The induction principle

Principle of mathematical induction
Raisonnement par récurrence
Vollständige Induktion oder Induktionsprinzip

The induction principle is the domino effect in mathematics! The domino effect is the chain reaction consisting of a row of falling dominoes. The dominoes are vertical and close enough to one another. One pushes the first domino of the row, and this falls onto the second domino, which falls onto the third domino and so on. In the end the whole row has fallen! You can find spectacular videos with thousands of falling dominoes...

In mathematics one uses the induction principle as a proof method.

- The dominoes are the cases of the proof. 'A domino has fallen' means that the case has been proven. When all dominoes have fallen, the proof is complete. In mathematics we can also consider infinitely many dominoes.
- **Base case:** One has to push the first domino, this means that *we need to prove the first case*.
- **Induction step:** One falling domino pushes the next because the dominoes are close together. *We have to prove that from one case we can deduce the next one.*

Some general facts if you want to apply the induction principle to solve a mathematical problem:

- *You need a statement that is a neat collection of cases* (possibly infinitely many). For example: Prove that for all natural numbers $n \geq 4$ we have $n^2 \leq 2^n$.
- *You need a base case.* For statement about the natural numbers this is usually the smallest number appearing in the statement. In the above example you would take $n = 4$.
- *You need an induction step.* One takes one case as known, and uses it to prove the next case. Supposing that the statement holds for n, we prove it for $n + 1$. The statement for n is also called *the induction hypothesis*.
  You should think where you have used the validity of case n: it is very suspicious if you did not use it at all!

Example: Let us prove that for all natural numbers $n \geq 4$ we have $n^2 \leq 2^n$.

Base case: For $n = 4$ the inequality $4^2 \leq 2^4$ holds (we have $4^2 = 16 = 2^4$).

Induction step (from n to $n+1$, with $n \geqslant 4$):
We take any $n \geqslant 4$, we fix it and we suppose that the statement holds for this n, i.e. $n^2 \leq 2^n$ holds. We need to prove the statement for $n+1$, i.e. that $(n+1)^2 \leq 2^{n+1}$ holds. We have:

$$2^{n+1} = 2 \cdot 2^n \geq 2 \cdot n^2 = n^2 + n \cdot n \geq n^2 + 4n = n^2 + 2n + 2n \geq n^2 + 2n + 2 \cdot 4 \geq n^2 + 2n + 1 = (n+1)^2$$

Check: We have used the statement for n to write $2 \cdot 2^n \geq 2 \cdot n^2$.

The Gauss sum formula: Let us prove by induction that the sum of all natural numbers from 1 to n equals $\frac{n(n+1)}{2}$ (notice that this fraction is an integer).

It is implicit that n is a natural number greater than or equal to 1. So the base case is $n = 1$: the sum of all numbers from 1 to 1 is 1, and this is the same as $\frac{1(1+1)}{2} = 1$.
Now consider the induction step $n \rightsquigarrow n+1$ for $n \geqslant 1$. We know that the sum of the numbers from 1 to n equals $\frac{n(n+1)}{2}$. To obtain the sum of the numbers up to $n+1$ we simply have to add the missing summand $n+1$:

$$\frac{n(n+1)}{2} + n + 1 = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

We thus obtained the statement for $n+1$, which says that the sum of all natural numbers from 1 to $n+1$ equals $\frac{(n+1)((n+1)+1)}{2}$. Check: In the induction step we have used the assertion for n to evaluate the sum of the first n numbers.

There are other methods for proving the formula. For example, we can evaluate twice the sum by writing the numbers as follows:

$$\begin{array}{ccccc} 1 & 2 & \dots & n-1 & n \\ n & n-1 & \dots & 2 & 1 \\ \hline n+1 & n+1 & \dots & n+1 & n+1 \end{array}$$

We have n columns and the sum in each column is $n+1$, thus twice the requested sum is $n(n+1)$.

Problems about the induction principle

1. **The sum of the first squares:** Prove by induction that the sum of the squares of the natural numbers from 1 to n equals
$$\frac{n(n+1)(2n+1)}{6}$$
(by the way, why is this fraction an integer?).

2. **The sum of the first odd numbers:** Guess a formula for the sum of the first n odd natural numbers, and prove it by induction.

3. **The Bernoulli inequality:** Prove that for all real numbers $x \geq -1$ and for all natural numbers $n \geq 1$ we have
$$(1+x)^n \geq 1 + nx\,.$$

4. **Find the flaw in the reasoning:** We prove by induction that all roses world-wide have the same color (every rose has one defined color). We consider $n \geq 1$ roses.

   - Base case: If $n = 1$ then clearly this one rose has just one color.
   - Induction step: If any n roses have the same color, then we prove that any $n+1$ roses have the same color.
     Take $n+1$ roses: The first n roses have the same color, and the last n roses have the same color. The two colors must be the same and hence all $n+1$ roses have the same color. (Visualize this reasoning for $n = 4$).

ANY QUESTION? JUST ASK!