Modular Forms Project: Eta Products

Henri Cohen,
Université de Bordeaux, Institut de Mathématiques de Bordeaux,
351 Cours de la Libération, 33405 TALENCE Cedex, FRANCE

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1 Introduction

This project is both an exercise on eta products, but also a computational introduction to the main theorems on modular forms of half-integral weight, viz., Shimura's theorem on lifts, and Waldspurger's theorem.

Recall that the *Dedekind eta function* is defined by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$$

with $q = e^{2\pi i \tau}$ (where by convention $q^{a/b} = e^{2\pi i (a/b)\tau}$). It is a modular form of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ with a complicated multiplier system formed of 24th roots of unity. Its 24th power is the discriminant function $\Delta(\tau)$.

An *eta product* is any function of the form $F(\tau) = \prod_{m \in I} \eta(m\tau)^{r_m}$, where I is a finite set of positive integers, and $r_m > 0$ for all $m \in I$ (if we allowed $r_m < 0$ this would be called an eta *quotient*, but we will not consider those here except in the very last exercise). It is immediate to see that $\eta(m\tau)$ is modular on $\Gamma_0(m)$ (with multiplier system), hence F is modular on $\Gamma_0(M)$, where M is the lowest common multiple of $m \in I$.

If we want F to belong to some $M_k(\Gamma_0(N), \chi)$ for some Dirichlet character χ and level N (and possibly half-integral weight k), it is evidently necessary that $F(\tau + 1) = F(\tau)$, which is clearly equivalent to $\sum_{m \in I} m r_m \equiv 0 \pmod{24}$. It can be shown that this necessary condition is in fact sufficient, and in addition that the character χ will have order 1 or 2 (i.e., be a real character), and that F will always be a cusp form (if I is nonempty, which we always assume). More precisely, we have the following theorem:

Theorem 1.1 *Let $F = \prod_{m \in I} \eta(m\tau)^{r_m}$ be an eta product, so that F has weight $k = \sum_{m \in I} r_m/2$, and denote by M the LCM of the elements of I. Then F belongs to some $M_k(\Gamma_0(N), \chi)$ if and only if $\sum_{m \in I} m r_m \equiv 0 \pmod{24}$, and if this is the case F will be a cusp form. One can choose for N the LCM of M and the denominator of $\sum_{m \in I} r_m/(24m)$, and $\chi = \chi_D$ (quadratic character of discriminant D) with $D = (-1)^k \prod_{m \in I} m^{r_m}$ if $k \in \mathbb{Z}$, or $D = 8 \prod_{m \in I} m^{r_m}$ if $k \in 1/2 + \mathbb{Z}$.*

Our first goal will be to find eta products which are (necessarily normalized) Hecke eigenforms, so in particular with $a(1) = 1$. We thus set the following:

Definition 1.2 *We will denote by $\mathcal{E}$ the set of eta products having a Fourier expansion $F(\tau) = \sum_{n\geq 1} a(n)q^n$ with $a(1) = 1$.*

It is clear that $F \in \mathcal{E}$ if and only if $\sum_{m\in I} mr_m = 24$, so by what we said above, F will be a cusp form of some level N and real character χ. Also, it is clear from this equality and the condition $r_m > 0$ that $m \leq 24$ and $r_m \leq 24$ so that $\mathcal{E}$ is a finite set.

2 Search for Hecke Eigenforms

Exercise 1.

(1) Show that there is a canonical bijection between the set of *partitions* of 24 and the set $\mathcal{E}$.

(2) Deduce the cardinality of $\mathcal{E}$.

In `Pari/GP` there are two built-in functions for handling partitions: one is simply `partitions(n)` which gives the vector of all partitions as a `Vecsmall` (for instance `Vecsmall([1,1,10,12])`), and the other is the *iterator* `forpart` which ranges through all partitions in a predefined ordering, and avoids storing them all.

Exercise 2.

(1) For each weight k integral or half-integral, compute the number of elements of $\mathcal{E}$ of weight k.

(2) We would like to find all eta products of integral weight which are Hecke *eigenforms*. It is clear that a necessary condition is that $a(mn) = a(m)a(n)$ when $\gcd(m, n) = 1$.

Find all 28 eta products of integral weight satisfying this condition for $mn \leq 100$, and write the list explicitly for each weight, together with their level N and character χ using Theorem 1.1. It can be shown (this is not trivial) that they are all indeed Hecke eigenforms.

Note that the eight forms in weight 2 have trivial character χ, hence all correspond to isogeny classes of elliptic curves defined over $\mathbb{Q}$, with the same level.

Exercise 3.

(1) We would now like to find all eta products of *half-integral* weight which are Hecke eigenforms. The definition is more subtle, but you only need to know that a necessary condition is that for any squarefree t such that $a(t) \neq 0$ we have $a(tm^2n^2)a(t) = a(tm^2)a(tn^2)$ when $\gcd(m, n) = 1$, and for any squarefree t such that $a(t) = 0$ we have $a(tn^2) = 0$ for all n. Find all 45 eta products of half-integral weight satisfying these conditions for $tm^2n^2 \leq 1000$, say.

(2) Once again, using Theorem 1.1, find their levels and characters.

Note: if you are lazy and program in `Pari/GP`, here are the commands which, given a `Vecsmall V` corresponding to an eta quotient, outputs the level, weight, and character:

```
? M=concat(Mat(Col(V)),Mat(vectorv(#V,j,1)));
? mfparams(mffrometaquo(M))[1..3]
```

3 Numerical Verification of the Theorems of Shimura and Waldspurger

There is an important theorem due to G. Shimura which states in our special case the following (you do not need the definition of $T(p^2)$, only the multiplicative property given in the above exercise).

Theorem 3.1 *Let $F = \sum_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N), \chi)$, where $k \in 1/2 + \mathbb{Z}$ (hence necessarily $4 \mid N$ and $\chi(-1) = 1$), and assume that F is an eigenform for the Hecke operators $T(p^2)$ for all $p \nmid N$ with eigenvalue λ_p. For any fundamental discriminant D such that $(-1)^{k-1/2}D > 0$ and $a(|D|) \neq 0$ set $S_D(F) = \sum_{n\geq 1} A_D(n)q^n$ (the* Shimura lift *of F at D) where*

$$A_D(n) = \sum_{d\mid n} \left(\frac{4D}{d}\right)\chi(d)d^{k-3/2}a((n/d)^2|D|)\ .$$

Then $S_D(F) \in M_{2k-1}(\Gamma_0(N/2), \chi^2)$, and it is an eigenform for the Hecke operators $T(p)$ for all $p \nmid N$ with the same eigenvalue λ_p.

In addition, if suitable conditions are satisfied (F should be a newform in the "Kohnen $+$-space") then all the $S_D(F)$ are proportional, and if $k \geq 5/2$ (and also often if $k = 3/2$) we have in fact $S_D(F) \in S_{2k-1}(\Gamma_0(N/4), \chi^2)$ (i.e., it is a cusp form and its level divides $N/4$).

Note that in this theorem you may replace the assumption that D is a fundamental discriminant by $D = (-1)^{k-1/2}t$ with t squarefree (which is the same if $D \equiv 1 \pmod 4$, and replaces D by $D/4$ if $D \equiv 0 \pmod 4$), but then the additional results for the Kohnen $+$-space will not be valid if $(-1)^{k-1/2}t \equiv 2, 3 \pmod 4$.

Exercise 4. For the 45 eta products F of half-integral weight k found in the previous exercise which seem to be Hecke eigenforms, do the following.

(1) Compute the beginning of the Fourier expansion of their Shimura lift for fundamental discriminants $D \leq 50$ (say) which are both *coprime to the level* and such that $(-1)^{k-1/2}D > 0$, and check if they are all proportional; store the corresponding indices in the vector of 45 eta products (you should find 28 such eta products). Note that if you are using `Pari/GP`, the Shimura lift is already preprogrammed (as `mfshimura`), so you can cheat if you want, or simply program the (easy) formula yourself.

(2) In our context (this is not the general definition), we will say that $F = \sum_{n\geq 1} a(n)q^n$ is in the Kohnen $+$-space if $n \equiv 2, 3 \pmod 4$ implies $a(n) = 0$ (in fact in our case we will also have $a(n) = 0$ if $n \equiv 0 \pmod 4$, but this is not part of the definition). Show that there are exactly five among the 45 that are in the Kohnen $+$-space, and give their indices.

The next exercise will illustrate the theorem of Waldspurger. Note that if $F = \sum_n a(n)q^n$ is a modular form of half-integral weight, Shimura's theorem gives precise links between $a(|D|)$ and $a(|D|n^2)$ for D a fundamental discriminant of suitable sign (more generally between $a(t)$ and $a(tn^2)$ for t squarefree). But it does not link the $a(|D|)$ (or the $a(|t|)$) together. This link is given by Waldspurger's theorem, which I will state very roughly and only in a special case, but sufficiently so that it may be verified on a computer.

Recall that if $L(G,s) = \sum_{n\geq 1} A(n)/n^s$ is an L-function, the *twist* of L by a Dirichlet character χ is the function $L(G \otimes \chi, s) = \sum_{n\geq 1} \chi(n)A(n)/n^s$.

Theorem 3.2 *Let $F = \sum_{n\geq 1} a(n)q^n \in S_k(\Gamma_0(N))$ be a modular form of half-integral weight (with no character) which is an eigenform for the Hecke operators $T(p^2)$. Denote by G one of the nonzero Shimura lifts of F to weight $2k-1$, and let $L(G,s)$ be the corresponding L-function. There exists a positive constant c (depending on F but not on D) such that for all fundamental discriminants D coprime to N (in particular $D \equiv 1 \pmod 4$) and such that $(-1)^{k-1/2}D > 0$, we have*

$$|D|^{k-1}L(G \otimes \chi_D, k-1/2) = c \cdot a(|D|)^2\ .$$

Exercise 5. For the 28 eta products F that you found in Exercise 4 whose Shimura lifts are proportional to some fixed G, check to see if this theorem applies as follows. For the fundamental discriminants $|D|$ of suitable sign and coprime to the level used in Exercise 4 and such that $a(|D|) \neq 0$ compute $D^{k-1}L(G \otimes \chi_D, k-1/2)/a(|D|)^2$, and check whether it is (approximately) constant. If you program in `Pari/GP` you may want to use the functions `[mf2,G]=mfshimura(F)` as above, then `LG=lfunmf(mf2,G)` which creates the corresponding L-function, then `L=lfuntwist(LG,D)` to twist, and finally `lfun(L,k-1/2)` to evaluate at $k-1/2$.

You should find six eta products among the 28 for which this is the case.

Exercise 6. If you (and I) performed the computations correctly, you will have found a single eta product of half-integral weight which (experimentally) satisfies the four conditions of the previous exercise: it is multiplicative (in the above sense), its Shimura lifts are proportional, it belongs to the Kohnen +-space, and the Waldspurger quotients are constant. This is $F(\tau) = \eta(4\tau)^4\eta(8\tau)$, and it belongs to $S_{5/2}(\Gamma_0(64))$.

(1) To visualize it better, compute the its Fourier expansion up to q^{30}.

(2) Compute the terms of its Shimura lift G for $D = 1$ (they are all proportional) up to q^{20}, which will be a generator of the 1-dimensional space $S_4^{\mathrm{new}}(\Gamma_0(16))$ by a theorem of Kohnen.

(3) Compute the "Waldspurger constant" c given in Exercise 5.

(4) (You will have to use `Pari/GP` to answer the present question, even if you did not for the previous ones). Show that (numerically) we have

$$c = \pi^2 \frac{< G, G >}{< F, F >}\ ,$$

which is a theorem of Kohnen–Zagier. To compute the Petersson scalar products, use `FS=mfsymbol(mfinit(F,1),F)` followed by `mfpetersson(FS,FS)`, and similarly for G.

Exercise 7. The aim of this exercise is to recognize the modular form $G \in S_4^{\mathrm{new}}(\Gamma_0(16))$ found in the previous exercise, and to show that it is in fact an *eta quotient* (same as an eta product except that we allow $r_m < 0$).

(1) Since we are in level 16, if G is an eta quotient it must be of the form $\prod_{m|16} \eta(m\tau)^{r_m}$. By identifying successively the Fourier coefficients of degree 2, 4, 8, and 16, express G as an (experimental) eta quotient.

(2) Using the notion of *Sturm bound*, *prove* that G is indeed that eta quotient.

References

[1] H. Cohen and F. Strömberg, *Modular Forms: A Classical Approach*, Graduate Studies in Math. **179**, American Math. Soc., (2017).

[2] D. Zagier, *Elliptic modular forms and their applications*, in *The 1-2-3 of modular forms*, Universitext, Springer (2008), pp. 1–103.