Varieties over Finite fields

- Varieties are reduced schemes of finite type over a field.
- Fix an algebraic closure k of $\mathbb{F}_q$. Recall $\mathrm{Frob} \in \mathrm{Gal}(k/\mathbb{F}_q)$, $\mathrm{Frob}(x) = x^q$
- If $X/\mathbb{F}_q$ is a variety, the $\mathrm{Frob}_{X,q} : X \to X$ is the identity on ~~the underlying top. space and sends $u \mapsto u^q$ at the level~~ the underlying top. space and sends $u \mapsto u^q$ at the level of sheaves.

$\mathrm{Frob}_{\overline{X},q} : \overline{X} \to \overline{X}$ is the extension of $\mathrm{Frob}_{X,q}$ to $\overline{X} := X \times_{\mathrm{Spec}\,\mathbb{F}_q} \mathrm{Spec}\,k$.

- $\mathrm{Frob}_{\overline{X},q^m} = \mathrm{Frob}_{\overline{X},q}^m$.
- ~~There are also The sets $X(\mathbb{F}_{q^m}) := \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{F}_q}(\mathrm{Spec}\,\mathbb{F}_{q^m}, X)$ and~~

~~Up~~

- ~~Let Γ_m denote the graph~~
- Let Γ_m denote the graph of $\mathrm{Frob}_{\overline{X},q}^m$ in $\overline{X} \times \overline{X}$ and $\Delta \subset \overline{X} \times \overline{X}$ the diagonal. Then Γ_m and Δ "intersect transversely" and the closed points of $\Gamma_m \cap \Delta$ can be identified with $X(\mathbb{F}_{q^m}) := \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{F}_q}(\mathrm{Spec}\,\mathbb{F}_{q^m}, X)$.

~~This means that $\Gamma_m \cap \Delta$~~

<u>Sketch</u>: $X = \mathbb{A}^n_{\mathbb{F}_q} \Rightarrow \overline{X} \times \overline{X} = \mathrm{Spec}\, k[x_1, \dots, x_n, y_1, \dots, y_n]$

Δ is cut by $(y_1 - x_1, \dots, y_n - x_n)$, Γ_m by $(y_1 - x_1^q, \dots, y_n - x_n^q)$

$\Rightarrow \Delta \cap \Gamma_m$ is cut by $(x_1 - x_1^q, \dots, x_n - x_n^q) \Rightarrow \Delta \cap \Gamma_m \cong \cancel{\mathrm{Spec}} \prod_{i=1}^n \mathrm{Spec}\, k[x_i]/(x_i - x_i^q)$

hence it is reduced.

$\boxed{\text{Def}}$: The zeta function of $X/\mathbb{F}_q$ is $Z(X,t) := \exp\left(\sum_{m \geq 1} |X(\mathbb{F}_{q^m})| \frac{t^m}{m} \right) \in \mathbb{Q}[[t]]$.

The Weil Conjectures

~~Suppose t~~ Let X be a smooth, geometrically connected, projective variety of dimension n defined over $\mathbb{F}_q$.

Rationality: $Z(X,t)$ is a rational function $\in \mathbb{Q}(t)$.

Functional Equation: If $E := (\Delta^2)$ is the self-intersection of the diagonal $\Delta \hookrightarrow X \times X$, then

$$Z\left(X, \frac{1}{q^n t}\right) = \pm q^{n\frac{E}{2}} t^E Z(X,t)$$

Riemann Hypothesis:

$$Z(X,t) = \frac{P_1(X,t)\,P_3(X,t)\cdots P_{2n-1}(X,t)}{P_0(X,t)\,P_2(X,t)\cdots P_{2n}(X,t)}$$

with $P_0(X,t) = 1-t$, $P_{2n}(X,t) = 1-q^n t$ and ~~for~~ $P_i(X,t) \in \mathbb{Z}[t]$.

Moreover, $P_i(X,t) = \prod_j (1-\alpha_{i,j} t)$ with $\alpha_{i,j}$ algebraic integers s.t. $|\alpha_{i,j}| = q^{\frac{i}{2}}$.

Betti numbers: Assuming the Riemann Hypothesis, define $b_i(X) := \deg P_i(X,t)$.

Then the following holds:

I) $E = \sum_{i=0}^{2n} (-1)^i b_i(X)$

II) If R is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, $\tilde{X}$ is a smooth projective scheme over $\operatorname{Spec} R$ ~~with~~, $p \in \operatorname{Spec} R$ s.t. $R/p \cong \mathbb{F}_q$ and $\tilde{X} \times_{\operatorname{Spec} R} \operatorname{Spec} R/p = X$, then $b_i(X) = \dim_{\mathbb{Q}} H^i\left(\left(\tilde{X} \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}\right)^{an}, \mathbb{Q}\right)$.

The Weil cohomology formalism

Let k be an algebraically closed field, K a characteristic 0 field.

All varieties are defined over k and assumed to be connected, projective, non-singular. $\mathrm{Var}(k)$ will denote the set of such varieties.

~~Definition: A contravariant functor X ↦~~

<u>Definition</u>: A <u>Weil cohomology theory</u> with coefficients in K consists of the following data:

I) A contravariant functor

$$\mathrm{Var}(k) \longrightarrow \text{Graded-Comm } K\text{-algebras}$$
$$X \longmapsto H^*(X) = \bigoplus_i H^i(X)$$

The product of $\alpha, \beta \in H^*(X)$ is denoted $\alpha \cup \beta$.

(<u>Graded-commutative</u>: α, β homogeneous $\Longrightarrow \alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$)

II) $\forall X \in \mathrm{Var}(k)$, there exists a linear map $\mathrm{Tr}_X : H^{2\dim(X)}(X) \longrightarrow K$.

III) $\forall X \in \mathrm{Var}(k)$ and for any closed irred subvariety $Z \subset X$ of codimension c, there exists a cohomology class $cl(Z) \in H^{2c}(X)$.

The above data is supposed to satisfy the following axioms:

a) $\forall X \in \mathrm{Var}(k)$, $H^i(X)$ is finite dimensional.

$H^i(X) = 0$ unless $0 \leq i \leq 2\dim(X)$. (<u>Finiteness</u> + <u>Cohomological dimension</u>)

b) (<u>Künneth isomorphism</u>) If $X, Y \in \mathrm{Var}(k)$ and if $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are the canonical projections, then the K-algebra homomorphism $H^*(X) \otimes_K H^*(Y) \longrightarrow H^*(X \times Y)$

$$\alpha \otimes \beta \longmapsto p_X^*(\alpha) \cup p_Y^*(\beta)$$

is in fact an isomorphism.

~~a) (Poincaré duality): $\forall X, Y \in Var(k)$, $Tr_{X\times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) =$~~

c) (Poincaré duality) $\forall X \in Var(k)$, $Tr_X : H^{2\dim(X)}(X) \longrightarrow K$ is an isomorphism and for every $i \in \{0, \dots, 2\dim(X)\}$, the bilinear map

$$H^i(X) \otimes_K H^{2\dim(X)-i}(X) \longrightarrow K, \ \alpha \otimes \beta \longmapsto Tr_X(\alpha \cup \beta)$$

is a perfect pairing.

d) $\forall X, Y \in Var(k)$, $Tr_{X\times Y}(p_X^*(\alpha) \cup p_Y^*(\beta)) = Tr_X(\alpha) Tr_Y(\beta)$

if $\forall \alpha \in H^{2\dim(X)}(X)$, $\forall \beta \in H^{2\dim(Y)}(Y)$.

e) (Exterior product of classes): Let $X, Y \in Var(k)$ and $Z \subseteq X$, $W \subseteq Y$ be closed irreducible subvarieties, then $cl(Z \times W) = p_X^*(cl(Z)) \cup p_Y^*(cl(W))$

f) (Push-forward): For any morphism $f : X \longrightarrow Y$, for any $Z \subseteq X$ closed and irreducible, if $\alpha \in H^{2\dim(Z)}(Y)$ then

$$Tr_X(cl(Z) \cup f^*(\alpha)) = \deg(Z/f(Z))\, Tr_Y(cl(f(Z)) \cup \alpha)$$

g) (Pull-back): For any $f : X \longrightarrow Y$, let $W \subseteq Y$ be irreducible and closed s.t.

i) all irred. comp. $W_1, \dots, W_r$ of $f^{-1}(Z)$ have pure dimension

$$\dim(Z) + \dim(X) - \dim(Y)$$

ii) Either f is flat in a nbhd of Z or $f^{-1}(Z)$ is generically smooth.

If $[f^{-1}(Z)] = \sum_{i=1}^{r} m_i W_i$, then $f^*(cl(Z)) = \sum_{i=1}^{r} m_i \, cl(W_i)$

h) If $x = \mathrm{Spec}(k)$, then $cl(x) = 1$ and $Tr_x(1) = 1$.

Proposition 1: Let X be a smooth, ~~connected~~ ~~contractible~~ connected, n-dimensional projective variety.

i) ~~The morphism $K \rightsquigarrow$~~ The structural morphism $K \to H^0(X)$ is an iso.

ii) $cl(X) = 1 \in H^0(X)$

iii) If $x \in X$ is closed, then $Tr_X(cl(x)) = 1$

iv) Let $f: X \to Y$ be (generically) finite, surjective of degree d, then $Tr_X(f^*(\alpha)) = d \, Tr_Y(\alpha)$, $\forall \alpha \in H^{2\dim(Y)}(Y)$. If $Y = X$, then f^* acts as multiplication by d on $H^{2\dim(X)}(X)$.

Proof:

i) Poincaré duality $+ \; H^{2\dim(X)}(X) \cong K \Rightarrow \dim_K H^0(X) = 1$.

(ii) The conditions in (g) are satisfied for $X \xrightarrow{f} \mathrm{Spec}\,k$ $\Rightarrow cl(X) = f^*(cl(\mathrm{Spec}\,k))$

$$= f^*(1) \quad \text{by } cl(\mathrm{Spec}\,k) = 1$$
$$= 1$$

(iii) Apply (f) to $X \xrightarrow{f} \mathrm{Spec}\,k$ with $Z = \{x\}$, $\alpha = 1 \in H^0(\mathrm{Spec}\,k) \Rightarrow \deg(Z/f(Z)) = 1$

$$\Rightarrow Tr_X(cl(\{x\}) \cup 1) = Tr_X(cl(\{x\}))$$
$$= 1 \cdot Tr_{\mathrm{Spec}\,k}(cl(\mathrm{Spec}\,k))$$
$$= 1 \quad \text{by } (h)$$

iv) ~~Let ... $q \in Y$, then with f^{-1}~~ Consider $q \in Y$. If $[f^{-1}(q)] = \sum_{i=1}^{r} m_i p_i$,

then $\sum_i m_i = d$. Generic flatness + (g) $\Rightarrow Tr_X(f^*(cl(q))) = Tr_X(\sum_i m_i \, cl(p_i))$

by (f) $\Rightarrow \; = d \, Tr_Y(cl(q))$

(f) $\Rightarrow Tr_X(cl(p_i)) = Tr_X(cl(p_i) \cup f^*(1)) = \deg(p_i/f(p_i)) \cdot Tr_Y(cl(q) \cup 1)$

Since $cl(q)$ generates $H^{2\dim(Y)}(Y)$, this finishes the proof.

~~Definition Pullback~~

<u>Push-forwards in cohomology</u>: Let $f: X \to Y$ be a morphism, $\dim X = m$ and $\dim Y = n$. Given $\alpha \in H^i(X)$, $\exists$ a unique $f_*(\alpha) \in H^{2n-2m+i}(Y)$ s.t. $Tr_Y(f_*(\alpha) \cup \beta) = Tr_X(\alpha \cup f^*(\beta))$, $\forall \beta \in H^{2m-i}(Y)$. f^* is K-linear.

<u>Proposition 2</u>: Let $f: X \to Y$ be as above.

i) (<u>Projection Formula</u>): $f_*(\alpha \cup f^*(\gamma)) = f_*(\alpha) \cup \gamma$

ii) If $g: Y \to Z$ is another morph, $(g \circ f)_* = g_* \circ f_*$.

iii) If Z is irred, closed in X, then $f_*(cl(Z)) = \deg(Z/f(Z)) \, cl(f(Z))$

Proof:

i) $Tr_Y(f_*(\alpha \cup f^*(\gamma)) \cup \beta) = Tr_X((\alpha \cup f^*(\gamma)) \cup f^*(\beta)) = Tr_X(\alpha \cup f^*(\gamma \cup \beta))$

$= Tr_Y(f_*(\alpha) \cup (\gamma \cup \beta))$

$= Tr_Y((f_*(\alpha) \cup \gamma) \cup \beta)$

iii) $Tr_Y(f_*(cl(Z)) \cup \beta) = Tr_X(cl(Z) \cup f^*(\beta)) = \underbrace{\deg(Z/f(Z))}_{d} \, Tr_Y(cl(f(Z)) \cup \beta)$

$= Tr_Y((d \, cl(f(Z))) \cup \beta)$

Proposition 3 : Let X, Y be connected, nonsingular, projective and consider the canonical projections $p: X\times Y \to X$, $q: X\times Y \to Y$. If $\alpha \in H^i(Y)$, then $p_*(q^*(\alpha)) = Tr_Y(\alpha)$ if $i = 2\dim(Y)$, and $p_*(q^*(\alpha)) = 0$, otherwise.

Proof:

$p_*(q^*(\alpha)) \in H^{2\dim(X) - 2\dim(X\times Y) + i}(X) = H^{i-2\dim(Y)}(X) \Longrightarrow p_*(q^*(\alpha)) = 0$ ~~since~~ if $i < 2\dim(Y)$

If $\alpha \in H^{2\dim(Y)}(Y)$, $\beta \in H^{2\dim(X)}(X)$, then

$$Tr_X(p_*(q^*(\alpha)) \cup \beta) = Tr_{X\times Y}(q^*(\alpha) \cup p^*(\beta)) \underset{(d)}{=} Tr_Y(\alpha)\, Tr_X(\beta) = Tr_X(Tr_Y(\alpha)\, \beta)$$

$$\cancel{\Rightarrow} \qquad \cancel{\forall Tr_X} \Longrightarrow p_*(q^*(\alpha)) = Tr_Y(\alpha)$$

Theorem (Trace Formula): If $\phi: X \to X$ is an endomorphism of the nonsingular, connected, projective variety X, and if $\Gamma_\phi, \Delta \subset X\times X$ denote the graph and the diagonal, then

$$(\Gamma_\phi \cdot \Delta) = \sum_{i=0}^{2\dim(X)} (-1)^i \operatorname{trace}(\phi^* | H^i(X))$$

If Γ_ϕ and Δ intersect transversely, the above computes $|\{x \in X \mid \phi(x) = x\}|$.

Proof:

~~Step 1~~ Claim 1: If $\alpha \in H^*(X)$, then $p_*(cl(\Gamma_\phi) \cup q^*(\alpha)) = \phi^*(\alpha)$.

Proof of claim: Consider the unique j making the following diagram commute

$$
\begin{tikzcd}
X \arrow[dr, dashed, "j"] \arrow[drr, bend left, "\phi"] \arrow[ddr, bend right, "Id_X"'] & & \\
& X \times X \arrow[r, "q"] \arrow[d, "p"] & X \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} k
\end{tikzcd}
$$

Then $p \circ j = Id_X$, $q \circ j = \phi$ and $\Gamma_\phi := j(X)$.

By Proposition 2 (iii) $\Longrightarrow$ $j_*(cl(X)) = \underbrace{\deg(X/j(X))}_{1} \, cl(j(X)) = cl(\Gamma_\phi)$

Hence, $p_*(cl(\Gamma_\phi) \cup q^*(\alpha)) = p_*(j_*(cl(X)) \cup q^*(\alpha))$ $(\star)$

Projection formula $\Longrightarrow$ $j_*(cl(X)) \cup q^*(\alpha) = j_*(cl(X) \cup j^* q^*(\alpha))$

$= j_*(1 \cup j^* q^*(\alpha))$

$= j_*(j^* q^*(\alpha))$

~~Apply p_* ...~~

Substituting into $(\star)$ $\Longrightarrow$ $p_*(cl(\Gamma_\phi) \cup q^*(\alpha)) = (p_* j_*)(j^* q^*(\alpha))$

$= \phi^*(\alpha)$

<u>Claim 2</u>: Let (e_i^r) be a basis of $H^r(X)$ and (f_i^{2n-r}) the dual basis of $H^{2n-r}(X)$ s.t. $Tr_X(f_\ell^{2n-r} \cup e_i^r) = \delta_{i,\ell}$. Then

$$cl(\Gamma_\phi) = \sum_{i,r} p^*(\phi^*(e_i^r)) \cup q^*(f_i^{2n-r}) \in H^{2n}(X \times X)$$

<u>Proof of Claim</u>:

<u>Künneth isomorphism</u> $\Rightarrow cl(\Gamma_\phi) = \sum_{\ell,s} p^*(a_{\ell,s}) \cup q^*(f_\ell^{2n-s})$ (♥) for unique elements $a_{\ell,s} \in H^s(X)$

<u>Claim 1</u> $\Rightarrow \phi^*(e_i^r) = p_*(cl(\Gamma_\phi) \cup q^*(e_i^r))$

by linearity of p_* + (♥) $= \sum_{\ell,s} p_*\left(p^*(a_{\ell,s}) \cup q^*(f_\ell^{2n-s}) \cup q^*(e_i^r)\right)$

$$= \sum_{\ell,s} p_*\left(p^*(a_{\ell,s}) \cup q^*(f_\ell^{2n-s} \cup e_i^r)\right)$$

by Projection formula $= \sum_{\ell,s} a_{\ell,s} \cup p_*\left(q^*(f_\ell^{2n-s} \cup e_i^r)\right)$

$p_*(q^*(f_\ell^{2n-s} \cup e_i^r)) = 0$, unless $r = s$, by <u>Prop. 3</u>, in which case it equals

$Tr_X(f_\ell^{2n-r} \cup e_i^r) = \delta_{i,\ell} \Rightarrow \phi^*(e_i^r) = a_{i,r}$.

Conclusion :

Apply Claim 2 to $cl(\Delta)$ and dual bases (f_ℓ^s) and $((-1)^s e_\ell^{2n-s})$

$$cl(\Delta) = \sum_{\ell,s} (-1)^s p^*(f_\ell^s) \cup q^*(e_\ell^{2n-s})$$

$$\Rightarrow \quad (\Gamma_\phi \cdot \Delta) \overset{!}{=} Tr_{X\times X}\left(cl(\Gamma_\phi) \cup cl(\Delta)\right)$$

$$= \sum_{i,r} Tr_X\left(\phi^*(e_i^r) \cup f_i^{2n-r}\right) Tr_X\left(f_i^{2n-r} \cup e_i^r\right)$$

$$= \sum_r (-1)^r \operatorname{trace}\left(\phi^* | H^r(X)\right)$$

! Fact from intersection theory : If X is above, $\alpha_i \in H^{2m_i}(X)$ with for $1 \leq i \leq r$ s.t. $\sum_i m_i = \dim X$, then

$$(\alpha_1 \cdots \alpha_r) = Tr_X\left(cl(\alpha_1) \cup \cdots \cup cl(\alpha_r)\right) \qquad \square$$

Corollary: For X as above,

$$(\Delta \cdot \Delta) = \sum_{i=0}^{2n} (-1)^i \dim_K H^i(X)$$

From now on, we assume the existence of a Weil cohomology for varieties over $k = \overline{\mathbb{F}}_p$ with coefficients in the field K, $\operatorname{char} K = 0$.

Recall that if X is a variety over $\mathbb{F}_q$, we have the morphism $\mathrm{Frob}_{X,q} : X \to X$. If $\overline{X} := X \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} k$, we have the morphism $F := \mathrm{Frob}_{X,q} \times \mathrm{Id}_X : \overline{X} \to \overline{X}$. We also have $\mathrm{Frob}_{X,q^m} \times \mathrm{Id}_X = F^m$

<u>Theorem</u>: If X is a nonsingular, geometrically connected, n-dimensional ~~variety~~ projective variety over $\mathbb{F}_q$, then

$$Z(X,t) = \frac{P_1(X,t)\cdot P_3(X,t)\cdot \ldots \cdot P_{2n-1}(X,t)}{P_0(X,t)\cdot P_2(X,t)\cdot \ldots \cdot P_{2n}(X,t)}$$

with $P_i(X,t) = \det(\mathrm{Id} - tF^* \mid H^i(\overline{X}))$. Hence, $Z(X,t) \in \mathbb{Q}(t)$.

<u>Proof</u>:

<u>Fact 1</u>: For every $\phi \in \mathrm{End}_K(V)$, where $\dim_K V < \infty$, we have

$$\det(\mathrm{Id} - t\phi) = \exp\left(-\sum_{m\geq 1} \operatorname{trace}(\phi^m \mid V)\frac{t^m}{m}\right)$$

<u>Fact 2</u>: Let L be a field, $f = \sum_{m\geq 0} a_m t^m \in L[[t]]$. If L'/L is a field extension, then f lies in $L(t) \Leftrightarrow f$ lies in $L'(t)$.

Set $N_m := |X(\mathbb{F}_{q^m})|$. Recall $N_m = |\{x \in \overline{X} \mid F^m(x) = x\}|$ and that $\Gamma_{F^m} \subset X \times X$ is transverse to the diagonal.

<u>Trace Formula</u> $\Longrightarrow$ $N_m = \Gamma_{F^m}\cdot\Delta = \sum_{i=0}^{2n}(-1)^i \operatorname{trace}((F^m)^* \mid H^i(\overline{X}))$

$$\Longrightarrow Z(X,t) = \exp\left(\sum_{m\geq 1}\sum_{i=0}^{2n}(-1)^i \operatorname{trace}((F^m)^* \mid H^i(\overline{X}))\frac{t^m}{m}\right)$$

Fact 1 $\Longrightarrow$ $\displaystyle = \prod_{i=0}^{2n} \det\left(Id - tF^* \mid H^i(\bar{X})\right)^{(-1)^{i+1}} \in K(t)$

Fact 2 $\Longrightarrow$ $Z(X,t) \in \mathbb{Q}(t)$.

Theorem (Functional Equation) Let X be a non-singular, geometrically connected, n-dimensional variety over $\mathbb{F}_q$, $E = \Delta\cdot\Delta$, where $\Delta \subset \bar{X}\times\bar{X}$ is the diagonal. Then $Z(X,t) = \pm\, q^{nE/2}\, t^E\, Z(X,t)$

Proof:

Fact : Let $\phi : V\times W \to K$ be a perfect pairing of vector spaces $/K$ of dimension r. If $\lambda \in K\setminus\{0\}$ and, $f\in End_K(V)$, $g\in End_K(W)$ are s.t. $\phi(f(v), g(w)) = \lambda\,\phi(v,w)$, $\forall v\in V$, $\forall w\in W$, then

$$\det(Id - tg \mid W) = \frac{(-1)^r\lambda^r t^r}{\det(f|V)} \det(Id - \lambda^{-1}t^{-1} f \mid V)$$

and

$$\det(g\mid W) = \frac{\lambda^r}{\det(f|V)}$$

~~Fact +~~ $\cancel{\phi_i : H^i(\bar{X})\otimes H^{2n-i}(\bar{X}) \to K,\ \phi_i(\alpha\otimes\beta) = Tr(\alpha\cup\beta)}$ ~~being perfect~~

Recall : $F : \bar{X} \to \bar{X}$ is finite of degree q^n.

Proposition 1 $\Longrightarrow$ $F^*(\alpha) = q^n\alpha$, $\forall \alpha \in H^{2n}(\bar{X})$

$$\Longrightarrow \phi_i(F^*(\alpha)\cup F^*(\beta)) = Tr_{\bar{X}}\big(F^*(\overset{H^{2n}(\bar{X})}{\overset{\in}{\alpha\cup\beta}})\big) = q^n\, Tr_{\bar{X}}(\alpha\cup\beta)$$
$$= q^n \phi_i(\alpha\otimes\beta)$$
$$\forall\alpha\in H^i(\bar{X}),\ \forall\beta\in H^{2n-i}(\bar{X})$$

Set $B_i := \dim_K H^i(\bar{X})$, $P_i(t) := \det(Id - tF^* \mid H^i(\bar{X}))$. Then

$$\underline{\text{Fact}} \implies \det\left(F^* \mid H^{2n-i}(\bar{X})\right) = \frac{q^{nB_i}}{\det\left(F^* \mid H^i(\bar{X})\right)} \quad \text{and}$$

$$P_{2n-i}(t) = \frac{(-1)^{B_i} q^{nB_i} t^{B_i}}{\det\left(F^* \mid H^i(\bar{X})\right)} P_i\left(1/q^n t\right) \qquad (*)$$

$$\underline{\text{Rationality}} \implies Z\left(1/q^n t\right) = \prod_{i=0}^{2n} P_i\left(1/q^n t\right)^{(-1)^{i+1}} =$$

$$E = \sum_{i=0}^{2n} (-1)^i B_i + (*) \implies \quad = \prod_{i=0}^{2n} P_{2n-i}(t)^{(-1)^{i+1}} \cdot \frac{(-1)^E q^{nE} t^E}{\prod_{i=0}^{2n} \det\left(F^* \mid H^i(\bar{X})\right)^{(-1)^i}}$$

$$= \pm Z(X,t) \cdot \frac{q^{nE} t^E}{q^{nE/2}} = \pm q^{nE/2} t^E Z(X,t)$$