Modular Forms Project: Atkin's Orthogonal Polynomials

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1 Answers and Programs

Exercise 1.

(1) and (2). Since the set of invariant meromorphic functions on $\mathfrak{H}$ is $\mathbb{C}(j)$, if $F \in V$ we can write $F = P(j)/Q(j)$ for some coprime polynomials P and Q. If Q is nonconstant, it has a root $z \in \mathbb{C}$, and since j induces an isomorphism between $\mathfrak{H}$ and $\mathbb{C}$ there exists $\tau \in \mathfrak{H}$ with $j(\tau) = z$. Since P and Q are coprime, z cannot be a root of P, hence τ is a pole of F, contradicting the assumption that F is holomorphic. It follows that Q is constant, in other words that $F \in \mathbb{C}[j]$, and of course conversely such an F does not have any pole in $\mathfrak{H}$ since j does not, so $F \in V$, hence $V = \mathbb{C}[j]$ as claimed. In particular, if N is the degree of P then F grows at most like q^{-N} as $\tau \to \infty$ since $j(\tau) = 1/q + O(1)$. Furthermore, since the Fourier coefficients of j are in $\mathbb{Q}$, so are the coefficients of the inverse function of $1/j = q + O(q^2)$, hence so are those of R such that $F = j^N R(1/j)$, hence those of P, the reciprocal polynomial of R, such that $F = P(j)$.

Exercise 2.

(1). We have $\log(\Delta) = \log(q) + 24\sum_{n\ge1}\log(1-q^n)$, hence

$$d\Delta/\Delta = dq/q(1 - 24\sum_{n\ge1} nq^n/(1-q^n)) = E_2 dq/q\ .$$

Furthermore we know that

$$(1/2\pi i)dE_4/d\tau - E_2E_4/3 = q dE_4/dq - E_2E_4/3$$

is a modular form of weight 6, hence proportional to E_6, hence by looking at the constant term equal to $-E_6/3$, we have $dE_4/E_4 = ((E_2 - E_6/E_4)/3)dq/q$. Furthermore $\log(j) = 3\log(E_4) - \log(\Delta)$, so

$$dj/j = 3dE_4/E_4 - d\Delta/\Delta = (E_2 - E_6/E_4)dq/q - E_2 dq/q = -(E_6/E_4)dq/q\ .$$

(2). Set $H = FG \in V$. If ϕ is any function on $\mathfrak{H}$ having a Fourier expansion of the type $\phi(\tau) = q + \sum_{n\ge2} a(n)q^n$ (such as Δ, $1/j$, or q itself), by the residue theorem the constant term of H as a Laurent series in ϕ is given by

$$\frac{1}{2\pi i}\int_{C_r} H(\tau)\frac{d\phi}{\phi}(\tau)$$

where C_r is any closed circle of radius $r < 1$ around $q = 0$, so the equivalence of the first three conditions follows from (1) which implies that

$$\frac{d\Delta}{\Delta} = E_2 \frac{dq}{q} = \frac{E_2 E_4}{E_6} \frac{d(j^{-1})}{j^{-1}} \ .$$

For the equivalence with the fourth, we choose the third formula which tells us (since $dq/q = 2\pi i d\tau$) that

$$< F, G >= \int_{C_r} F(\tau)G(\tau)E_2(\tau)d\tau \ .$$

Now $\tau \in C_r$ is equivalent to $|q| = r$, i.e., to $\Im(\tau) = T$ for some $T = -\log(r)/(2\pi i)$, so

$$< F, G >= \int_{-1/2}^{1/2} (FGE_2)(x + iT)\, dx \ .$$

Let us choose $T > 1$. Since F, G and E_2 are holomorphic, the integral of FGE_2 around the truncated fundamental domain $|\tau| \geq 1$, $|x| \leq 1/2$, $y \leq T$ is equal to 0, and since the three functions are periodic of period 1, the integrals on the vertical sides are equal, which shows that $< F, G >= \int_B (FGE_2)(\tau)d\tau$, where B is the bottom side of the fundamental domain, i.e., the arc $|\tau| = 1$, $|x| \leq 1/2$. If we set $\rho = e^{2\pi i/3}$, we thus have

$$< F, G >= \int_\rho^i (FGE_2)(\tau)d\tau + \int_i^{-1/\rho} (FGE_2)(\tau)d\tau \ .$$

In the first integral we set $\tau' = -1/\tau$. Since F and G are invariant by Γ and $E_2(-1/\tau) = \tau^2 E_2(\tau) + (12/(2\pi i))\tau$, we deduce that

$$\int_\rho^i (FGE_2)(\tau)d\tau = -\int_i^{-1/\rho} (FG)(\tau)(E_2(\tau) + (12/2\pi i)/\tau)d\tau \ ,$$

hence

$$< F, G >= -(12/2\pi i) \int_i^{-1/\rho} FG(\tau)d\tau/\tau \ ,$$

so setting $\tau = e^{i\theta}$ we obtain

$$< F, G >= -(6/\pi) \int_{\pi/2}^{\pi/3} (FG)(e^{i\theta})d\theta \ ,$$

proving (d), and it is clear that the reasoning can be inverted to show that (d) implies (c).

(3). (a). Write

$$j(\tau) = \sum_{n \geq -1} c(n)q^n = \sum_{n \geq -1} c(n)e^{2\pi i n\tau} \ .$$

Since the Fourier coefficients $c(n)$ are real, we have

$$\overline{j(\tau)} = \sum_{n \geq -1} c(n)e^{-2\pi i n\overline{\tau}} = j(-\overline{\tau}) \ .$$

In particular, if $|\tau| = 1$ (and $\tau \in \mathfrak{H}$) we have $\overline{\tau} = 1/\tau$, so we deduce that $\overline{j(\tau)} = j(-1/\tau) = j(\tau)$, so $j(\tau)$ is indeed real when $\tau = e^{i\theta}$, $0 < \theta < \pi$.

(3). (b). It follows that if $F \in V(\mathbb{R})$ is a polynomial in j with real coefficients we also have that $F(e^{i\theta})$ is real for $0 < \theta < \pi$, hence $F^2(e^{i\theta}) \geq 0$, and if F is not identically zero the integral from $\pi/3$ to $\pi/2$ of $F^2(e^{i\theta})$ will be strictly positive, so defines a positive definite scalar product.

Exercise 3.

(1). We must have $P_0 = 1$. Assume that the P_k exist (and are unique) for $k < n$. Since they are monic of degree k they form a basis of the space of polynomials of degree $< n$. It follows that there exist constants $\mu_{n,k}$ such that $P_n = X^n + \sum_{0\leq k\leq n-1} \mu_{n,k} P_k$, and since P_n must be orthogonal to all the P_k and the P_k are pairwise orthogonal, we deduce that $0 =< P_n, P_k >=< X^n, P_k > + \mu_{n,k} < P_k, P_k >$, so $\mu_{n,k} = - < X^n, P_k > / < P_k, P_k >$ exists and is unique.

(2). By (1) we indeed have $P_1 = X- < X, 1 > / < 1, 1 >$. As already mentioned, the P_k for $k < n$ form a basis of the space of polynomials of degree $< n$, so P_n is indeed orthogonal to all such polynomials.

Now consider the polynomial $P_{n+1}(X) - XP_n(X)$. Since the P_k are monic, this is a polynomial of degree at most n, so we can write $P_{n+1}(X) - XP_n(X) = \sum_{0\leq k\leq n} \lambda_k P_k$, and since P_{n+1} is orthogonal to all the P_k for $k \leq n$, we have $\lambda_k = - < XP_n, P_k > / < P_k, P_k >$. However, since the scalar product is of the form $< F, G >= \phi(FG)$, we have $< XP_n, P_k >=< P_n, XP_k >$, and since XP_k has degree $k + 1$ it follows that $\lambda_k = 0$ when $k \leq n - 2$, and

$$\lambda_{n-1} = - < P_n, XP_{n-1} > / < P_{n-1}, P_{n-1} >= - < P_n, P_n > / < P_{n-1}, P_{n-1} >$$

since $P_n - XP_{n-1}$ has degree strictly less than n so is orthogonal to P_n, and $\lambda_n = - < XP_n, P_n > / < P_n, P_n >$, giving the desired recurrence.

Exercise 4.

(1). Since the coefficient of X^{-1} in $F(X)G(X)\Phi(X)$ is bilinear in F and G, it is sufficient to prove this for $F(X) = X^m$ and $G(X) = X^n$, but in that case $< X^m, X^n >= \phi(X^{m+n})$ while the coefficient of X^{-1} in $X^{m+n}\Phi(X)$ is also $w_{m+n} = \phi(X^{m+n})$.

(2). Since $\Phi(X) = w_0/X + O(X^{-2})$, it is clear that $P_n(X)\Phi(X) = Q_n(X) + O(X^{-1})$, where $Q_n(X)$ is the polynomial part of the product, thus of degree exactly equal to one less than P_n, i.e., $n - 1$.

Write

$$P_n(X)\Phi(X) = Q_n(X) + \sum_{m\geq 0} a(m)X^{-m-1} .$$

For any $m < n$ we know that P_n is orthogonal to X^m, so by (1) the coefficient of X^{-1} in $X^m P_n(X)\Phi(X)$ vanishes. Since this coefficient is $a(m)$, we thus have $a(m) = 0$ for $m < n$, hence $P_n(X)\Phi(X) = Q_n(X) + O(X^{-n-1})$ as claimed.

(3). Multiplying the recursion for P_n by $\Phi(X)$, we deduce that $Q_{n+1}(X) = (X - a_n)Q_n(X) - b_n Q_{n-1}(X) + O(X^{-n})$, and since the $Q_k(X)$ are polynomials, this implies that the $O(X^{-n})$ is in fact equal to 0, so the Q_n satisfy the same recursion. In addition, since $P_0 = 1$ we have $Q_0 = 0$, and since $P_1 = X - a$ for some constant a and $\Phi(X) = w_0/X + O(1/X^2)$ we have $Q_1 = w_0$.

(4). Dividing $P_n(X)\Phi(X) = Q_n(X) + O(X^{-n-1})$ proved above by $P_n(X)$ and using the fact that P_n has degree exactly n we deduce that

$$P_n(X)/Q_n(X) = \Phi(X)+O(X^{-2n-1}) = w_0/X+\cdots+w_{2n-1}/X^{2n}+O(X^{-2n-1}) .$$

Exercise 5.

(1). Assume by induction that for $k < n$ we have $P_k^*(-X) = (-1)^k P_k^*(X)$. Since P_n^* has degree n the polynomial $(-1)^n P_n^*(-X)$ is monic. In addition by our induction hypothesis, if $k < n$ we have

$$< (-1)^n P_n^*(-X), P_k^*(X) >=< (-1)^{n+k} < P_n^*(-X), P_k^*(-X) >= (-1)^{n+k}\phi^*(F(-X)) ,$$

where $F(X) = P_n^*(X)P_k^*(X)$. By definition $\phi^*(G) = \phi(G^*)$, but since $F(-X)+F(X) = F(X)+F(-X)$ we deduce that $\phi^*(F(-X)) = \phi^*(F(X))$, so $(-1)^n P_n^*(-X)$ is orthogonal to all the $P_k^*(X)$ for $k < n$ and is monic, so by uniqueness must be equal to $P_n^*(X)$.

The recursion of Exercise 3 is of the form $P_{n+1}^*(X) = (X - d_n)P_n^*(X) - c_n P_{n-1}^*(X)$. In this recursion, we change X into $-X$ and multiply by $(-1)^{n+1}$. Using what we have just proved shows that $P_{n+1}^*(X) = (X + d_n)P_n^*(X) - c_n P_{n-1}^*(X)$, so that $d_n = 0$ as claimed, and as in Exercise 3 we have $c_n =< P_n^*, P_n^* > / < P_{n-1}^*, P_{n-1}^* >$.

(2). (a). Once again, assume by induction that for $k < n$ we have $P_{2k}^*(X) = P_k(X^2)$. By definition of ϕ^* any polynomial in X^2 is orthogonal to any odd power of X. On the other hand, let X^{2k} be any even power of X with $k < n$. By parity proved in (1), it is a linear combination of the $P_{2j}^*(X)$ for $j \le k$ (the $P_{2j+1}^*(X)$ cannot occur), so to prove that $P_n(X^2)$ is orthogonal to all the X^{2k} it is sufficient to prove that it is orthogonal to all the $P_{2j}^*(X)$. However, by our induction hypothesis we have $P_{2j}^*(X) = P_j(X^2)$, and $\phi^*(P_n(X^2)P_j(X^2)) = \phi(P_n(X)P_j(X)) = 0$ by assumption.

(2). (b). We have $P_{2n+1}^*(X) = XP_{2n}^*(X) - c_{2n}P_{2n-1}^*(X)$ and $P_{2n+2}^*(X) = XP_{2n+1}^*(X) - c_{2n+1}P_{2n}^*(X)$, so

$$P_{2n+2}^*(X) = (X^2 - c_{2n+1})P_{2n}^*(X) - c_{2n}XP_{2n-1}^*(X) ,$$

and $P_{2n}^*(X) = XP_{2n-1}^*(X) - c_{2n-1}P_{2n-2}^*(X)$, so

$$\begin{aligned} P_{2n+2}^*(X) &= (X^2 - c_{2n+1})P_{2n}^*(X) - c_{2n}(P_{2n}^*(X) + c_{2n-1}P_{2n-2}^*(X)) \\ &= (X^2 - (c_{2n} + c_{2n+1}))P_{2n}^*(X) - c_{2n-1}c_{2n}P_{2n-2}^*(X) . \end{aligned}$$

Using $P_{2n}^*(X) = P_n(X^2)$ and identifying with the recursion for P_n gives $a_n = c_{2n} + c_{2n+1}$ and $b_n = c_{2n-1}c_{2n}$.

(3). If we denote by u_n/v_n the nth convergent of the continued fraction $w_0/(X - c_1/(X - c_2/(X - \cdots - c_n/X)))$ we have $u_0 = 0$, $v_0 = 1$, $u_1 = w_0$, $v_1 = X$, and the recursion $u_{n+1} = Xu_n - c_n u_{n-1}$ and similarly for v. These are exactly the recursions satisfied by Since P_n^* and Q_n^* with the same initial conditions since $P_1^*(X) = X-\phi^*(X)/\phi^*(1) = X$, so that $u_n = Q_n^*$ and v_n^*, giving the first continued fraction, and the second is trivially obtained by dividing by X each fraction.

(4). With evident notation, by definition we have $w^*_{2n} = w_n$, $w^*_{2n+1} = 0$, hence

$$\Phi^*(T) = \sum_{n\ge0} w^*_n T^{-n-1} = \sum_{n\ge0} w_n T^{-2n-1} = T\Phi(T^2)\ .$$

Since by definition $P^*_{2n}(X)\Phi^*(X) = Q^*_{2n}(X) + O(X^{-1})$, and we proved that $P^*_{2n}(X) = P_n(X^2)$, we have $P_n(X^2)X\Phi(X^2) = Q^*_{2n}(X) + O(X^{-1})$, but on the other hand $P_n(X^2)\Phi(X^2) = Q_n(X^2) + O(X^{-2})$ so we deduce that $Q^*_{2n}(X) = XQ_n(X^2)$.

Replacing n by $2n-1$ in the second continued fraction obtained above we thus obtain

$$XQ_n(X^2)/P_n(X^2) = w_0X^{-1}/(1 - c_1X^{-2}/(1 - c_2X^{-2}/(1 - \cdots c_nX^{-2})))\ ,$$

so multiplying by X^{-1} and replacing X^2 by T^{-1} gives

$$Q_n(T^{-1})/P_n(T^{-1}) = w_0T/(1 - c_1T/(1 - c_2T/(1 - \cdots c_nT)))\ ,$$

and since by the previous exercise we know that $Q_n(T^{-1})/P_n(T^{-1}) = \Phi(T^{-1}) + O(T^{2n-1})$, we obtain the required identity

$$w_0T + w_1T^2 + w_2T^3 + \cdots = w_0T/(1 - c_1T/(1 - c_2T/(1 - c_3T/(1 - \cdots))))\ .$$

Note: One can of course directly compute the c_i by successively inverting a power series and subtracting 1, so knowing the moments w_k it is immediate to compute the c_j, hence by (2) the a_n and b_n. However, there is a very simple, useful and faster algorithm called the *quotient-difference algorithm* which converts a power series into a continued fraction of the above type. Independently of the Atkin scalar product setting, this is usually by far the fastest and simplest way to obtain the recursions for general orthogonal polynomials.

Exercise 6.

(1). By condition (b) of Exercise 2, $w_n =< X^n, 1 >$ is equal to the constant term of $j^nE_2E_4/E_6$ as a Laurent series in j^{-1}, in other words to the coefficient of j^{-n-1} in $E_2E_4/(jE_6)$. Thus, with the notation of Exercise 4 we have

$$\Phi(j) = \sum_{n\ge0} w_n j^{-n-1} = \frac{E_2E_4}{jE_6}\ .$$

Now we can easily find the q-expansion of $1/j$, and by reverting this expansion we find the $1/j$-expansion of q (in `Pari/GP` the command is

`Q=serreverse(1/ellj(x+O(x^N)))`

where N is the number of terms that we want), and we can then replace in the known q-expansions of E_2, E_4, and E_6.

(1) For instance, with $N = 21$ (which is what we will need below) we can write:

```
Q=serreverse(1/ellj(x+O(x^22)))
E2=1-24*sum(n=1,16,n*x^n/(1-x^n),O(x^22));
E4=1+240*sum(n=1,16,n^3*x^n/(1-x^n),O(x^22));
E6=1-504*sum(n=1,16,n^5*x^n/(1-x^n),O(x^22));
Phi=x*subst(E2*E4/E6,x,Q)
```

This outputs a power series in $x = 1/j$ starting with

```
x + 720*x^2 + 911520*x^3 + 1301011200*x^4 + 1958042030400*x^5 +...
```

and w_n is the coefficient of x^{n+1}.

It is clear that $A_0(X) = 1$ and $A_1(X) = X - w_1/w_0 = X - 720$. If we write $A_2(X) = X^2 + aX + b$, we have $< A_2, 1 >= w_2 + aw_1 + bw_0$ and $< A_2, X >= w_3 + aw_2 + bw_1$, so using the above expansion of Φ and solving for a and b gives $A_2(X) = X^2 - 1640X + 269280$, and similarly writing $A_3(X) = X^3 + cX^2 + dX + e$ and solving in the same manner for c, d, and e gives $A_3(X) = X^3 - (12576/5)X^2 + 1526958X - 107765856$.

(3). Although we have mentioned above that there is a very efficient quotient-difference algorithm to compute the c_n, we will do it naively since we only need 20 values. We can write the following program, where the input is the power series `Phi` in the variable `x` computed above:

```
C=vector(20); S=x/Phi; /* S=1-c_1x/(1-c_2x/(1-...)) */
for(n=1,20,C[n]=-polcoeff(S,1); S=C[n]*x/(1-S)); C
```

This little program outputs

[720, 546, 374, 475, 2001/5, 2294/5, 410, 903/2, 2491/6, 1342/3, 4602/11, 4891/11, 5467/13, 40290/91, 14774/35, 8827/20, 28785/68, 22454/51, 24182/57, 8349/19]

The denominator of $C[14]$ is $91 = 7 \cdot 13$, so maybe we can see better by multiplying $C[n]$ by $n(n-1)$: the vector of $n(n-1)C[n]$ is equal to

[0, 1092, 2244, 5700, 8004, 13764, 17220, 25284, 29892, 40260, 46020, 58692, 65604, 80580, 88644, 105924, 115140, 134724, 145092, 166980]

At least the denominators have disappeared. Following the advice of the exercise we first consider the even terms:

[1092, 5700, 13764, 25284, 40260, 58692, 80580, 105924, 134724, 166980]

It is easy to spot that this is a quadratic polynomial (in `Pari/GP` simply use `polinterpolate`), and we find that it is the polynomial $1728x^2 - 576x - 60 = 12(144x^2 - 48x - 5) = 12(12x - 5)(12x + 1)$, and since $x = n/2$ we can conjecture that for n even we have $c_n = 12(6n - 5)(6n + 1)/(n(n - 1))$.

Similarly, we now consider the odd terms and ignore the first:

[2244, 8004, 17220, 29892, 46020, 65604, 88644, 115140, 145092]

Once again, we see that it is $1728x^2 + 576x - 60 = 12(12x + 5)(12x - 1)$, and since $x = (n - 1)/2$ we can conjecture that for n odd, $n \geq 3$ we have $c_n = 12(6n - 1)(6n - 7)/(n(n - 1))$, We can put both formulas together by conjecturing that $c_n = 12(6n + (-1)^n)(6n - 6 + (-1)^n)/(n(n - 1))$, in other words

$$c_n = 12 \left(6 + \frac{(-1)^n}{n}\right) \left(6 + \frac{(-1)^n}{n-1}\right) .$$

This formula is indeed *proved* in the paper of Kaneko–Zagier.

(4). To compute the constants a_n and b_n for $n \leq 2$ we come back to the computation of A_2 and A_3 done above. By definition, we have $A_2(X) = (X -$

$a_1)A_1(X) - b_1A_0(X)$, hence $X^2 - 1640X + 269280 = (X - a_1)(X - 720) - b_1$. Identifying the term in X gives $-1640 = -a_1 - 720$, so $a_1 = 920$, and identifying constant terms gives $269280 = 720 \cdot 920 - b_1$, so $b_1 = 393120$. We can do a similar computation for a_2 and b_2, but being a mathematician I am lazy so I will first do the next question, which will give me the answer to this one.

(5). By the previous exercises, we have $a_n = c_{2n} + c_{2n+1}$ and $b_n = c_{2n-1}c_{2n}$. Thus, using the conjectural formula found above we deduce that for $n \geq 1$ we have

$$\begin{aligned} a_n &= 12((6 + 1/(2n))(6 + 1/(2n - 1)) + (6 - 1/(2n + 1))(6 - 1/(2n))) \\ &= 24(144n^2 - 29)/((2n - 1)(2n + 1)) \end{aligned}$$

(which indeed gives $a_1 = 920$), and for $n \geq 2$ we have

$$\begin{aligned} b_n &= 144(6 + 1/(2n))(6 + 1/(2n - 1))(6 - 1/(2n - 1))(6 - 1/(2n - 2)) \\ &= 36(12n + 1)(12n - 5)(12n - 7)(12n - 13)/(n(n - 1)(2n - 1)^2)\ , \end{aligned}$$

while for $n = 1$, since $c_1 = 720$ and $c_2 = 546$, we have $b_1 = 720 \cdot 546 = 393120$, which is what we found above.

To finish the previous question, using these conjectural formulas (which are true up to $n = 10$ since we computed c_n for $n \leq 20$), we find that $a_2 = 4376/5$ and $b_2 = 177650$.

(6). Using the above (conjectural) values of a_n and b_n, we find immediately the following table:

$$\begin{aligned} A_0(X) &= 1 \\ A_1(X) &= X - 720 \\ A_2(X) &= X^2 - 1640X + 269280 \\ A_3(X) &= X^3 - (12576/5)X^2 + 1526958X - 107765856 \\ A_4(X) &= X^4 - 3384X^3 + 3528552X^2 - 1133263680X + 44184000960 \\ A_5(X) &= X^5 - (12752/3)X^4 + 6276237X^3 - 3725740832X^2 \\ &\quad + 743683026790X - 18343724398560 \\ A_6(X) &= X^6 - (56280/11)X^5 + (107473392/11)X^4 - 8530590848X^3 \\ &\quad + 3313730346654X^2 - 451680528901680X + 7674347243833920\ . \end{aligned}$$

Exercise 7.

(1). Recall that we have shown that $b_n =< A_n, A_n > / < A_{n-1}, A_{n-1} >$. Since $< A_0, A_0 >= w_0 = 1$, we deduce that $< A_1, A_1 >= b_1 = 393120$, and for $n \geq 2$:

$$\begin{aligned} < A_n, A_n > &= 36^{n-1}b_1 \prod_{2 \leq j \leq n} (12j + 1)(12j - 5)(12j - 7)(12j - 13)/(j(j - 1)(2j - 1)^2) \\ &= 36^{n-1}b_1/(13 \cdot 7 \cdot 5 \cdot (-1))12^{4n} \cdot \\ &\quad \cdot \prod_{1 \leq j \leq n} (j + 1/12)(j - 5/12)(j - 7/12)(j - 13/12)/(n!(n - 1)! \prod_{1 \leq j \leq n} (2j - 1)^2) \\ &= -12 \cdot 144^n \cdot 12^{4n}(13/12)_n(7/12)_n(5/12)_n(-1/12)_n/((2n)!(2n - 1)!) \\ &= -12^{6n+1}(13/12)_n(7/12)_n(5/12)_n(-1/12)_n/((2n)!(2n - 1)!) \end{aligned}$$

since $\prod_{1\le j\le n}(2j-1)=(2n)/(2^n n!)$.

(2). By Exercise 5 we know that $A_n(0)=A^*_{2n}(0)$, and the recursion for $A^*_{2n}(0)$ is simply $A^*_{2n}(0)=-c_{2n-1}A^*_{2n-2}(0)$, in other words $A_n(0)=-c_{2n-1}A_{n-1}(0)$. Since $A_0(0)=1$, we deduce from the formula $c_1=720$ and $c_{2n-1}=6(12n-7)(12n-13)/((2n-1)(n-1))$ that

$$\begin{aligned}A_n(0)&=(-1)^n720/(6\cdot5\cdot(-1))6^n12^{2n}\cdot\\&\quad\cdot\prod_{1\le j\le n}(j-7/12)(j-13/12)/((n-1)!(2n)!/(2^n n!))\\&=(-1)^{n+1}\cdot12^{3n+1}(5/12)_n(-1/12)_n/(2n-1)!\ .\end{aligned}$$

(3). Set $u_n=A_n(1728)$ and $d_n=u_n/u_{n-1}$. Using the computations done above we compute that the values of d_n for $1\le n\le 6$ are

$$[1008,418,2139/5,430,2585/6,23718/55]\ .$$

As before the denominator 55 suggests that we multiply by $(n-1)(2n-1)$, and the values of $(n-1)(2n-1)d_n$ are now $[0,1254,4278,9030,15510,23718]$. Once again, ignoring the term for $n=1$ we easily find that the other coefficients can be interpolated by $6(12n-5)(12n-13)$, so it is reasonable to conjecture that $d_1=1008$ and $d_n=6(12n-5)(12n-13)/((n-1)(2n-1))$ for $n\ge2$. If this is the case, since $u_0=1$ we have as above

$$\begin{aligned}A_n(1728)=u_n&=6^{n-1}(d_1/(7\cdot(-1)))\cdot\\&\quad\cdot\prod_{1\le j\le n}(12j-5)(12j-13)/((n-1)!(2n)!/(2^n n!)\\&=-12^{3n+1}(7/12)_n(-1/12)_n/(2n-1)!\ .\end{aligned}$$