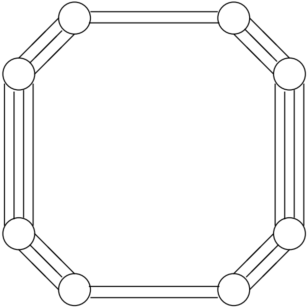
$\mathfrak{sl}(2)$, $\mathfrak{sp}(4)$, $\mathfrak{so}(5)$, $g = 2$ cases

Case $\mathfrak{sl}(2)$ Spectral curve:
$\lambda^2 p + H_0 + xH_1 + x^2 H_2 = 0$
has 4 branching points.
By Riemann–Hurwitz $\hat{g} = 5$.
Involution $\sigma : \lambda \to -\lambda$ is a rotation by π around the vertical axis of the picture.



Fixed points=branching points with $\lambda = 0$

Case $\mathfrak{sp}(4)$ Spectral curve:
$\lambda^4 + \lambda^2 p + q = 0$,
$p = H_0 + xH_1 + x^2 H_2$, $q = H_3 + xH_4 + \ldots x^4 H_7 + yH_8 + xyH_9$,
has 24 branching points.
By Riemann–Hurwitz $\hat{g} = 17$.
Involution $\sigma : \lambda \to -\lambda$ is a reflection in the vertical axis.



Case $\mathfrak{so}(5)$ Spectral curve is the same as for $\mathfrak{sp}(4)$