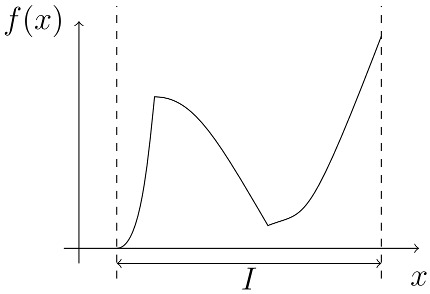
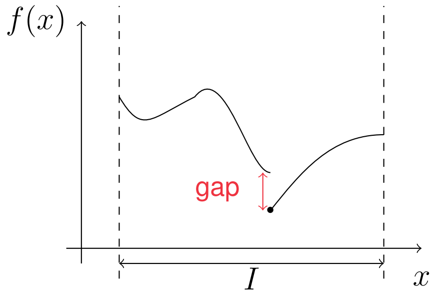
Continuity

1. Let $a = -\infty$ or a real number and b a real number or $b = +\infty$ such that $a < b$. The set of real numbers x verifying $a \leq x \leq b$ (resp. $a \leq x < b$, $a < x \leq b$ or $a < x < b$) is denoted $[a, b]$ (resp. $[a, b[$, $]a, b]$ or $]a, b[$). By definition, any set of this form is an *interval* of $\mathbb{R}$.

2. Let I be an interval of real numbers. A function f defined on I sending x in I to $f(x)$ is *continuous* if one can draw its graph without lifting the pencil. In other words, if there is no gap in the graph of the function. For instance :

   continuous example

   non-continuous example

   It is an intuitive definition. Although we will not write it to avoid technicalities, there is a more rigorous way to define continuous functions.

3. Classical functions such as polynomials (that we have seen last month), trigonometric functions ($\cos$, $\sin$) are continuous. For those who know them, the logarithm and exponential functions ($\ln$ and $\exp$) are continuous as well.

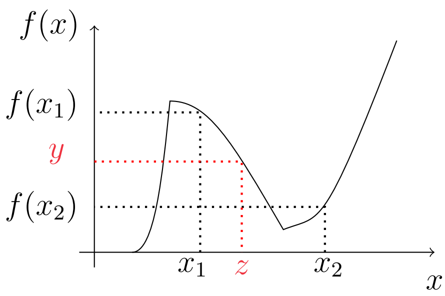
4. The sum/difference/product/quotient of two continuous functions on I is continuous on I (in the latter case, the denominator must not vanish on I for the quotient to be defined).

5. Let f be a function defined on I and g be a function defined on some interval J. Assume that for any x in I, $f(x)$ belongs to J. Then, we define the *composition* of f by g to be the function $g \circ f$: defined on I sending x in I to $g(f(x))$, the image of $f(x)$ by g.[1] If f and g are continuous then $g \circ f$ is continuous.

1. For example, with $f(x) = x^2$ and $g(y) = y + 1$ we find $g \circ f(x) = x^2 + 1$.

6. Using these two facts, you can prove that a function is continuous without drawing it. Most examples of functions that you have seen are continuous functions. For instance, justify that the function sending x to $\frac{2x+1}{x+5}$ is continuous on $]0, +\infty[$ [2].

7. **The intermediate value theorem** Let f be a continuous function on some interval I. Let x_1, x_2 be elements in I and let y be a real number such that $f(x_1) \leq y \leq f(x_2)$. Then there exists a real number z between x_1 and x_2 such that $y = f(z)$. For instance :



The non-continuous function that we have drawn in point 1 does not satisfy the intermediate value theorem, can you see why ?

8. The following consequence of the intermediate value theorem. Let f be a continuous function on some interval I. If the equation $f(x) = 0$ has no solutions on I then the sign of f is constant on I (that is to say either for all x in I, $f(x) > 0$ or for all x in I, $f(x) < 0$). This is usually how you should use continuity of functions in IMO problems.

2. Since both functions $x \mapsto 2x + 1$ and $x \mapsto x + 5$ are polynomials, they are continuous. Remarking that the only root -5 of $x + 5$ is negative and therefore not in $]0, +\infty[$, we have that the quotient of these functions is continuous.

Problem 1 (Prove Fact 8)**.** Let I be an interval and let f be a continuous function on I such that the equation $f(x) = 0$ has no solutions on I. Prove that the sign of f is constant on I.

Problem 2. Let a, b be real numbers such that $a < b$ and let f be a continuous function on $[a, b]$. We assume that $f(a)f(b) < 0$.

1. Show that the equation $f(x) = 0$ has at least one solution on $[a, b]$.
2. If $c := \frac{a+b}{2}$, justify that the equation $f(x) = 0$ has at least one solution on $[a, c]$ or $[c, b]$. Use the sign of $f(c)$ to decide which of these two intervals contains a solution to the equation $f(x) = 0$.
3. Describe an algorithm that takes a continuous function f on $[0, 1]$ such that $f(0)f(1) < 0$ and returns an approximation of a solution to the equation $f(x) = 0$ on $[0, 1]$.

Problem 3. Let f and g be continuous function from $\mathbb{R}$ to $\mathbb{R}$ verifying $f \circ g = g \circ f$ [3]. Suppose furthermore that the equation $f(x) = g(x)$ has no solution. Prove that the equation $f(f(x)) = g(g(x))$ has no solution.

Problem 4. Let f be a continuous function on $[0, 1]$ such that $f(0) = f(1)$. Find x in $\left[0, \frac{4}{5}\right]$ such that $f\left(x + \frac{1}{5}\right) = f(x)$.

Problem 5 (more challenging)**.** Let f be a continuous function on $[0, 1]$ such that for any real number $0 \leq x \leq 0.7$, $f(x + 0.3) \neq f(x)$ and $f(0) = 0 = f(1)$.

1. Show that the equation $f(x) = 0$ has at least seven solutions.
2. Draw an example of such function f.

ANY QUESTION ? JUST ASK !

3. This is the composition of functions we defined in Fact 4.