Solutions: Problems from mathematical contests

1. **Manhattan Mathematical Olympiad 2003**
   Prove that from any set of one hundred integers, one can choose either one number which is divisible by 100, or several numbers whose sum is divisible by 100.

Hint: Label the numbers in the set $x_1, \dots, x_{100}$, consider the 100 subsets

$$\{x_1\}, \{x_1, x_2\}, \dots, \{x_1, \dots, x_{100}\}$$

and for each of these subsets, compute its sum.

Proof. As suggested by the hint, we label the 100 numbers by $x_1, \dots, x_{100}$, and consider the 100 subsets $\{x_1\}, \{x_1, x_2\}, \dots, \{x_1, \dots, x_{100}\}$. For each of these subsets, we compute the sum of its elements, that is, for each $1 \leq n \leq 100$, we set

$$S_n := x_1 + \dots + x_n \,.$$

If for some n, S_n is divisible by 100, then the proof is done. Now let us assume that S_n is NOT divisible by 100, for ANY $n = 1, \dots, 100$, and we define 99 boxes of integers by setting ($j = 1, 2, \dots, 99$)

$$B_j = \{ z : \ z \text{ is an integer such that } z - j \text{ is divisible by } 100 \} \,.$$

Therefore by *Pigeonhole principle*, we can find $1 \leq n < m \leq 100$ such that S_n, S_m lie in the same box B_j for some j. In other words, both $S_n - j$ and $S_m - j$ are divisible by 100, so is $S_m - S_n$. Note that

$$S_m - S_n = x_{n+1} + \dots + x_m$$

so that the proof is completed. □

2. **Japan 1997*** Prove that among any ten points located on a circle with diameter 5, there exist at least two at a distance less than 2 from each other.

Hint: Inscribe a regular 9-gon in a circle, and consider the length of an arc.

Proof. Inscribe a regular 9-gon in this circle, whose diameter is 5 (so its circumference is 5π) and this 9-gon will divide the circle into 9 equal arcs, each of which has length $\frac{5\pi}{9} = 1.745... < 2$. Since a straight segment connecting two points is the shortest path that connects them,

any two points on the same arc are within distance less than 2. $(\star)$

Now let us think about the 9 arcs as our boxes, on which the given ten points are located. It follows from the *pigeonhole principle* that one of the arcs contains at least two of the points and the proof is concluded by the observation $(\star)$. □

3. **IMO 1959** Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n.

Hint: Observe that $3(14n+3) - 2(21n+4) = 1$.

Proof. Arguing by contradiction, we assume that $\dfrac{21n+4}{14n+3}$ is reducible for some natural number n, that is, $21n+4$, $14n+3$ have common factor $p > 1$. It follows that one can find two natural numbers k, ℓ such that $21n+4 = pk$, $14n+3 = p\ell$ implying that

$$2 \times 21n + 2 \times 4 = 2pk\,, \quad 3 \times 14n + 3 \times 3 = 3p\ell$$

that is, $42n + 8 = 2pk$ and $42n + 9 = 3p\ell$. Subtracting two equations gives us $1 = (3\ell - 2k)p$, which is a contradiction. □

4. **IMO 1981***** Determine the maximum value of m^2+n^2, where m and n are integers in the range $1, 2, \ldots, 1981$ satisfying

$$(n^2 - mn - m^2)^2 = 1\,.$$

Hints:

(1) Show that $m \leq n \leq 2m$ if n and m positive integers.

(2) Show that m and n are relatively primes.

(3) Show first the pair (m, n) satisfies (#) if and only if the pair $(n-m, m)$ satisfies (#).

(4) Show that starting with a pair (m, n) $(m, n > 0)$, then after finitely many steps we reach $(1, 1)$.

(5) Show that (m, n) is of the form (F_k, F_{k+1}), where F_k's are the Fibonacci numbers defined inductively by $F_1 = 1, F_2 = 1, F_k = F_{k-1} + F_{k-2}$ for every $k \geq 3$

Solution: Now we verify (1) of the hints. If $m > n \geq 1$, then $m^2 + mn - n^2 > mn > 1$, so that (m, n) can not satisfy (#).

Similarly, if $n > 2m \geq 1$, then

$$n^2 - mn + m^2 = n(n - m) + m^2 > m^2 \geq 1.$$

Hence (#) cannot be satisfied. Thus $m \leq n \leq 2m$ if $m, n \geq 1$.

If d is a common divisor of m and n, then d^4 divides the right-hand side of (#), and thus, d^4 divides 1. Hence, m, n must be relatively primes.

Now let us verify (3) of the hints, assuming first that $m \leq n$:

$$\left(n^2 - mn - m^2\right)^2 = 1 \iff \left(n(n - m) + m^2\right)^2 = 1 \iff$$
$$\left((n - m)^2 + m(n - m) + m^2\right)^2 = 1 \iff \left[m^2 - m(n - m) - (n - m)^2\right]^2 = 1\,,$$

this gives us the equivalence stated in (3).

We know that $m \leq n \leq 2m$ for every $n \geq 1, m \geq 1$, thus $n - m \geq 1$, if $n \neq 1$. Since the pair $(n - m, m)$ satisfies (#) and $n - m, m \geq 1$, we get from (1) that, $n - m \geq m \geq 2(n - m)$. Given any positive pair (m, n) of solutions with $n \neq m$, applying (1) we get positive solutions $(n - m, m)$, $(2m - n, n - m)$, $\ldots$

Since m, n are relatively primes, thus $n - m$ and m are also relatively primes. If they are not equal then we can continue the process. Since the pairs (n, m), $(n - m, m)$, $(2m - n, n - m)$, $\ldots$ are decreasing in each term. After finitely many steps the process must finish. Thus, there exists a pair (l, l) and l must be positive and relatively prime to itself, thus $l = 1$.

This argument shows that if we start with an arbitrary pair (m, n) , $n, m \geq 1$, then following the previous argument after finitely many steps we reach the pair $(1, 1)$.

Now consider the first few (m, n) pairs.

$$(m, n) = (1, 1) = (F_1, F_2),$$
$$(m, n) = (1, 2) = (F_2, F_3),$$
$$(m, n) = (2, 3) = (F_3, F_4),$$
$$(m, n) = (3, 5) = (F_4, F_5),$$
$$(m, n) = (5, 8) = (F_5, F_6),$$
$$(m, n) = (8, 13) = (F_6, F_7),$$
$$\ldots\ldots$$

where F_k are the Fibonacci numbers defined inductively by $F_1 = 1, F_2 = 1, F_k = F_{k-1} + F_{k-2}$ for every $k \geq 3$. Clearly, if $(n-m, m) = (F_k, F_{k+1})$, then $(m, n) = (F_{k+1}, F_{k+2})$. Hence we can say that all solutions (m, n) with natural numbers m, n is of the form (F_k, F_{k+1}).

Because we are restricted to consider only $1 \leq m \leq n \leq 1981$, by direct computation, we have $F_{16} = 987$, $F_{17} = 1597$ and $F_{18} = 2594$. Hence the desired maximum value will be

$$987^2 + 1597^2 = 3524578.$$

□