Modular Forms Project: Atkin's Orthogonal Polynomials

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1 Introduction

The present project is entirely copied from part of a paper of M. Kaneko and D. Zagier [2] that you can find on Zagier's web page. It is almost entirely theoretical, with only a little experimental aspect.

Definition 1.1 (1) *We let V be the $\mathbb{C}$-vector space of* holomorphic *functions on the upper half-plane $\mathfrak{H}$ which are invariant under the full modular group.*

(2) *For any subfield K of $\mathbb{C}$ (in practice $K = \mathbb{R}$ or $K = \mathbb{Q}$) we denote by $V(K)$ the K-vector space of elements of V whose Fourier expansion has coefficients in K.*

Exercise 1. Recall that the set of all Γ-invariant *meromorphic* functions on $\mathfrak{H}$ is $\mathbb{C}(j)$, where j is the elliptic j-function, and that j induces an isomorphism between $\mathfrak{H}$ and $\mathbb{C}$.

(1) Show that $V = \mathbb{C}[j]$, polynomials in j, and that $V(K) = K[j]$ for any subfield K of $\mathbb{C}$.

(2) If $F \in V$ show that F grows at most like q^{-N} for a suitable N as $\tau \to i\infty$ (i.e., $q \to 0$).

Recall that we choose $q = e^{2\pi i\tau}$ as uniformizer at infinity, but that we could choose more generally any power series in q of the form $a(1)q + a(2)q^2 + \cdots$ with $a(1) \neq 0$, so in particular $\Delta = q - 24q^2 + \cdots$, $j^{-1} = q - 744q^2 + \cdots$, etc...

Exercise 2.

(1) Show that $d\Delta/\Delta = E_2 dq/q$, that $dE_4/E_4 = ((E_2 - E_6/E_4)/3)dq/q$, and that $dj/j = -(E_6/E_4)dq/q$.

(2) Let F and G be in $V(\mathbb{R})$. Show that the following four expressions are equal:

   (a) The constant term of FG as a Laurent series in Δ.
   (b) The constant term of FGE_2E_4/E_6 as a Laurent series in j^{-1}.

(c) The constant term of FGE_2 as a Laurent series in q.

(d) The integral

$$\frac{6}{\pi}\int_{\pi/3}^{\pi/2} F(e^{i\theta})G(e^{i\theta})\,d\theta\ .$$

(3) Show that $j(e^{i\theta})$ is real for $0 < \theta < \pi$, and deduce from the last condition that the common value in (2) defines a positive definite scalar product on $V(\mathbb{R})$.

Definition 1.2 *For F, G in V we define the* Atkin scalar product $< F, G >$ *as the common value of the above expressions.*

2 Orthogonal Polynomials

The following is well-known, but perhaps needs a refresher.

Exercise 3. Let $W = \mathbb{R}[X]$, let ϕ be a linear form from W to $\mathbb{R}$, and assume that $< F, G >= \phi(FG)$ defines a positive definite Euclidean scalar product on W.

(1) Prove that there *exists* a *unique* sequence of *monic* polynomials $P_n \in W$ of degree exactly equal to n which are orthogonal with respect to the scalar product.

(2) We have $P_0 = 1$, $P_1 = X - < X, 1 > / < 1, 1 >= X - \phi(X)/\phi(1)$. After explaining why P_n is orthogonal to all polynomials of degree less than or equal to $n - 1$, prove that for $n \geq 1$ the polynomials P_n satisfy a second order linear recurrence

$$P_{n+1}(X) = (X - a_n)P_n(X) - b_nP_{n-1}(X)$$

for suitable constants a_n and b_n given by

$$a_n = \frac{< XP_n, P_n >}{< P_n, P_n >} \quad \text{and} \quad b_n = \frac{< P_n, P_n >}{< P_{n-1}, P_{n-1} >}\ .$$

Unfortunately, the above formulas are not very useful in practice to compute a_n and b_n. The following exercises give a usually much better method to compute them.

Exercise 4. Define the nth *moment* w_n by $w_n =< X^n, 1 >= \phi(X^n)$, and let $\Phi(T)$ be the formal power series in T^{-1} defined by

$$\Phi(T) = \sum_{n\geq 0} w_nT^{-n-1}\ .$$

(1) Show that $< F, G >$ is the coefficient of X^{-1} in the Laurent expansion of $F(X)G(X)\Phi(X)$.

(2) Note that $P_n(X)\Phi(X) = Q_n(X) + O(X^{-1})$ for some polynomial Q_n of degree exactly equal to $n-1$. Show that in fact

$$P_n(X)\Phi(X) = Q_n(X) + O(X^{-n-1})\ ,$$

in other words that the coefficients of X^{-m} in $P_n(X)\Phi(X)$ vanish for $1 \le m \le n$.

(3) Deduce that Q_n satisfies exactly the same recursion as P_n, and that $Q_0 = 0$ and $Q_1 = w_0 =< 1, 1 >$.

(4) Show that

$$\frac{Q_n(X)}{P_n(X)} = \Phi(X) + O(X^{-2n-1}) = \frac{w_0}{X} + \frac{w_1}{X^2} + \cdots + \frac{w_{2n-1}}{X^{2n}} + O(X^{-2n-1})\ .$$

Exercise 5. Let ϕ^* be the linear form on W deduced from ϕ defined by $\phi^*(X^{2n+1}) = 0$, $\phi^*(X^{2n}) = \phi(X^n) = w_n$, and extended by linearity (equivalently, if $F(X) + F(-X) = 2F^*(X^2)$ we define $\phi^*(F) = \phi(F^*)$), and let P_n^* be the corresponding orthogonal polynomials as in the previous exercise.

(1) Show that $P_n^*(-X) = (-1)^n P_n^*(X)$, and deduce that the recursion of the previous exercise is of the form

$$P_{n+1}^*(X) = XP_n^*(X) - c_n P_{n-1}^*(X)$$

with $c_n =< P_n^*, P_n^* > / < P_{n-1}^*, P_{n-1}^* >$ (with the new scalar product).

(2) Show that $P_{2n}^*(X) = P_n(X^2)$, and deduce that the coefficients a_n and b_n of the recursion for P_n are given in terms of the c_n by

$$a_n = c_{2n} + c_{2n+1} \quad \text{and} \quad b_n = c_{2n-1}c_{2n}\ .$$

(3) Denote by Q_n^* the corresponding Q-polynomials given by the previous exercise. Noting that the above recursions are exactly those for continued fractions, show that

$$\frac{Q_{n+1}^*(X)}{P_{n+1}^*(X)} = \cfrac{w_0}{X - \cfrac{c_1}{X - \cfrac{c_2}{X - \ddots - \cfrac{c_n}{X}}}} = \cfrac{w_0 X^{-1}}{1 - \cfrac{c_1 X^{-2}}{1 - \cfrac{c_2 X^{-2}}{1 - \ddots - c_n X^{-2}}}}\ .$$

(4) Show that $Q_{2n}^*(X) = XQ_n(X^2)$ and deduce from the previous exercise that we have the following formal identity giving a direct link between the moments w_n and the coefficients c_n:

$$\Phi(T^{-1}) = w_0 T + w_1 T^2 + w_2 T^3 + \cdots = \cfrac{w_0 T}{1 - \cfrac{c_1 T}{1 - \cfrac{c_2 T}{1 - \ddots}}}\ .$$

Together with the formulas obtained above expressing a_n and b_n in terms of the c_n, this allows to compute these quantities directly from the w_n.

3 The Atkin Polynomials

Exercise 6. We now restrict to the Atkin scalar product on $V(\mathbb{R})$. We denote by $A_n(X)$ the corresponding orthogonal polynomials.

(1) Let $Q(T)$ be the *reverse* of the series expressing $1/j$ in terms of q, so that $1/j(Q(T)) = T$. Keeping the notation of the previous exercises, show that

$$\Phi(1/T) = \sum_{n\geq 0} w_n T^{n+1} = TE_2(Q(T))E_4(Q(T))/E_6(Q(T)) ,$$

where by abuse of notation $E_k(q)$ is the Fourier expansion of E_k.

(2) After computing explicitly w_n for $0 \leq n \leq 6$ using this formula, compute $A_n(X)$ for $0 \leq n \leq 3$.

(3) Using a computer, compute the constants c_n of the previous exercise for $n \leq 20$, and then guess an (easy) conjectural formula for c_n. (Hints: first guess (a multiple of) the denominator of c_n, for instance by looking at c_{14}, then setting aside $c_1 = 720$ and separating n even and n odd guess a formula for the numerator.)

(4) Compute the constants a_n and b_n of the recursion for $n = 1$ and $n = 2$.

(5) Using the conjectural formula for c_n and the previous exercise, give a conjectural formula for a_n and b_n.

(6) Using this, compute $A_n(X)$ for $n \leq 6$.

The reader can refer to [2] for a *proof* of these formulas as well as many more important results. In particular, with a suitable definition of the Hecke operators one can show that they are self-adjoint with respect to the Atkin scalar product. Also, the Atkin polynomials themselves are closely linked to the *supersingular polynomials*, and this link was Atkin's initial motivation for introducing them.

Exercise 7. Using the previous exercises (and the fact that all the conjectural formulas are indeed true thanks to Kaneko–Zagier), do the following:

(1) Using the general formulas for orthogonal polynomials, find a recursion for $< A_n, A_n >$ and then compute it. It will be useful to use the standard *Pochhammer symbol*

$$(x)_n = x(x+1)\cdots(x+n-1) = \Gamma(x+n)/\Gamma(x) .$$

(2) Using the formula $P^*_{2n}(X) = P_n(X^2)$ proved above, find a recursion for $A_n(0)$, and deduce its value.

(3) Using the values of $A_n(X)$ that were computed above for $n \leq 6$, compute $A_n(1728)/A_{n-1}(1728)$ for $1 \leq n \leq 6$, guess a formula (very similar to the one for $A_n(0)/A_{n-1}(0)$), and then *prove* that your guess is correct using the recursion for $A_n(X)$.

References

[1] H. Cohen and F. Strömberg, *Modular Forms: A Classical Approach*, Graduate Studies in Math. **179**, American Math. Soc., (2017).

[2] M. Kaneko and D. Zagier, *Supersingular j-invariants, hypergeometric series, and Atkin's orthogonal polynomials*, in *Proceedings of the Conference on Computational Aspects of Number Theory,* AMS/IP Studies in Advanced Math. **7**, International Press, Cambridge (1997), pp. 97–126.

[3] D. Zagier, *Elliptic modular forms and their applications*, in *The 1-2-3 of modular forms*, Universitext, Springer (2008), pp. 1–103.