Modular Forms Project: Zero Fourier Coefficients of Eta Quotients

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June 20, 2018

1 Introduction

The present project is extracted from an unpublished preprint of the author [1], which will be on the author's web page at the end of July 2018. It is almost entirely *experimental*: mainly computer experiments, and a few exercises.

1.1 Definitions

Recall the *Dedekind eta function*

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

with $q = e^{2\pi i\tau}$ (where by convention $q^{a/b} = e^{2\pi i(a/b)\tau}$). It is a modular form of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ with a complicated multiplier system formed of 24th roots of unity. Its 24th power is the discriminant function $\Delta(\tau)$.

There are two `GP` syntaxes for η: either simply `eta(x+O(x^m))`, which computes the Fourier expansion of η to m terms, omitting the factor $q^{1/24}$ which cannot be represented in the `GP` language. Or `eta(z,1)`, which computes $\eta(z)$ with z in the upper half-plane numerically (including of course the $e^{2\pi iz/24}$). We will not need this second syntax for the present project.

However, for very large values of m (such as $m = 10^7$ or 10^8), the available `GP` function is extremely slow. Fortunately, there exists a `Pari` library function which is highly optimized for other uses, and which in addition provides directly $\eta(v\tau)$ (corresponding to `eta(x^v)`), and I strongly suggest using it. For this, write `install(eta_inflate_ZXn,LL)`, which gives `GP` access to this function, whose syntax is `eta_inflate_ZXn(m,v)` which provides the power series expansion of `eta(x^v)` to `m` terms.

Definition 1.1 (1) *If m, r and s are in $\mathbb{Z}$, we define the function*

$$F_m(r, s) = \eta(\tau)^r \eta(m\tau)^s \ ,$$

and we set $k = (r + s)/2 \in (1/2)\mathbb{Z}$, its weight.

(2) *Set* $(r+ms)/24 = e/d$ *with* $\gcd(e,d)=1$ *and* $d \geq 1$*, and let*

$$\mathcal{E}(d) = \{n \geq e,\ n \equiv e \pmod{d}\}\ .$$

We define the Fourier coefficients $b(n) = b_m(r,s;n)$ *by*

$$F_m(r,s) = \sum_{n \in \mathcal{E}(d)} b(n) q^{n/d}\ .$$

(3) *We define the set* $\mathcal{Z}(F)$ *of* Fourier zeros *of* F *as the set of* $n \in \mathcal{E}(d)$ *such that* $b(n) = 0$.

(4) *The (normalized) counting function of the Fourier zeros is defined by*

$$N(F;X) = d \cdot |\{n \in \mathcal{Z}(F),\ n \leq X\}|\ ,$$

and the Fourier zero density as $z(F) = \liminf_{X\to\infty} N(F;X)/X$.

Some remarks: first, since $\eta(\tau) \in q^{1/24}\mathbb{Z}[[q]]$, we have $\eta(m\tau) \in q^{m/24}\mathbb{Z}[[q^m]]$, so

$$F_m(r,s) \in q^{(r+ms)/24}\mathbb{Z}[[q]] = q^{e/d}\mathbb{Z}[[q]]\ ,$$

so the only coefficients to consider are indeed those of the form $q^{e/d+\lambda} = q^{(e+\lambda d)/d}$ with $\lambda \in \mathbb{Z}_{\geq 0}$, i.e., $q^{n/d}$ with $n = e + d\mathbb{Z} \in \mathcal{E}(d)$.

Second, we normalize the counting function by multiplying by d since the density of $\mathcal{E}(d)$ is equal to $1/d$. For the same reason, it is clear that $z(F) \leq 1$.

Since $\eta(\tau)$ is modular on Γ (with a complicated multiplier system), $\eta(m\tau)$ and hence also $F_m(r,s)$ is modular on $\Gamma_0(m)$ (with a complicated multiplier system), and since η has weight $1/2$, $F_m(r,s)$ has weight $k = (r+s)/2$. The following proposition says when $F_m(r,s)$ is holomorphic:

Proposition 1.2 *$F_m(r,s)$ is a modular form (i.e., is holomorphic on $\mathfrak{H}$ and at the cusps) if and only if $r \geq \max(-s/m, -sm)$, in which case it will be modular of weight $k = (r+s)/2 \geq 0$ on $\Gamma_0(m)$ with a complicated multiplier system.*

Exercise 0. (Not part of the project.)

(1) Let $\gamma \in \Gamma$ and $m \in \mathbb{Z}_{\geq 1}$. Show that there exist $\delta \in \Gamma$ and some upper-triangular matrix T such that $\left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right)\gamma = \delta T$.

(2) Deduce that the order of vanishing of $\eta(m\tau)$ at a cusp a/c is equal to $\gcd(m,c)^2/(24m)$.

(3) Using this, prove the proposition.

In view of the fundamental difference between modular forms of integral and half-integral weight, it is clear that there will be a sharp distinction between $r+s$ even (integral weight), and $r+s$ odd (half-integral weight).

1.2 The Cases $rs = 0$

Note that if $r = 0$ we have $F_m(0, s) = \eta(m\tau)^s = F_1(s, 0)(m\tau)$, so without loss of generality we may assume that $s = 0$, so we are dealing with a pure eta power $F(r) := \eta(\tau)^r$. In the case r even (integral weight), we have the following nice result of Serre [3]:

Definition 1.3 *We say that F is* lacunary *if $N(F; Z) \sim X$, or equivalently $z(F) = 1$.*

Theorem 1.4 (Serre) *If $r \geq 2$ is even, η^r is lacunary if and only if $r = 2$, 4, 6, 8, 10, 14, and 26, and in all these cases it is a linear combination of Hecke theta series.*

No need to define these theta series: simply note that this means that the there is an efficient explicit formula for the Fourier coefficients.

Conjecture 1.5 *For r even not in Serre's list we have $N(\eta^r, X) = O(\log\log(X))$.*

Experiment 1. (Don't waste time on this negative result). Show that in fact $N(\eta^r, X) = 0$ for $r \leq 30$ *even* and not in Serre's list for X as large as you can (I went up to $X = 5 \cdot 10^7$), or find an example of a nonzero Fourier coefficient (I did not find any, but it seems reasonable to expect that there are some).

For r odd (half-integral weight) the problem is much more interesting. First note that η and η^3 have very simple explicit lacunary expansions, so we assume $r \geq 5$ odd and set the following definition:

Definition 1.6 *Let $F = F_m(r, s)$ be a form of half-integral weight. We will say that $a \cdot u^2$ for a fixed integer a are* Shimura zeros *if for all $n \in \mathcal{E}(d)$ of the form $n = au^2$ we have $n \in \mathcal{Z}(f)$, i.e., $b(n) = 0$.*

The name comes from the construction of Shimura *lifts* from modular forms of half-integral weight $k \in 1/2 + \mathbb{Z}$ to forms of integral weight $2k - 1$, which implies in particular that if $b(n) = 0$ then $b(nu^2) = 0$ if suitable conditions are satisfied. Note the trivial fact that if there exist at least one Shimura zero then $N(F, X) \geq C(F)X^{1/2}$ for a suitable positive constant $C(F)$. In fact, we can formulate a very weak analogue of Conjecture 1.5 in the case of half-integral weight:

Conjecture 1.7 *For r odd we have $N(\eta^r, X) = o(X)$, or equivalently $z(\eta^r) = 0$.*

In fact a more rash conjecture would be that $N(\eta^r, X) = O(X^{1/2+\varepsilon})$ for all $\varepsilon > 0$, or even $N(\eta^r, X) = O(X^{1/2}\log(X))$.

Experiment 2.

(1) For $r = 5$, find as many Shimura zeros as you can (for instance $a = 37445$, 43253,...) Can you find a pattern, or some estimate of the number of a up to some bound ? (I do not know).

(2) For $r = 7$, find at least one Shimura zero (it exists, and I found only one).

(3) For $r = 15$, find at least one Shimura zero (it exists, and I found only one).

(4) For $r = 9$, 11, 13, 17,... I have not been able to find any, but maybe you can.

In this experiment, as well as all the others, one asks only to check whether it seems conjecturally true. For instance, when we say that $a = 37445$ is a Shimura zero, this means that one should at least check that $b(n) = 0$ for $n = au^2$ with $u = 1$, 5, and 7 say (recall that we need that $n \in \mathcal{E}(d)$). In particular checking a single zero would not be sufficient.

For the cases $r = 5$ and $r = 7$, you may be helped by the following proposition which follows trivially from the explicit expansions of η and η^3, and which can avoid handling power series with millions of coefficients:

Proposition 1.8 *We have*

$$\eta^5(\tau) = \frac{1}{8} \sum_{(x,y,z)\in\mathbb{Z}^3} \left(\frac{12}{xy}\right)\left(\frac{-4}{z}\right) z q^{(x^2+y^2+3z^2)/24} \quad \text{and}$$
$$\eta^7(\tau) = \frac{1}{8} \sum_{(x,y,z)\in\mathbb{Z}^3} \left(\frac{12}{x}\right)\left(\frac{-4}{yz}\right) yz q^{(x^2+3y^2+3z^2)/24} \ .$$

2 The Case $m = 2$

We now study the case $m = 2$, i.e., functions $F_2(r, s) = \eta(\tau)^r \eta(2\tau)^s$. Here, several new types of families of zeros occur. In particular, there are many cases where the density $z(F)$ is not 0 nor 1 (which conjecturally cannot happen for pure eta powers, see Conjectures 1.5 and 1.7), so we give a formal definition of such zeros:

Definition 2.1 *We say that F has* congruence zeros *if there exists (u, v) with $d \mid v$ and $u \equiv e \pmod d$ such that $n \equiv u \pmod v$ implies $b(n) = 0$, in other words $u + v\mathbb{Z} \subset \mathcal{Z}(F)$, and we will simply say that $u \bmod v$ is a congruence zero.*

Clearly if such congruence zeros exist then $z(F) \geq 1/(v/d)$. Note that congruence zeros may exist both in integral and half-integral weight.

The lacunary functions $F_2(r, s)$ have been classified (there are 45 in all), so in the following experiments we restrict to $z(F) < 1$.

2.1 Implementation Remarks

You may skip this section, but it will allow to go much further in the experiments.

We will need to compute the Fourier expansions of $F(\tau) = \eta(\tau)^r \eta(2\tau)^s$ for different values of r and s, certainly at least to 10^6 terms, but if possible to 10^7 terms or more. When r and s are positive, one can do this naively (although there are better methods). In `Pari/GP` you would write for instance:

```
L=10^6; /* for instance */
E=eta(x+O(x^L)); /* or eta_inflate_ZXn(L,1) */
ED=eta(x^2+O(x^L));  /* or eta_inflate_ZXn(L,2) */
   /* better and much faster than substituting x for x^2 in E. */
F=E^r*ED^s
```

On the other hand, if r or s is negative, computing the inverse of a series with 10^6 terms will be extremely expensive, and the coefficients very large. Since we are only looking for zero coefficients and not the coefficients themselves, the natural idea is to work modulo p for a sufficiently large prime p (of course we may get false positives if a coefficient is divisible by p, but then first we can start again with a different p, and second, in view of the incredible *regularity* of the appearance of the zeros that we will find, any deviation would be immediately spotted). This already solves the problem of explosion of coefficients, but not the problem of inverting a series. Solving the latter problem is specific to the package that you use. The following implementation exercise assumes that you are working with `Pari/GP`.

Implementation Exercise 1.

(1) For $L = 10^m$ with $4 \leq m \leq 7$, compare the times taken by the `GP` function `eta(x+O(x^L))` and the installed library function `eta_inflate_ZXn(L,1)`; you can also compare with your own homemade `GP` script.

(2) For $L = 10^m$ with $4 \leq m \leq 5$ (not $m \geq 6$, it would be too long), look at the time to compute $1/\eta$ using the simple command `1/E`, where `E` has been computed beforehand (either with the command `eta`, or with `eta_inflate_ZXn`).

(3) Let us choose $p = 10^9+7$ (`nextprime(10^9)`), which is the prime I usually choose because $p^2 < 2^{63}$, and set `Ep=Mod(E,p)`. For $L = 10^4$ (only!), look at the time taken for `1/Ep`. Much longer than `1/E`, so why did we reduce modulo p?

(4) However, there is a very efficient program in the `Pari` library for inverting a series, called `ser_inv`. To be able to use it, you must type `install(ser_inv,G)` (don't worry about the syntax). Now look at the time taken for `ser_inv(Ep)` for $L = 10^m$ for $4 \leq m \leq 6$ (even $m = 7$ if you want): considerably faster than `1/E`. Note: `ser_inv` is really useful modulo p, but not otherwise: `ser_inv(E)` is only 2 or 3 times faster than `1/E`.

Implementation Exercise 2.

So as to considerably simplify the experiments that we are going to do, write the following three small programs:

(1) A precomputation program which, given a number `L0` of coefficients and a prime `p0` such as $10^9 + 7$, puts these values in global variables `L` and `p` (declaration: `global`), and computes modulo `p` to `L` terms and stores in global variables the four power series `eta(x), 1/eta(x), eta(x^2)`, and `1/eta(x^2)` (corresponding of course to $\eta(\tau)$, $1/\eta(\tau)$, $\eta(2\tau)$, and $1/\eta(2\tau)$ as modular forms), using of course the indications given in the previous exercise.

(2) A program computing $F_2(r,s)$ modulo p to L terms.

(3) A program to compute the list of zero Fourier coefficients of $F_2(r,s)$ modulo p to L terms, as elements of $\mathcal{E}(d)$ as given by Definition 1.1.

As a check on your implementation, compute the (Shimura) zeros of η^5 for $L = 10^4$, say (which will correspond to $n \le 24L + 5$).

Experiment 3.

(1) For $F = F_2(-8, 24)$ find a congruence zero. Apparently there is only one, and it would imply $z(F) = 1/4$. Can you prove it ? (I did not).

(2) For $F = F_2(r,s)$ with $(r,s) = (-5,13)$, $(1,7)$, $(7,1)$, and $(13,-5)$, (all of weight 4), look for congruence zeros, and show that apparently odd powers of 31 enter the picture. This would imply $z(F) = 1/32$. True ?

(3) For $F = F_2(r,s)$ with $(r,s) = (-3,13)$, $(1,9)$, $(9,1)$, and $(13,-3)$, (all of weight 5), do the same, and show that apparently odd powers of 1223 enter the picture, so that $z(F) = 1/1224$. True ?

(4) Can you find any other $F_2(r,s)$ of *integral weight* such that $0 < z(F) < 1$? I could not.

Exercise 1. The goal of this exercise is to *prove* that $b(n) = 0$ when $v_{31}(n)$ is odd for one of the forms in (2) above, for instance for $(r,s) = (1,7)$. It will be useful to use the latest version of `Pari/GP` for this, but can of course use another package if you prefer.

(1) Let $F(\tau) = \eta(8\tau)\eta(16\tau)^7 = F_2(1,7)(8\tau)$. Show that $F \in S_4(\Gamma_0(128), \chi_8)$, and that it is a newform (in `GP`, you can use the functions `F=mffrometaquo([8,1;16,7]);`, `mfparams(F)`, and `mf=mfinit(F,0);`).

(2) Compute a basis of eigenforms of the newspace, and express F as a linear combination of these eigenforms. You can do this using linear algebra, but more amusingly, using Petersson products.

(3) Compute the eigenvalues of T_{31} on these eigenforms, and deduce that $b(n) = 0$ when $v_{31}(n)$ is odd.

Experiment 4. Similar to the above, but now in half-integral weight. One can show that $F = F_2(r,s)$ is lacunary if and only if $(r,s) = (-2,5)$, $(-1,2)$, $(2,-1)$, and $(5,-2)$, so we exclude those.

Find all $F_2(r,s)$ of half-integral weight outside of these four which have congruence zeros. I found exactly 10, and they all have $z(F) = 1/6$, and the congruence zeros are all of the form $a \cdot 2^{2k-2} \pmod{d \cdot 2^{2k+1}}$. Find all these forms and the corresponding values of a. Are there more ?

Experiment 5. Find non-lacunary $F_2(r,s)$ having Shimura zeros (I found 12 in weight 5/2, 13 in weight 7/2, 6 in weight 9/2, 1 in weights 11/2 and 13/2, and none in higher weight). Can you find them, and can you find more ?

We finish with still another kind of Fourier zero.

Definition 2.2 *We say that F has a* Hecke zero *(for the prime p) if $b(n) = 0$ whenever n is of the form $n = a \cdot p^j$, $n = a \cdot p^{2j}$, or, in the case where $k \in 1/2 + \mathbb{Z}$, $n = a \cdot p^{4j}$, for some fixed a and all j.*

Note that this is the weakest kind of zero: if F has a Hecke zero it implies only that $N(F; X) \geq C(F) \log(X)$ for a suitable positive constant $C(F)$.

Experiment 6. Assume that $z(F) = 0$ (in particular F is not lacunary and has no congruence zeros). Find all forms which have Hecke zeros. I found 8 for $p = 2$ (all in even weight), and 8 for $p = 31$ (all in weight 8 in that case), and no other, in particular none in odd or half-integral weight. Can you reproduce this, and/or do better ? For instance, in view of Experiment 3, there are perhaps Hecke zeros for $p = 1223$?.

Problem (Difficult, I needed J.-P Serre's help!) Prove the following result: $n = 2^j$ is a Hecke zero for $F_2(-16, 68)$; in other words, if we set

$$\eta(\tau)^{-16} \eta(2\tau)^{68} = \sum_{n \geq 5} b(n) q^n \ ,$$

then for all j we have $b(2^j) = 0$.

Experiment ∞ Repeat the same experiments for $F_m(r, s)$ for $m \geq 3$.

References

[1] H. Cohen, *Experimental observations on eta quotients*, preprint, 27p.

[2] H. Cohen and F. Strömberg, *Modular Forms: A Classical Approach*, Graduate Studies in Math. **179**, American Math. Soc., (2017).

[3] J.-P. Serre, *Sur la lacunarité des puissances de η*, Glasgow Math. J. **27** (1985), 203–221.

[4] D. Zagier, *Elliptic modular forms and their applications*, in *The 1-2-3 of modular forms*, Universitext, Springer (2008), pp. 1–103.