A GENERALIZATION OF INJECTIVE MODULES

SEBASTIANO TRONTO

Abstract. The underlying abelian group of the field of rational numbers $\mathbb{Q}$ has an interesting property: it is divisible, which means that for every $x \in \mathbb{Q}$ and every positive integer n there is a $y \in \mathbb{Q}$ such that $ny = x$. On the other hand, if we only care about dividing by the powers of a certain prime, then also the underlying abelian group of the ring $\mathbb{Z}[p^{-1}]$ has a similar property: it is p-divisible, that is for every $x \in \mathbb{Z}[p^{-1}]$ there is $y \in \mathbb{Z}[p^{-1}]$ such that $py = x$. If one tries to generalize these concepts to modules over a general (associative, unitary) ring R, things may not work so well, among other things due to the possible presence of zero-divisors in the base ring. There is however a natural (or categorical) concept that works well over any ring, which is injectivity. Indeed an abelian group is divisible if and only if it is injective as a $\mathbb{Z}$-module. What is in this setting a suitable generalization for p-divisibility? Is there a more general property that includes divisibility and p-divisibility as special cases, and that also works well for R-modules? In this talk I will propose a definition that provides a positive answer to the two questions above. If time permits I will also show an analogue of Morita duality using this more general definition.

1. Divisible abelian groups and injective modules

Consider the abelian group $\mathbb{Q}$. If $x \in \mathbb{Q}$ and $n \in \mathbb{Z} \setminus \{0\}$, then there is $y \in \mathbb{Q}$ such that $ny = x$; namely, we can take $y = \frac{x}{n}$. This holds also, for example, for the abelian group $\mathbb{Q}/\mathbb{Z}$. In general, an abelian group satisfying this property is called *divisible*.

Definition 1.1. An abelian group A is called *divisible* if for every $x \in A$ and every $n \in \mathbb{Z} \setminus \{0\}$ there is $y \in A$ such that $ny = x$.

For modules over a general ring R this definition might not scale so well. For example, taking $R = \mathbb{Z} \times \mathbb{Z}$, the R-module $M = \mathbb{Q} \times \mathbb{Q}$ (with action of R given by multiplication component-wise) does not satisfy the property above for every $x \in M$: if $x = (1, 1)$ and $r = (0, 1)$ then there is clearly no $y \in M$ such that $ry = x$.

There is however a property that plays the same role in many circumstances.

Definition 1.2. An R module Q is called *injective* if for every injective R-module homomorphism $i : M \hookrightarrow N$ and every R-module homomorphism $f : M \to Q$ there is an R-module homomorphism g such that $g \circ i = f$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "i"'] & Q \\ N \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

For $\mathbb{Z}$-modules being injective is equivalent to being divisible.

Proposition 1.3. *A $\mathbb{Z}$-module is injective if and only if it is divisible as an abelian group.*

Proof. Let A be an abelian group and assume that it is injective as a $\mathbb{Z}$-module. Let $x \in A$ and $n \in \mathbb{Z} \setminus \{0\}$. Consider the inclusion $i : n\mathbb{Z} \hookrightarrow \mathbb{Z}$ and the map $f : n\mathbb{Z} \to A$ which sends n to x. Then since A is injective f extends to a map $g : \mathbb{Z} \to A$ which sends n to x, so letting $y = g(1)$ we have $ny = x$, as required.

Assume now that A is divisible and let $J : M \hookrightarrow N$ be and injective homomorphism of abelian groups and $f : M \to A$ any homomorphism. In order to extend f to a map $g : N \to A$ we will use Zorn's Lemma. Let S be the set of pairs (N', φ) with N' a sugroup of N containing M and φ a homomorphism $N' \to A$ that extends f. The set S admits a partial order

$$(N', \varphi) \leqslant (N'', \psi) \iff N' \subseteq N'' \text{ and } \psi|_{N'} = \varphi$$

Every chain in S has an upper bound. Namely, if $C \subseteq S$ is a chain, i.e. a totally ordered subset of S, then we can take $\mathcal{N}'$ to be the union of all N' for $(N', \varphi) \in C$ and we let

$$\begin{array}{rcc} \Phi : \mathcal{N}' & \to & A \\ x & \mapsto & \varphi(x), \text{ if there is any } (N', \varphi) \in C \text{ with } x \in N' \end{array}$$

which is well-defined because C is totally ordered (which means that if x belongs to N' and to N'' for $(N', \varphi) \in C$ and $(N'', \psi) \in C$, then either $(N', \varphi) \leqslant (N''\psi)$ or $(N'', \psi) \leqslant (N', \psi)$, and in any case φ and ψ are compatible on x).

By Zorn's lemma there is then a maximal element $(N', \varphi) \in S$, and we want to show that $N' = N$, so that f extends to the whole N. Assume that $N' \neq N$ and let $x \in N \setminus N'$; if we manage to extend φ to $\varphi_+ : N' + \langle x \rangle \to A$ this will yield a contradiction with the maximality of (N', φ), and thus we would be able to conclude that indeed $N' = N$.

If $\langle x \rangle \cap N' = 0$, we may simply define $\varphi_+(x) = 0$. Otherwise $\langle x \rangle \cap N'$ contains some $nx \neq 0$ for some positive integer n which we may assume minimal with respect to this property. Since A is divisible there is $y \in A$ such that $ny = \varphi(nx)$, and one easily checks that defining φ_+ as $\varphi_+(x) = y$ is compatible with φ. As explained above, this concludes the proof. $\square$

For a prime number p, the abelian groups $\mathbb{Z}[p^{-1}]$ and $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ have a property similar to the divisibility of $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$, but only if we restrict to dividing by (powers of) p.

Definition 1.4. Let p be a prime number. An abelian group A is called p-divisible if for every $x \in A$ there is $y \in A$ such that $py = x$.

Is there any property of R-modules that generalizes p-divisibility, in a way similar to how injectivity generalizes divisibility?

2. Division in modules

Fix for this and the following sections a unitary ring R.

Definition 2.1. If $M \subseteq N$ are left R-modules and I is an ideal of R, we call the R-submodule of N

$$(M :_N I) := \{x \in N \mid Ix \subseteq M\}$$

the *I-division module of M (in N)*.

Notice that $(M :_N 0) = N$ and $(M :_N R) = M$. If $I' \supseteq I$ we have $(M :_N I') \subseteq (M :_N I)$.

In general we might want to work with (possibly infinite) unions of division modules. For example if $R = \mathbb{Z}$ we are interested in working with

$$\bigcup_{k \geqslant 0} \left(M :_N (p^k) \right)$$

or with

$$\bigcup_{n \geqslant 1} (M :_N (n))$$

So it makes sense to give the following definition.

Definition 2.2. An *ideal filter* of R is a non-empty set J of two-sided ideals of R such that:

(1) If $I, I' \in J$ then $I \cap I' \in J$ and
(2) If $I \in J$ and $I' \triangleleft R$ contains I, then $I' \in J$.

If J is an ideal filter of R and $M \subseteq N$ are R-modules, we let

$$(M :_N J) := \bigcup_{I \in J} (M :_N I)$$

which we call the J-division module of M in N, and

$$M[J] := (0 :_M J)$$

which we call the J-torsion submodule of M.

Notice that if the zero ideal belongs to an ideal filter J, then every ideal of R belongs to J, that is J is the maximal ideal filter. We will denote this ideal filter by 0, and we will denote the minimal ideal filter $\{R\}$ by 1. We have $(M :_N 0) = N$ and $(M :_N R) = M$, and if $J' \subseteq J$ we have $(M :_N J') \subseteq (M :_N J)$.

Given a set of ideals S of R, we may consider the ideal filter J generated by S, that is the minimal (with respect to inclusion) ideal filter of R that contains S. If $S = \{I\}$ we have $(M :_N J) = (M :_N I)$.

Example 2.3. For any unitary ring R, there are two interesting examples: the ideal filter generated by the powers of a given prime number p

$$p^\infty := \{I \triangleleft R \mid I \supseteq p^n R \text{ for some } n \in \mathbb{N}\}$$

and the one generated by all non-zero integers

$$\widehat{n} := \{I \triangleleft R \mid I \supseteq nR \text{ for some } n \in \mathbb{N}_{>0}\} \ .$$

Notice that some power of p is equal to 0 in R (respectively $n = 0$ for some $n \in \mathbb{N}_{n>0}$) then p^∞ (resp. $\widehat{n}$) is simply the set of all two-sided ideals of R.

Thus ideal filters allow us to consider the possibly infinite unions of division modules mentioned above. We would also like to have a way to distinguish those ideal filters that describe a complete iteration of the division process, as p^∞ and $\widehat{n}$ do and (n) or (p^k) do not. We propose two definition that might capture this concept, and we show that, under certain condition, one is stronger than the other.

Definition 2.4. We call an ideal filter J of R:

- *Complete* if for every left R-module N and every submodule $M \subseteq N$ we have
$$((M :_N J) :_N J) = (M :_N J) \ .$$
- *Product-closed* if for any $I, I' \in J$ we have $II' \in J$.

Proposition 2.5. *Let J be a product-closed ideal filter of R. If every ideal in J is finitely generated, then J is complete.*

Proof. Let J be a product-closed ideal filter of R and let $M \subseteq N$ be left R-modules. The inclusion $(M :_N J) \subseteq ((M :_N J) :_N J)$ is always true, so in order to show that equality holds we need to prove the other inclusion. Let $x \in N$ be such that there is $I \in J$ with $Ix \subseteq (M :_N J)$. Let $\{y_1, \dots y_n\}$ be a set of generators for I. Then for every $i = 1, \dots n$ there is an ideal $I_i \in J$ such that $I_i y_i x \subseteq M$. By definition of ideal filter we have $I' := \bigcap_{i=1}^n I_i \in J$ and since J is product-closed we have $I'I \in J$. But we also have $I'Ix \subseteq M$, which shows that J is complete. □

The ideal filters introduced in Example 2.3 are both product-closed. If, for example, R is Noetherian, then they are also complete.

3. J-injective modules

Fix for this section a unitary ring R and a complete ideal filter J of R.

Definition 3.1. An injective R-module homomorphism $i : M \hookrightarrow N$ such that $(i(M) :_N J) = N$ is called a *J-extension.*

We can finally give our definition of J-injective module. In words, one can say that an injective module is one that admits extensions of maps into it along any injective map. A J-injective module is one that admits extensions of maps into it along J-extensions.

Definition 3.2. A left R-module Q is called J-injective if for every J-extension $i : M \hookrightarrow N$ and every R-module homomorphism $f : M \to Q$ there exists a homomorphism $g : N \to Q$ such that $g \circ i = f$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, hook, "i", "(J\text{-extension})"'] & Q \\ N \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Notice that in case $J = 0$ the definition of J-injective module coincides with that of injective module, because any injective homomorphism is a 0-extension. Moreover, if J' is an ideal filter of R such that $J' \subseteq J$, then a J-injective module is also J'-injective, because every J'-extension is also a J-extension.

The following Proposition is an analogue of the well-known Baer's criterion in the classical case of injective modules, and the proof is almost identical to the classical case.

Proposition 3.3. *A left R-module Q is J-injective if and only if for every $I \in J$ and every R-module homomorphism $f : I \to Q$ there is an R-module homomorphism $g : R \to Q$ that extends f.*

Proof. The "only if" part is trivial, because any two-sided ideal of R is also a left R-module. For the other implication, let $i : M \hookrightarrow N$ be a J-extensions and let $f : M \to Q$ be any R-module homomorphism. By Zorn's Lemma there is a submodule N' of N and an extension $g' : N' \to Q$ of f to N' that is maximal in the sense that it cannot be extended to any larger submodule of N. If $N' = N$ we are done, so assume that $N' \neq N$ and let $x \in N \setminus N'$.

Let I be the two-sided ideal of R generated by $\{r \in R \mid rx \in N'\}$. Since $i(M) \subseteq N'$ and $(i(M) :_N J) = N$ there is $I' \in J$ such that $I'x \subseteq N'$, which implies $I' \subseteq I$, so also $I \in J$. By assumption the map $I \to Q$ that sends $y \in I$ to $g'(yx)$ extends to a map $h : R \to Q$. Since $\ker(R \to Rx)$ is contained in $\ker(h)$, the map h gives rise to a map $h' : Rx \to Q$ by sending $rx \in Rx$ to $h(r)$. By definition the restrictions of g' and h' to $N' \cap Rx$ coincide, so we can define a map $g'' : N' + Rx \to Q$ that extends both. This contradicts the maximality of g', so we conclude that $N' = N$. □

Remark 3.4. One can adapt the proof or Proposition 1.3 to show that an abelian group is p-divisible if and only if it is p^∞-injective (see Example 2.3) as a $\mathbb{Z}$-module.

Let $J = 0$ be the maximal ideal filter of R and assume that $J' = J \setminus 0$ is an ideal filter; this amounts to say that no two non-zero ideals of R have trivial intersection. Using Proposition 3.3 one can easily show that an R-module Q is J-injective if and only if it is J'-injective. Indeed, one implication holds, as remarked above, because $J \subseteq J'$, and for the other it is enough to notice that the only map $0 \to Q$ can always be extended to the zero map on R.

One advantage of using J' instead of J is that the J'-torsion submodule may be different from $M[0] = M$.

Example 3.5. Let M be an abelian group, let p be a prime and let $J = p^\infty$ be the ideal filter of $\mathbb{Z}$ introduced in Example 2.3. Then the localization $M[p^{-1}]$ is a J-injective $\mathbb{Z}$-module. Indeed if $i : N \hookrightarrow P$ is a J-extension and $f : N \to M[p^{-1}]$ is any homomorphism then for every $x \in P$ there is $k \in \mathbb{N}$ such that $p^k x \in N$, and one can define $g(x) := \frac{f(p^k x)}{p^k}$. It is easy to check that g is then a well-defined group homomorphism such that $g \circ i = f$.

4. Injective hulls and J-hulls

Definition 4.1. A map of R-modules $i : M \hookrightarrow N$ is called an *essential extension* if for every nonzero submodule P of N we have $P \cap i(M) \neq 0$.

It is a well-known fact of commutative algebra that every R-module M admits an injective hull $\iota : M \hookrightarrow \Gamma$, which is an essential extension such that Γ is injective. Such an extension, which is unique up to a not-necessarily-unique isomorphism that is the identity on M, may be as well characterized by either of the following two properties:

(1) It is the largest essential extension of M, that is to say if $i : M \hookrightarrow N$ is an essential extension then there is an (injective) R-module homomorphism $j : N \hookrightarrow \Gamma$ such that $\iota \circ i = j$ (the injectivity of j follows from the injectivity of ι and the fact that $i : M \hookrightarrow N$ is an essential extension).
(2) It is the smallest injective extension of M, that is to say if $i : M \hookrightarrow N$ is an injective R-module homomorphism and N is injective, then there is an *injective* R-module homomorphism $j : \Gamma \hookrightarrow N$ such that $j \circ \iota = i$ (the existence of a map $\Gamma \to N$ that commutes with i follows from the injectivity of N, but the fact that this map is injective does not).

As an example, the standard map $\mathbb{Z}^n \hookrightarrow \mathbb{Q}^n$ is an injective hull of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

There is an analogue construction for J-injectivity.

Definition 4.2. Let J be a complete ideal filter of R and let M be a left R-module. A J-extension $\iota : M \hookrightarrow \Omega$ is called a *J-hull* of M if it is an essential extension and Ω is J-injective.

The following theorem is not a replacement for the classical one, since it relies on it.

Theorem 4.3. *Every left R-module M admits a J-hull, which is unique up to a not-necessarily-unique isomorphism that is the identity on M.*

Sketch of proof. Let $\iota : M \hookrightarrow \Gamma$ be an injective hull of M and let $\Omega := (\iota(M) :_\Gamma J)$. One can show that ι maps M into Ω and $\iota : M \hookrightarrow \Omega$ is indeed a J-hull of M, and that for any other J-hull $\iota' : M \hookrightarrow \Omega'$ there is an isomorphism $j : \Omega \xrightarrow{\sim} \Omega'$ such that $j \circ \iota = \iota'$. ☐

Example 4.4. Let M be an abelian group, let p be a prime and let $J = p^\infty$ be the ideal filter of $\mathbb{Z}$ introduced in Example 2.3. Write M as

$$M = \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}/p^{e_i}\mathbb{Z} \oplus M[n]$$

where n is a positive integer coprime to p and the e_i's are suitable exponents. Let

$$\Gamma = (\mathbb{Z}[p^{-1}])^r \oplus (\mathbb{Z}[p^{-1}]/\mathbb{Z})^k \oplus M[n]$$

and

$$\begin{array}{rccc} \iota: & \Gamma & \to & M \\ & (z,(s_i \bmod p^{e_i})_i,t) & \mapsto & \left(\frac{z}{1},\left(\frac{s}{p^{e_i}} \bmod \mathbb{Z}\right)_i,t\right) \end{array}$$

Then $\iota:\Gamma\to M$ is a J-hull. To see this it is enough to show that $f:\mathbb{Z}^r\hookrightarrow(\mathbb{Z}[p^{-1}])^r$ and $g_i:\mathbb{Z}/p^{e_i}\mathbb{Z}\hookrightarrow\mathbb{Z}[p^{-1}]/\mathbb{Z}$ for every $i=1,\dots,k$ are J-hulls, and that $M[n]$ is J-injective, being trivially an essential extension of itself. The assertions about f and $M[n]$ follow from Example 3.5, noticing that multiplication by p is an automorphism of $M[n]$ and that $\mathbb{Z}^r\hookrightarrow(\mathbb{Z}[p^{-1}])^r$ is an essential J-extension.

So we are left to show that for every positive integer e the map $g:\mathbb{Z}/p^e\mathbb{Z}\hookrightarrow\mathbb{Z}[p^{-1}]/\mathbb{Z}$ defined by $(s \bmod p^e)\mapsto(\frac{s}{p^e} \bmod \mathbb{Z})$ is a J-hull. It is a J-extension, because the Prüfer group $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ itself is J-torsion, and it is also essential because every subgroup of $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ is of the form $\frac{1}{p^d}\mathbb{Z}$, so it intersects the image of g in $\frac{1}{p^{\min(e,d)}}\mathbb{Z}$.

Finally, $\mathbb{Z}[p^{-1}]/\mathbb{Z}$ is divisible as an abelian group, so in particular it is J-injective, since in this case it is equivalent to being p-divisible.

We can draw an interesting parallel between the J-hull of an R-module M and the algebraic closure $\overline{k}$ of a field k. Indeed $\overline{k}$ is at the same time the smallest *algebraically closed* extension and the largest *algebraic* extension of k. Similarly to J-hulls, an algebraic closure is unique up to a not-necessarily-unique isomorphism that fixes the base field.

5. Morita duality

Consider the following well-know fact about vector spaces and linear maps.

Proposition 5.1. *Let V be a finite dimensional vector space over a field k and let $f_1,\dots,f_n:V\to k$ be linear functions. If $g:V\to k$ is a linear function such that $\ker(g)\supseteq\bigcap_{i=1}^n\ker(f_i)$, then g is a linear combination of the f_i.*

Proof. Consider the map

$$F:=(f_1,\dots,f_n):V\to k^n$$
$$x\mapsto(f_1(x),\dots,f_n(x))$$

and notice that $K:=\ker(F)=\bigcap_{i=1}^n\ker(f_i)$. Then both g and F factor through $V/\ker(F)$ as $\overline{g}:V/\ker(F)\to k$ and $\overline{F}:V/\ker(F)\to k^n$ respectively, and $\overline{F}$ is injective. By extending a basis of $\operatorname{Im}(F)\subseteq k^n$ to a basis of k^n one can find a linear map

$$\lambda:k^n\to k$$
$$(x_1,\dots,x_n)\mapsto e_1x_1+\dots+e_nx_n$$

such that $\lambda\circ\overline{F}=\overline{g}$, which implies $\lambda\circ F=g$. Then for every $v\in V$ we have

$$g(v)=\lambda(F(v))=\lambda(f_1(v),\dots,f_n(v))=e_1f_1(v)+\dots+e_nf_n(v)$$

which shows that g is a linear combination of the f_i. □

We can give a much more general version of this result. Fix a ring R, a complete ideal filter J of R and R-modules M and T, with T being J-injective and J-torsion (i.e. $T[J]=T$). Let $E=\operatorname{End}_R(T)$, and notice that $\operatorname{Hom}_R(M,T)$ is an E-module.

For every submodule M' of M we will identify $\operatorname{Hom}_R(M/M',T)$ with the E-submodule $\{f\in\operatorname{Hom}_R(M,T)\mid\ker(f)\supseteq M\}$ of $\operatorname{Hom}_R(M,T)$, and we will denote $\bigcap_{f\in V}\ker(f)$ by $\ker(V)$ for any subset $V\subseteq\operatorname{Hom}_R(M,T)$.

Proposition 5.2. *If V is a finitely generated E-submodule of* $\mathrm{Hom}_R(M,T)$ *we have* $V = \mathrm{Hom}_R(M/\ker(V),T)$.

Proof. Notice that the inclusion $V \subseteq \mathrm{Hom}_R(M/\ker(V),T)$ is obvious. For the other inclusion we want to show that every homomorphism $g : M \to T$ with $\ker(g) \supseteq \ker(V)$ belongs to V. Let then g be such a map and let $\overline{g} : M/\ker(V) \to T$ be its factorization through the quotient $M/\ker(V)$. Let $\{f_1, \dots, f_n\}$ be a set of generators for V as an E-module and let

$$\begin{aligned} \varepsilon : M &\to T^n \\ x &\mapsto (f_1(x), \dots, f_n(x)) \end{aligned}$$

We have $\ker(\varepsilon) = \ker(V)$, so that ε factors as an injective map $\overline{\varepsilon} : M/\ker(V) \to T^n$. Since T is J-torsion, so is T^n, hence $\overline{\varepsilon}$ is a J-extension. Since T is J-injective there is an R-linear map $\lambda : T^n \to T$ such that $\lambda \circ \overline{\varepsilon} = \overline{g}$, or equivalently $\lambda \circ \varepsilon = g$.

$$\begin{tikzcd}
& & T \\
M \arrow[r, two heads] \arrow[urr, bend left, "g"] \arrow[drr, bend right, "\varepsilon"'] & M/\ker(V) \arrow[ur, "\overline{g}"] \arrow[dr, hook, "\overline{\varepsilon}"'] & \\
& & T^n \arrow[uu, dashed, "\lambda"']
\end{tikzcd}$$

Since $\mathrm{Hom}_R(T^n,T) \cong \bigoplus_{i=1}^n \mathrm{End}_R(T)$, there are elements $e_1, \dots, e_n \in \mathrm{End}_R(T)$ such that $\lambda(t_1, \dots, t_n) = e_1(t_1) + \dots + e_n(t_n)$ for every $(t_1, \dots, t_n) \in T^n$. Then for $x \in M$ we get

$$\begin{aligned} \lambda(\varepsilon(x)) &= \lambda(f_1(x), \dots, f_n(x)) \\ &= e_1(f_1(x)) + \dots + e_n(f_n(x)) \end{aligned}$$

which means that $g = e_1 \circ f_1 + \dots + e_n \circ f_n \in V$ because V is an E-module. $\square$

From a different point of view, we have two maps

$$\begin{array}{rcc} k : \{E\text{-submodules of } \mathrm{Hom}_R(M,T)\} & \to & \{R\text{-submodules of } M\} \\ V & \mapsto & \ker(V) \end{array}$$

and

$$\begin{array}{rcc} h : \{R\text{-submodules of } M\} & \to & \{E\text{-submodules of } \mathrm{Hom}_R(M,T)\} \\ M' & \mapsto & \mathrm{Hom}_R(M/M',T) \end{array}$$

and the previous proposition shows that the restriction $k'$of k to the subset of *finitely generated* E-submodules satisfies $h \circ k' = \mathrm{id}$. It is natural to ask whether the two maps are inverse of each other, possibly after restricting h to a suitable subset.

Definition 5.3. We say that T is a *cogenerator* for an R-module N if

$$\bigcap_{f \in \mathrm{Hom}_R(N,T)} \ker(f) = 0\,.$$

Using this definition, we may formulate the following duality statement.

Theorem 5.4. *Let R be a unitary ring and let J be a complete ideal filter on R. Let T be a J-injective and J-torsion left R-module and let M be a left R-module. Assume that T is a cogenerator for every quotient of M and that $\operatorname{Hom}_R(M,T)$ is Noetherian as an $\operatorname{End}_R(T)$-module. The maps*

$$\begin{array}{ccc} \{R\text{-}\textit{submodules of } M\} & \to & \{\operatorname{End}_R(T)\text{-}\textit{sumbodules of } \operatorname{Hom}_R(M,T)\} \\ M' & \mapsto & \operatorname{Hom}_R(M/M',T) \\ \ker(V) & \leftarrow\!\shortmid & V \end{array}$$

define an inclusion-reversing bijection between the the set of R-submodules of M and that of $\operatorname{End}_R(T)$-submodules of $\operatorname{Hom}_R(M,T)$.

Proof. The maps are clearly inclusion-reversing and the fact that they are inverse of each other follows from Proposition 5.2 combined with the Noetherianity of M and from the assumption that T is a cogenerator for every quotient of M. ☐

Department of Mathematics, University of Luxembourg, 6 av. de la Fonte, 4364 Esch-sur-Alzette, Luxembourg

Email address: `sebastiano.tronto@uni.lu`