Modular Forms Project: L-Functions of Quadratic Characters at Negative Integers

Henri Cohen,
Université de Bordeaux, Institut de Mathématiques de Bordeaux,
351 Cours de la Libération, 33405 TALENCE Cedex, FRANCE

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1 Introduction

The present project is extracted from an unpublished preprint of the author [1], which will be on the author's web page at the end of July 2018. It is mostly experimental.

Let D be a fundamental discriminant, i.e., the discriminant of a quadratic number field (so either $D \equiv 1 \pmod 4$ and squarefree, or $D \equiv 8,\ 12 \pmod{16}$ and $D/4$ squarefree). To such a discriminant is associated a Dirichlet character χ_D of conductor $|D|$ such that $\chi_D(n) = \left(\frac{D}{n}\right)$, and a corresponding L-function $L(\chi_D, s) = \sum_{n\ge1} \left(\frac{D}{n}\right)/n^s$ for $\Re(s) > 1$, and extended by analytic continuation to the whole of $\mathbb{C}$ with functional equation $\Lambda(\chi_D, 1-s) = \Lambda(\chi_D, s)$ with

$$\Lambda(\chi_D, s) = |D|^{s/2}\Gamma_{\mathbb{R}}(s+e)L(\chi_D, s)\ ,$$

where as usual $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $e = 0$ or 1 according to $D > 0$ or $D < 0$. This functional equation implies that for $k \ge 2$ we have $L(\chi_D, 1-k) = 0$ if $k \not\equiv e \pmod 2$, and on the other hand it is easy to show that the values $L(\chi_D, 1-k)$ for $k \equiv e \pmod 2$ are *rational numbers* with completely controlled (small) denominators.

We are interested in computing these values, with emphasis on small values of k (say $2 \le k \le 12$) and large values of $|D|$. Since the denominators are controlled and small, computing exactly is equivalent to computing approximately to a reasonable accuracy.

There are (at least) four methods for computing these values.

(1) Direct method: they can be expressed as a finite sum of $O(|D|)$ values in terms of Bernoulli numbers.

(2) Using the functional equation directly: when k is not too small (e.g., $k \ge 12$), we can use an approximation to $L(\chi_D, k)$ and use the functional equation.

(3) Using the *approximate* functional equation. This is a method which takes time $O(|D|^{1/2+\varepsilon})$ for any $\varepsilon > 0$, but with a large implicit constant in the $O()$ constant because of the need to compute higher transcendental

functions. Nonetheless, apart from methods coming from modular forms, it is almost always the best method if $|D|$ is large.

(4) Using formulas coming from the theory of modular forms.

The aim of this project is to *discover* both theoretically and experimentally the formulas coming from modular forms, to estimate their *cost*, and to deduce a reasonable algorithm for computing $L(\chi_D, 1-k)$. Note that all the formulas take time $O(D^{1/2+\varepsilon})$ but with a *very small* implicit constant of the $O()$, and the project consists in finding the smallest possible constants.

2 Some Theory: Case k Even

2.1 The Basic Theorem

The basic theorem which lead to modular form methods for computing $L(\chi_D, 1-k)$ is due to C.-L. Siegel, and refined by D. Zagier and the author.

Theorem 2.1 *Let $D > 1$ be a fundamental discriminant and $k \geq 2$ be an even integer. Set*

$$F_{k,D}(\tau) = -\frac{B_k}{4k}L(\chi_D, 1-k) + \sum_{n\geq 1} c_{k,D}(n)q^n \quad \textit{with}$$

$$c_{k,D}(n) = \sum_{d\mid n} d^{k-1}\left(\frac{D}{d}\right) \sum_{\substack{|s|<(n/d)\sqrt{D} \\ s\equiv(n/d)D \pmod 2}} \sigma_{k-1}\left(\frac{(n/d)^2D - s^2}{4}\right) .$$

Then $F_{k,D} \in M_{2k}(\Gamma)$, i.e., it is a modular form of weight $2k$ on the full modular group.

The proof of this theorem is in fact relatively simple: in the space of *Hilbert modular forms* in two variables over the real quadratic field $\mathbb{Q}(\sqrt{D})$ one defines natural Eisenstein series, here called *Hecke–Eisenstein* series, one computes their Fourier expansion in a manner completely similar to the one variable case, and one then restricts to the diagonal $\tau_2 = \tau_1$.

Consider the example of $k = 2$. The theorem says that there exists a modular form in $M_4(\Gamma)$ whose Fourier expansion begins by

$$F_{2,D}(\tau) = -\frac{L(\chi_D, -1)}{48} + q \sum_{\substack{|s|<\sqrt{D} \\ s\equiv D \pmod 2}} \sigma_1\left(\frac{D - s^2}{4}\right) + O(q^2) .$$

However $M_4(\Gamma)$ has dimension 1 generated by $E_4 = 1 + 240q + O(q^2)$, so $F_{2,D}$ must be proportional to E_4, and we deduce that

$$L(\chi_D, -1) = -\frac{1}{5} \sum_{\substack{|s|<\sqrt{D} \\ s\equiv D \pmod 2}} \sigma_1\left(\frac{D - s^2}{4}\right) .$$

This is our first modular form formula for $L(\chi_D, 1-k)$.

Definition 2.2 (1) *To avoid including conditions such as* $|s| < \sqrt{D}$ *and* $s \equiv D \pmod 2$, *we* define $\sigma_k(x) = 0$ *for* $x \notin \mathbb{Z}_{\geq 1}$.

(2) *We define*

$$S_k(m, N) = \sum_{s \in \mathbb{Z}} \sigma_{k-1}\left(\frac{m - s^2}{N}\right)$$

(warning: because of the relation with modular forms, it is σ_{k-1} *and not* σ_k*).*

Thus, with this notation we have shown that $L(\chi_D, -1) = -S_2(D, 4)/5$.

Exercise 1. The goal of this exercise is to compute $S_2(m, 4) = \sum_{s \in \mathbb{Z}} \sigma_1((m - s^2)/4)$ for all positive integers m and not only for fundamental discriminants. Since $s^2 \equiv 0, 1 \pmod 4$, it is clear that $S_2(m, 4) = 0$ if $m \equiv 2, 3 \pmod 4$, so we assume that $m \equiv 0, 1 \pmod 4$.

(1) Using the above theorem, show that if $D > 1$ is a fundamental discriminant we have

$$\sum_{d \mid n} d\left(\frac{D}{d}\right) S_2((n/d)^2 D) = -5L(\chi_D, -1)\sigma_3(n) \ .$$

(2) Show that for any positive integer $m \equiv 0, 1 \pmod 4$ one can write uniquely $m = Df^2$ for a fundamental discriminant D (including $D = 1$) and some integer f. Using the Möbius inversion formula, compute $S_2(m)$ in terms of $L(\chi_D, -1)$ when m is not a square.

(3) We now assume that m is a square. Proving the correct formula would require too much work, so we will do this experimentally. Call $F(n)$ the difference between the right and left-hand side of the formula of (1) for $D = 1$, in other words since $L(\chi_1, -1) = \zeta(-1) = -1/12$:

$$F(n) = \frac{5}{12}\sigma_3(n) - \sum_{d \mid n} d S_2((n/d)^2) \ .$$

Compute $F(n)/\sigma_1(n)$ for a few consecutive values of n, and deduce a conjectural formula for $F(n)$.

(4) As before, using Möbius inversion deduce the (conjectural) value of $S_2(m)$ when m is a square, hence the general formula using the symbol $\delta(\sqrt{m})$ equal to 1 if m is a square and to 0 otherwise. In addition, using Möbius inversion but now in reverse, show that the factor of $\delta(\sqrt{m})$ is a linear function of m.

Note: probably the simplest way to *prove* the formula for m square is as follows. One first shows that there exists a constant c such that $E_2(4\tau)\theta(\tau) - c\theta'(\tau) \in M_{5/2}(\Gamma_0(4))$. Now this space is of dimension 2 generated by two Eisenstein series whose Fourier coefficients can be computed explicitly (with some difficulty), and identifying Fourier coefficients gives the above formulas including the one for m square, whose contribution comes from the presence of $\theta'(\tau)$ above.

Exercise 2.

(1) Using the explicit description of $M_{2k}(\Gamma)$, express $L(\chi_D, 1-k)$ for $k = 4$, 6, 8, and 10 in terms of $S_k(D, 4)$ and $S_k(D, 1)$.

(2) Using the fact that for $k \geq 4$ even E_k is an eigenfunction of the Hecke operators, show that

$$\sigma_{k-1}(4n) = (4^{k-1}+1)\sigma_{k-1}(n) - 4^{k-1}\sigma_{k-1}(n/4)\ .$$

(3) Show directly that the above relation is in fact true for all $k \in \mathbb{C}$.

(4) Deduce that once $S_k(D, 4)$ computed, one can speed up the computation of $S_k(D, 1)$ by writing

$$S_k(D,1) = \sum_{s \not\equiv D \pmod 4} \sigma_{k-1}(D-s^2) + (4^{k-1}+1)S_k(D,4) - 4^{k-1}S_k(D,16)\ .$$

2.2 The General Theorem

The more general theorem which will both allow us to obtain faster formulas in many cases, and also treat the case of $D < 0$ (hence k odd) is as follows (unpublished as far as I know):

Theorem 2.3 *Let $D > 1$ be a fundamental discriminant, let $k \geq 2$ be any integer, let ψ be a primitive character modulo F such that $\psi(-1) = (-1)^k$, let N be a squarefree integer, and assume that $\gcd(F, ND) = 1$. The parameters ψ and N being implicit, set*

$$c_{k,D}(0) = \frac{L(\psi, 1-k)L(\psi\chi_D, 1-k)}{4} \prod_{\substack{p \mid N \\ p\ prime}} (1 - \psi\chi_D(p)p^{k-1})$$

and for $n \geq 1$:

$$c_{k,D}(n) = \sum_{\substack{d \mid n \\ gcd(d,N)=1}} \psi\chi_D(d)d^{k-1} \sum_{s \in \mathbb{Z}} \sigma_{k-1,\psi}\left(\frac{(n/d)^2 D - s^2}{4N}\right)\ ,$$

where we set $\sigma_{k-1,\psi}(m) = 0$ if m is not a positive integer, and otherwise $\sigma_{k-1,\psi}(m) = \sum_{d \mid m} \psi(d)d^{k-1}$.

Then

$$\sum_{n \geq 0} c_{k,D}(n)q^n \in M_{2k}(\Gamma_0(FN), \psi^2)\ .$$

Note that Theorem 2.1 is the special case $F = N = 1$ (hence ψ trivial) of this theorem. Once again, since $M_{2k}(\Gamma_0(FN), \psi^2)$ is finite-dimensional, this allows us to obtain other formulas for L-functions at negative integers. However, this can become tedious, so it is preferable to use the following corollary:

Corollary 2.4 *For any $k \geq 2$* even *and $N \geq 1$ with $4 \mid N$, there exist t, and constants $(c_n)_{1 \leq n \leq t}$ depending only on $\left(\frac{D}{p}\right)$ for primes $p \mid N$ or $p \leq t$, such that whenever D is a square modulo N we have*

$$L(\chi_D, 1-k) = \sum_{1 \leq n \leq t} c_n S_k(n^2 D, N)\ .$$

For example, the formula that we have seen above says that $L(\chi_D, -1) = (-1/5)S_2(D, 4)$.

Exercise 3.

(1) Using either your own program for computing directly $L(\chi_D, 1-k)$ in terms of Bernoulli numbers, or the built-in GP function `lfun(D,1-k)`, *experiment* with the above corollary for $k = 2$ and $8 \le N \le 120$, $4 \mid N$ (the case $N = 4$ has been done), and find the smallest possible t which seems to give a valid formula (I do not ask for any proof, simply a reasonable confidence). To get you started, check (again experimentally) that for $N = 12$ we have two formulas: if $D \equiv 0 \pmod 3$ then $L(\chi_D, -1) = -2S_1(D, 12)$, and if $D \equiv 1 \pmod 3$ then $L(\chi_D, -1) = -S_1(D, 12)$.

(2) Write two programs which, given any one of the formulas that you find, gives the computational *cost* (as a rational number times $D^{1/2}$), and the probability that the formula applies. Warning: these probabilities are not what you may believe without thinking.

Examples: our initial formula for $L(\chi_D, -1)$ has cost $D^{1/2}/2$ and applies with probability 1; the formula $L(\chi_D, -1) = -2S_1(D, 12)$ for $D \equiv 0 \pmod 3$ has cost $D^{1/2}/6$ and applies with probability $1/4$; the formula $L(\chi_D, -1) = -S_1(D, 12)$ for $D \equiv 1 \pmod 3$ has cost $D^{1/2}/3$ and applies with probability $3/8$.

(3) By ordering the formulas by increasing cost, and for simplicity keeping only Legendre conditions involving $(D/2)$ and $(D/3)$, write an algorithm for computing $L(\chi_D, -1)$ valid for all D (our initial formula, with cost $D^{1/2}/2$, will be the most costly but it applies to all D), and estimate the average cost of your algorithm.

Note: doing all the computations of this exercise may take you very long. If you are pressed for time, I suggest that you do only part of the exercise (for instance by finding only a few more experimental formulas), and that you go directly to Exercise 5 which treats $L(\chi_D, 1-k)$ for $D < 0$ and k odd, the corresponding formulas being difficult (if not impossible) to find in the literature.

Exercise 4.

(1) Do the same exercise for $k = 4$, i.e., to compute $L(\chi_D, -3)$. Note: all the formulas that I found have a worse cost than the initial formula found in Exercise 2 (1), maybe you can do better.

(2) Same for $k = 6$ (test $4 \le N \le 60$, $4 \mid N$, here there are again several useful formulas).

(3) Same for $k = 8$ (only two useful formulas ?) and $k = 10$ (no other than the one obtained in Exercise 2 ?).

3 The Case k Odd

Thanks to the more general Theorem 2.3, although Hilbert modular forms in two variables are only for *real* quadratic fields, thus with discriminant $D > 0$, with a suitable choice of ψ (typically $\psi = \chi_{-4}$), it can also be used to compute $L(\chi_D, 1-k)$ for $D < 0$, hence k odd. In fact, we need a similar additional theorem which we do not give to obtain good formulas, and we state directly the analogue of Corollary 2.4, but first introduce some notation.

Definition 3.1 (1) *We set*

$$\sigma_k^{(1)}(m) = \sum_{d \mid m} \left(\frac{-4}{d}\right) d^k \quad \textit{and} \quad \sigma_k^{(2)}(m) = \sum_{d \mid m} \left(\frac{-4}{m/d}\right) d^k \ ,$$

with the usual understanding that $\sigma_k^{(j)}(m) = 0$ *if* $m \notin \mathbb{Z}_{\geq 1}$.

(2) *We set*

$$S_k^{(j)}(m, N) = \sum_{s \in \mathbb{Z}} \sigma_{k-1}^{(j)}\left(\frac{m - s^2}{N}\right) \ .$$

Corollary 3.2 *Assume that* $D < 0$ *is a fundamental discriminant. For any* $k \geq 3$ odd *and* $N \geq 1$*, there exist* r*,* t*, and constants* $(c_n^{(j)})_{1 \leq n \leq t}$ *for* $j = 1$*, 2 depending only on* $\left(\frac{D}{p}\right)$ *for primes* $p \leq r$ *such that whenever* D *is a square modulo* N *we have*

$$L(\chi_D, 1-k) = \sum_{1 \leq n \leq t} \sum_{1 \leq j \leq 2} c_n^{(j)} S_k^{(j)}(n^2 |D|, N) \ .$$

Exercise 5.

(1) Do the same as the previous exercises, but now for $D < 0$ and $k = 3$, 5, 7, and 9, and again compute the costs. For simplicity, only look for formulas such that the coefficients c_n depend only on $\left(\frac{D}{2}\right)$, i.e., simply distinguish the cases $D \equiv 1 \pmod 8$, $D \equiv 5 \pmod 8$, and $D \equiv 0 \pmod 4$, but now only for $1 \leq N \leq 4$. To get you started, check (experimentally) that for $k = 3$ and $D \equiv 1 \pmod 8$ we have $L(\chi_D, -2) = S_2^{(1)}(|D|, 2)/7$.

(2) Set $S_k^{(3)}(m) = \sum_{s \in \mathbb{Z}} \sigma_{k-1}^{(1)}(m - 4s^2)$. Show that $S_k^{(1)}(4m, 2) = S_k^{(1)}(m, 2) + S_k^{(3)}(m)$, and deduce that when $D \equiv 1 \pmod 4$ one can reduce the cost for $k = 7$ and 9.

(3) Similarly, set $S_k^{(4)}(m) = \sum_{s \in \mathbb{Z},\ s \text{ odd}} \sigma_{k-1}^{(1)}(4m - s^2)$. Show that $S_k^{(1)}(4m, 1) = S_k^{(1)}(m, 1) + S_k^{(4)}(m)$, and deduce that when $D \equiv 0 \pmod 4$ one can reduce the cost for $k = 9$.

Exercise ∞ (for courageous people). The formulas with $D < 0$ were all obtained from Theorem 2.3 using $\psi = \chi_{-4}$. Find analogous formulas using $\psi = \chi_{-3}$ or more general quadratic characters. Have any of these formulas smaller "cost" than those that you obtained in Exercise 4 ?

References

[1] H. Cohen, *Computing L-functions of Dirichlet Characters at Negative Integers*, preprint, 27p.

[2] H. Cohen and F. Strömberg, *Modular Forms: A Classical Approach*, Graduate Studies in Math. **179**, American Math. Soc., (2017).