Functional equations

Many exercises in math contests consist of functional equations : here you find some first tricks and some easy examples !

You are used to equations where the solution is a real number, e.g. $x + 3 = 5$. Possibly there is more than one solution ($x^2 - 9 = 0$) and possibly there is no solution ($x^2 + 9 = 0$). But, in mathematics, you can also write an equation where the solutions are functions. This means that, instead of looking for all real numbers having a certain property, you now look for all functions having a certain property.
For simplicity, we consider only functions $\mathbb{R} \to \mathbb{R}$, so real functions which are defined for all real numbers. Easy examples of such functions are the functions whose graph is a line ($x \mapsto 2x + 5$), and more generally polynomial functions ($x \mapsto 2x^7 + 5x^2 + 1$). In these families you also have the constants ($x \mapsto 5$).

Checking if a function is a solution. With a real number, you know that it is very easy to check if it is a solution to a given equation : you just have to plug the number into the equation and check if you have a true identity or not (2 is a solution to $x^3 - 3 = 5$, but not 7). Similarly, it is very easy to check if a function is a solution to a functional equation : you plug the function into the equation and check if you have a true identity or not.

For example, consider the functional equation

$$2f(x) = f(2x)$$

You can easily check that the constant function zero, and that the function $x \mapsto 7x$ are solutions ($2 \cdot 0 = 0$ and $2 \cdot 7x = 7 \cdot 2x$ are true identities) but the function $x \mapsto x + 1$ is not a solution ($2 \cdot (x + 1) = 2x + 1$ is not a true identity).

On the sets of solutions. Sometimes there are no solutions, for example

$$f(x)^2 = -1$$

has no solution (square numbers of real numbers are non-negative). In this case the set of solutions is empty.
Sometimes there is exactly one solution, for example

$$f(x)^2 = 0$$

implies that $f(x) = 0$ for every x and hence the only possible candidate is the constant function zero. We check that this is a valid candidate ($0^2 = 0$) so there is exactly one solution, namely the constant function zero.

Sometimes there is more than one solution, for example if we take

$$f(x)^2 = 1$$

you can guess correctly (and check) that the constant function 1 and the constant function -1 are both solutions. The set of the solutions contains infinitely many functions, namely those functions whose values are either 1, -1 or both ± 1. For example the function that takes value 1 at the rational numbers and -1 at the irrational numbers is also a solution. Or the function taking value 1 at the numbers ≥ 0 and -1 at the numbers < 0 is also a solution.

Finding all solutions. By various tricks (depending on the functional equation) we write down conditions which tell us that the set of solutions is small : *all solutions must be of a certain form*. To determine the set of solutions exactly we also have to check that *all functions of that specific form are solutions*. Of course, the conditions can be so restrictive as to imply that there is no solution, and in that case no further verification is required.

Trick : renaming the variables. You can rename the variables, for example you can set $t = x + 1$. This helps simplifying the functional equation

$$f(x+1) = x^2 + 2x + 5 \qquad \leftrightarrow \qquad f(t) = t^2 + 4$$

So the solution is the function $x^2 + 4$. Recall that all functions we consider are defined over $\mathbb{R}$, otherwise we should take care of the domain of the function after such a substitution.

Trick : special cases give easier conditions. If you consider a functional equation for a function $f(x)$ then considering $x = 0$ will probably give you information about the value $f(0)$. For example, for

$$f(x) + f(2x) = 2$$

the special case $x = 0$ will give you $f(0)+f(0) = 2$ and hence $f(0) = 1$. But, for example, with

$$f(x) = f(-x)$$

we have absolutely no information on $f(0)$.
This special cases are especially interesting if more variables appear in the functional equation (which, for simplicity, we suppose can vary over all real numbers $\mathbb{R}$) : for example with

$$f(x) + f(y) = x + 3y$$

you can try the special cases $x = 0$, but also $x = y$ and so on... In the special case $x = y$ you have $f(x) + f(x) = 4x$ and hence the solution must be of the form $f(x) = 2x$. Careful, this is not a solution because $2x+2y = x+3y$ does not hold for all real numbers x, y (in fact, it only holds if $x = y$).

Which special case is helpful depends on the functional equation, and with the time you may develop some sort of intuition. Sometimes it may be even helpful to try the special case $x = f(y)$, as for example in

$$f(y) = x + 2y$$

This gives you $f(y) = f(y) + 2y$ and hence $y = 0$, which cannot be because y varies over all real numbers (in the special case $x = f(y)$ the variable x is determined from y but y is free). So there are no solutions !

Trick : symmetry. For example with

$$f(x) + f(y) = x + 3y$$

the left-hand side is invariant by swapping x and y, so the right-hand side should also have this property, and we get $x + 3y = y + 3x$. This is only possible if $x = y$ but since x and y are free to vary over $\mathbb{R}$ we deduce that there is no solution to the functional equation.

1. Check if $f(x) = 2x + 5$ ($f : \mathbb{R} \to \mathbb{R}$) is a solution of the equation

   $$f(x) + 2 = f(x + 1)\,.$$

   Are there solutions $f(x)$ such that $f(0) = f(1) = 0$?
2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x + y) = x + f(y)$.
3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(x + 2y) = x + 1 + 2f(y)$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

   $$f(x + y) + f(x + f(y)) = f(f(x + y))$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying

   $$f(x^2 + y) + f(1) = 2f(x)y + f(y - 1) + f(x^2)$$

   and such that $f(0) = 1$. What is $f(2)$?

ANY QUESTION ? JUST ASK !